Chapter six

Sequences and Series

6.1 Sequences

Definition (6.1.1)

Let X be a nonempty set. By a sequence on X we mean a function $f: \mathbb{N} \longrightarrow X$ such that $f(n) \in X \ \forall n \in \mathbb{N}$. We denote to sequence by $\{f(n)\}$, $\{a_n\}$, $a(n)$ or a_n.

Example (1)

Write all terms of the sequence in the following:

① $a_n = \frac{1}{n}$ ② $a_n = n-1$ ③ $a_n = (-1)^n$

Solve

① $a_n = \frac{1}{n}$, $\forall n \in \mathbb{N}$

$\Rightarrow a_n = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3}$, ----, $a_n = \frac{1}{n}$, ...

$\Rightarrow \langle a_n \rangle = \langle 1, \frac{1}{2}, \frac{1}{3}, ---, \frac{1}{n}, --- \rangle$

② $a_n = n-1 \quad \forall n \in \mathbb{N}$

$a_1 = 0 \;,\; a_2 = 1 \;,\; a_3 = 2 \;, \dots, a_n = n-1, \dots$

$\langle a_n \rangle = \langle 0, 1, 2, \dots, n-1, \dots \rangle$

③ $a_n = (-1)^n \quad \forall n \in \mathbb{N}$

$a_1 = -1 \;,\; a_2 = 1, \dots, a_n = (-1)^n, \dots$

$\langle a_n \rangle = \langle -1, 1, -1, 1, \dots, (-1)^n, \dots \rangle$

– The graph of the sequence

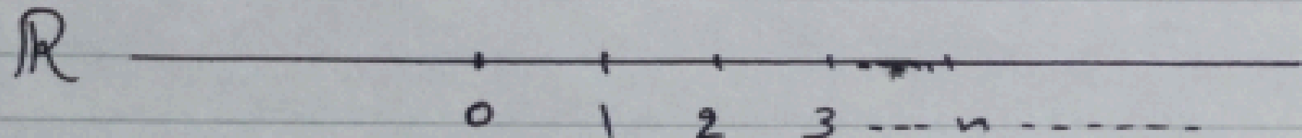
There are two ways to represent the sequence as we shown in the following example

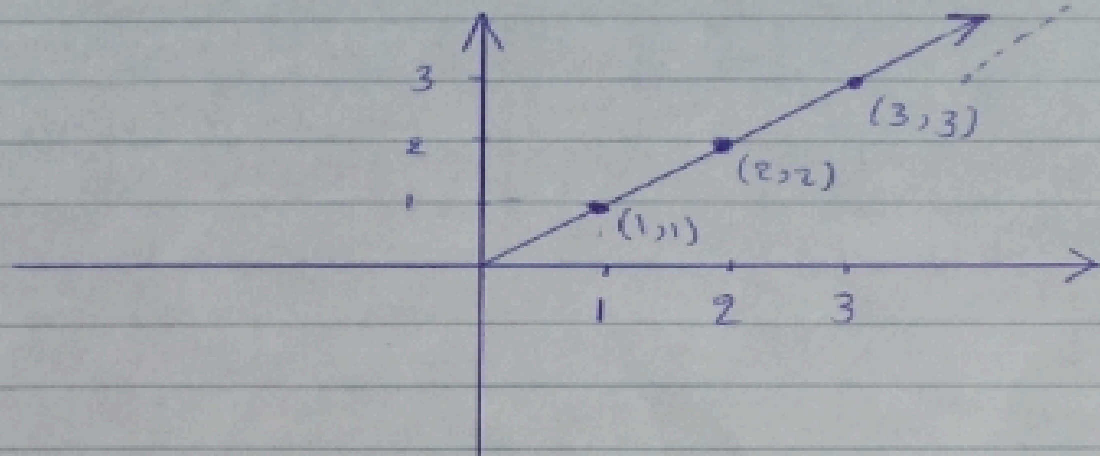
Example (2)

Gragh the following sequences:

① $a_n = n \quad \forall n \in \mathbb{N}$

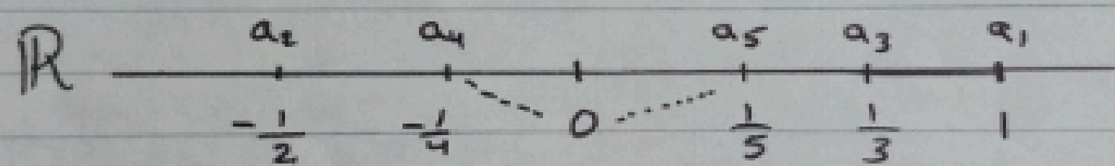
$\langle a_n \rangle = \langle 1, 2, 3, \dots, n, \dots \rangle$

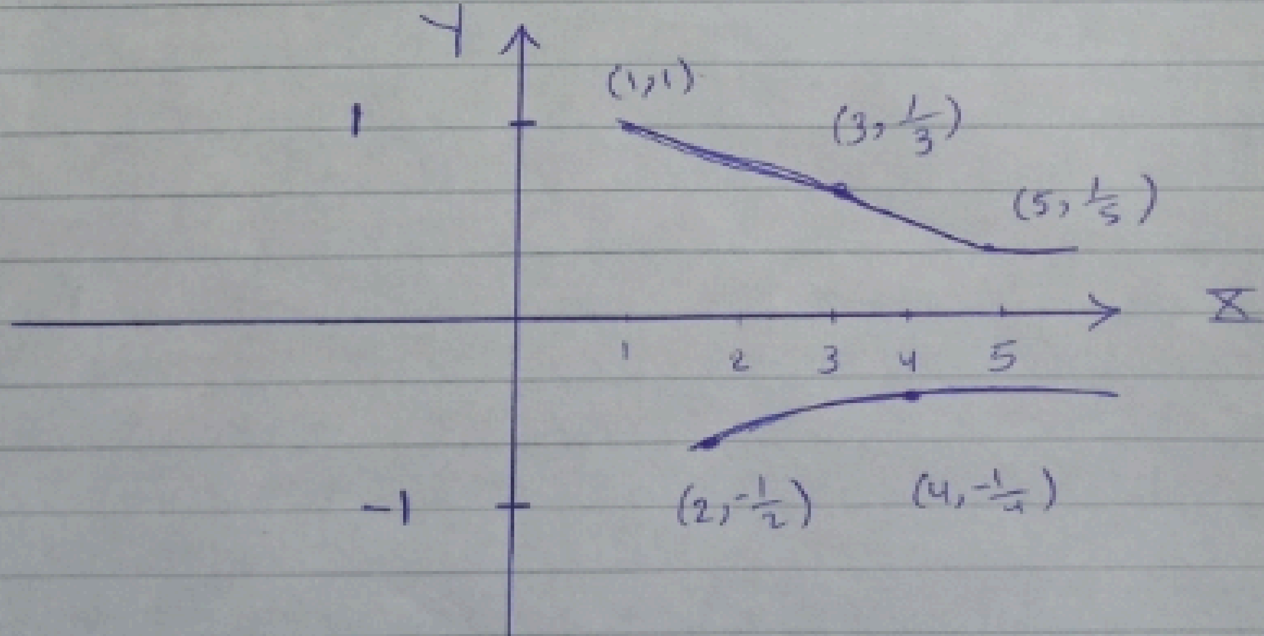




② $a_n = (-1)^{n+1} \frac{1}{n} \quad \forall n \in \mathbb{N}$

$\langle a_n \rangle = \langle 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots\ldots \rangle$





- Convergent and Divergent Sequences

Definition (6.1.2)

Let $\langle a_n \rangle$ be a sequence of real numbers and let L be a constant, then $\langle a_n \rangle$ is said to be convergent to L if :

$$\forall \epsilon > 0, \exists K \in \mathbb{N} \ni |a_n - L| < \epsilon \quad \forall n > K$$

and it is denoted by : $a_n \longrightarrow L$

That means $\lim\limits_{n \to \infty} a_n = L$

and L is called the limite of the sequence

If the limite of the sequence is not exist then we say that $\langle a_n \rangle$ is divergent.

Theorems (6.1.3)

① The limite of the sequence is unique.

② If $\langle a_n \rangle$, $\langle b_n \rangle$ and $\langle c_n \rangle$ are three

Sequences, such that $\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L$

and $a_n \leq b_n \leq c_n$, then $\lim_{n \to \infty} b_n = L$.

③ Let $f: D \longrightarrow E$ be a continuous function where $D, E \subseteq \mathbb{R}$, and let $\langle a_n \rangle$ be a sequence on D such that $a_n \longrightarrow L$, $L \in D$, then

$f(a_n) \longrightarrow f(L)$.

④ If f and g are two differentiable functions at the point x_0, then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} .$$

This rule is called the Lhopetal's rule.

This rule is used when the following terms appear : $\frac{\infty}{\infty}$, $\frac{0}{0}$, $\infty - \infty$, 0^0, 1^∞, ∞^0, ...

— Properties of the convergent sequences:

Theorem (6.1.4)

Let $\langle a_n \rangle$, $\langle b_n \rangle$ be two convergent sequences such that:

$\lim_{n \to \infty} a_n = A$ & $\lim_{n \to \infty} b_n = B$, then

① $\lim_{n \to \infty} (a_n \mp b_n) = A \mp B$

② $\lim_{n \to \infty} (K a_n) = K \lim_{n \to \infty} a_n = KA$

③ $\lim_{n \to \infty} (a_n b_n) = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n = AB$

④ $\lim_{n \to \infty} \dfrac{a_n}{b_n} = \dfrac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n} = \dfrac{A}{B}$

Examples about convergence and divergence

① $a_n = \frac{1}{n}, \forall n \in \mathbb{N}$

$$\langle a_n \rangle = \langle 1, \frac{1}{2}, \frac{1}{3}, ----, \frac{1}{n}, --- \rangle$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = \frac{1}{\infty} = 0$$

$$\Rightarrow a_n \longrightarrow 0$$

② $a_n = \frac{1}{2^{n-1}}, \forall n \in \mathbb{N}$

$$\langle a_n \rangle = \langle 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ---- \rangle$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{2^{n-1}} = \lim_{n \to \infty} \frac{1}{2^n \cdot 2^{-1}}$$

$$= \lim_{n \to \infty} \frac{2}{2^n} = \frac{2}{2^\infty} = \frac{2}{\infty} = 0$$

$$\therefore a_n \longrightarrow 0$$

③ $a_n = n - 1, \forall n \in \mathbb{N}$

$$\langle a_n \rangle = \langle 0, 1, 2, 3, --- \rangle$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} n - 1 = \infty - 1 = \infty$$

$\therefore \langle a_n \rangle$ is divergent

④ $\langle a_n \rangle = \dfrac{n^2-n}{3n^2+n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2-n}{3n^2+n} = \lim_{n\to\infty} \frac{\frac{n^2}{n^2}-\frac{n}{n^2}}{\frac{3n^2}{n^2}+\frac{n}{n^2}}$$

$$= \lim_{n\to\infty} \frac{1-\frac{1}{n}}{3+\frac{1}{n}} = \frac{1-\frac{1}{\infty}}{3+\frac{1}{\infty}} = \frac{1-0}{3+0} = \frac{1}{3}$$

$\therefore \quad a_n \longrightarrow \frac{1}{3}$

⑤ $a_n = 1+\dfrac{(-1)^n}{n}, \ \forall n \in \mathbb{N}$

$$\langle a_n \rangle = \left\langle 0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{4}{5}, \ldots \right\rangle$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1+\frac{(-1)^n}{n} = \lim_{n\to\infty} \begin{cases} 1+\frac{1}{n} & n \text{ even} \\ 1-\frac{1}{n} & n \text{ odd} \end{cases}$$

$$= \begin{cases} 1+\frac{1}{\infty} & n \text{ even} \\ 1-\frac{1}{\infty} & n \text{ odd} \end{cases}$$

$$= \begin{cases} 1+0 & n \text{ even} \\ 1-0 & n \text{ odd} \end{cases}$$

$\therefore \quad \lim_{n\to\infty} a_n = 1$

$\therefore \quad a_n \longrightarrow 1$

Some Special Cases

1. $\lim_{n \to \infty} \sqrt[n]{n} = 1$

2. $\lim_{n \to \infty} (1 \mp \frac{x}{n})^n = e^{\mp x}$

3. $\lim_{n \to \infty} x^n = 0 \; ; \quad |x| < 1$

4. $\lim_{n \to \infty} \frac{\ln n}{n} = 0$

5. $\lim_{n \to \infty} \sqrt[n]{x} = 1 \quad , \; x > 0$

6. $\lim_{n \to \infty} \frac{x^n}{n!} = 0$

Excercises

1. Find $\lim_{n \to \infty} \frac{\cos(n)}{n}$. (Use theorem(6.1.3)(2))

2. Find $\lim_{n \to \infty} \sqrt{\frac{n+1}{n}}$. (Use theorem(6.1.3)(3))

3. Find $\lim_{n \to \infty} \frac{\ln n}{n}$, (Use Lhopetal's rule)

4. Find $\lim_{n \to \infty} \sqrt[n]{2}$. (Use theorem(6.1.3)(3))

5. Find $\lim_{n \to \infty} (\ln n - \ln(n+1))$. (Use Lhopetal's rule)

DEFINITIONS

Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots \tag{1}$$

is called an **infinite series.** The number a_n is called the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \end{aligned} \tag{2}$$

is the **sequence of partial sums** of the series, the number s_n being the ***n*th partial sum.** If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L. \tag{3}$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges.**

Examples: Show that the following series are convergent or divergent

① $a_n = 1, \forall n$

solution

$S_1 = a_1 = 1$

$S_2 = a_1 + a_2 = 1 + 1 = 2$

$S_3 = a_1 + a_2 + a_3 = 1 + 1 + 1 = 3$

$\vdots$

$S_n = a_1 + a_2 + \cdots + a_n$

$= \underbrace{1 + 1 + \cdots + 1}_{n\text{-time}}$

$\therefore S_n = n$

The sequence of partial sum $\{S_n\}$

$\lim_{n \to \infty} S_n = \lim_{n \to \infty} n = \infty$

$\therefore \{S_n\}$ is divergent

$\therefore \sum_{n=1}^{\infty} a_n$ is divergent

(2) $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{3}{(10)^n}$

solution

$S_1 = a_1 = \frac{3}{(10)^1} = \frac{3}{10}$

$S_2 = a_1 + a_2 = \frac{3}{(10)^1} + \frac{3}{(10)^2}$

$S_3 = a_1 + a_2 + a_3 = \frac{3}{10} + \frac{3}{(10)^2} + \frac{3}{(10)^3}$

$\vdots$

$S_n = a_1 + a_2 + \cdots + a_n$

$S_n = \frac{3}{10} + \frac{3}{(10)^2} + \cdots + \frac{3}{(10)^n} \quad \text{---} (1)$

$\frac{1}{10} S_n = \frac{3}{(10)^2} + \frac{3}{(10)^3} + \cdots + \frac{3}{(10)^{n+1}}$

$S_n - \frac{1}{10} S_n = \frac{3}{10} - \frac{3}{(10)^{n+1}}$

$\left(1 - \frac{1}{10}\right) S_n = \frac{3}{10}\left(1 - \frac{1}{10^n}\right)$

$(10 - 1) S_n = 3\left(1 - \frac{1}{10^n}\right)$

$9 S_n = 3\left(1 - \frac{1}{10^n}\right)$

$S_n = \frac{3}{9}\left(1 - \frac{1}{(10)^n}\right)$

$\therefore S_n = \frac{1}{3}\left(1 - \frac{1}{(10)^n}\right)$

$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1}{3}\left(1 - \frac{1}{(10)^n}\right) = \frac{1}{3}$

$\therefore \{S_n\}$ convergent $\Rightarrow \sum_{n=1}^{\infty} \frac{3}{(10)^n} = \frac{1}{3}$ conv

③ $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$S_1 = a_1 = 1 - \frac{1}{2}$

$S_2 = a_1 + a_2 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3}$

$S_3 = a_1 + a_2 + a_3 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4}$
$= 1 - \frac{1}{4}$

$S_n = a_1 + a_2 + \cdots + a_n = 1 - \frac{1}{n+1}$

$\{S_n\}$ sequence of partial sum

$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1$

$\therefore$ $\{S_n\}$ is convergent sequence

$\therefore$ $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent series

and $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

Geometric Series

Geometric series are series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \tag{4}$$

in which a and r are fixed real numbers and $a \neq 0$. The **ratio** r can be positive, as in

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} + \cdots, \tag{5}$$

or negative, as in

$$1 - \frac{1}{3} + \frac{1}{9} - \cdots + (-1)^{n-1}\frac{1}{3^{n-1}} + \cdots. \tag{6}$$

If $r = 1$, the nth partial sum of the geometric series in (4) is

$$s_n = a + a(1) + a(1)^2 + \cdots + a(1)^{n-1} = na$$

and the series diverges because $\lim_{n\to\infty} s_n = \pm\infty$. If $r \neq 1$, we can determine the convergence or nonconvergence of the series in the following way. We multiply the nth partial sum

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

by r, obtaining

$$rs_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n.$$

We then subtract s_n from rs_n. Most of the terms on the right cancel when we do this, leaving only

$$s_n - rs_n = a - ar^n \quad \text{or} \quad s_n(1 - r) = a(1 - r^n). \tag{7}$$

We solve for s_n, obtaining

$$s_n = \frac{a(1 - r^n)}{(1 - r)}, \quad (r \neq 1). \tag{8}$$

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$ (as we saw in Section 9.1), and $s_n \to a/(1 - r)$. In other words, the series converges to $a/(1 - r)$. If $|r| > 1$, then $|r^n| \to \infty$ and the series diverges.

> If $|r| < 1$, the geometric series converges and
>
> $$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r}. \tag{9}$$
>
> If $|r| \geq 1$, the series diverges.

Examples: Determine which of the following convergent and which of them divergent:

1) $3+6+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\dots$

$= 3+5+\underbrace{1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\dots}_{\text{Geometric series}} \quad \text{---} (1)$

$a=1$ & $r=\frac{1}{3}$

$\Rightarrow |r| = |\frac{1}{3}| < 1 \Rightarrow$ convergent

$$\sum_{n=1}^{\infty} ar^n = \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{1-\frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

$$\Rightarrow 3+5+\underbrace{1+\frac{1}{3}+\frac{1}{3^2}+\dots}_{\frac{3}{2}}$$

$$= 3+5+\frac{3}{2} = 9\frac{1}{2}$$

$\therefore\ 3+6+\frac{1}{3}+\frac{1}{(3)^2}+\dots$ is convergent series.

② $1+(2)(7)+(2)(7)^2+(2)(7)^3+\cdots$

$=(-1+1)+1+(2)(7)+(2)(7)^2+(2)(7)^3+\cdots$

$=-1+\underbrace{2+(2)(7)+(2)(7)^2+(2)(7)^3+\cdots}$

Geometric Series

$a=2$ & $r=7$

$\Rightarrow |r|=|7|=7>1$

$\therefore \sum_{n=1}^{\infty} ar^n = \sum_{n=1}^{\infty} 2(7)^n$ is divergent series

$\Rightarrow -1+\sum_{n=1}^{\infty} 2(7)^n$ is divergent series

③ $\sum_{n=0}^{\infty} \frac{2^n}{5^n} = 1+\frac{2}{5}+\frac{2^2}{5^2}+\frac{2^3}{5^3}+\cdots$

$=\underbrace{1+\frac{2}{5}+\left(\frac{2}{5}\right)^2+\left(\frac{2}{5}\right)^3+\cdots}$

Geometric Series

$a=1$ & $r=\frac{2}{5}$, $|r|=\frac{2}{5}<1$

$\therefore \sum_{n=0}^{\infty} \frac{2^n}{5^n}$ is convergent series

and $\sum_{n=0}^{\infty} \frac{2^n}{5^n} = \frac{1}{1-\frac{2}{5}} = \frac{1}{\frac{3}{5}} = \frac{5}{3}$

(4) $1-2+4-8+\ldots$

Geometric series

$a=1$ & $r=-2$

$|r|=|-2|=2>1$

$\therefore$ $1-2+4-8+\ldots$ is divergent series

6.3 Positive series Convergence test

Definition (6.3.1)

Let $\sum_{n=0}^{\infty} a_n$ be an infinite series, $\sum_{n=0}^{\infty} a_n$ is said to be positive series if $a_n \geqslant 0 \ \forall n \in \mathbb{N}$.

Therefore, the partial sumes sequence is called an increasing sequence, where

$S_1 = a_0$, $S_2 = a_0 + a_1$, $S_3 = a_0 + a_1 + a_2$, ----

$S_1 \leq S_2 \leq S_3 \leq \text{---} \leq S_n \leq \text{---}$.

- Some special series

① Harmonic Series

This series is always divergent and its general form is $\sum_{n=1}^{\infty} \frac{1}{n}$. Therefore, we can conclude that the following series are divergent:

$\sum_{n=1}^{\infty} \frac{3}{n}$, $\sum_{n=1}^{\infty} \frac{1}{9n}$, $\sum_{n=1}^{\infty} \frac{2}{5n}$, ----

② P - Series

The general form of this series is $\sum_{n=1}^{\infty} \frac{1}{n^p}$;

p is a positive real number.

The convergence and divergence of this series depend on the value of p, where:

* If $p > 1$, then the series is convergent.
* If $p < 1$, then the series is divergent.
* If $p = 1$, then the series is Harmonic, and so is divergent.

Examples

① $\sum_{n=1}^{\infty} \frac{1}{13n}$

$\Rightarrow \quad \frac{1}{13} \sum_{n=1}^{\infty} \frac{1}{n}$ ← Harmonic series (divergent)

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{13n}$ is divergent

② $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ ← P-series ; $P = \frac{1}{2} < 1$ (divergent)

- Convergence Tests

1- Comparison Test

Let $\sum a_n$ be a positive infinite series :

* $\sum a_n$ is convergent if there exists a convergent positive infinite series $\sum c_n$ such that

  $a_n \leqslant c_n \quad \forall n \in \mathbb{N}$.

* $\sum a_n$ is divergent if there exists a divergent positive infinite series $\sum b_n$ such that

  $a_n \geqslant b_n \quad \forall n \in \mathbb{N}$.

- Compasison test by the limit

Let $\sum a_n$, $\sum b_n$ be two positive infinite Series

such that $\quad \lim\limits_{n \to \infty} \dfrac{a_n}{b_n} = t \neq 0$, t real number

then:

① $\sum a_n$ is convergent $\Leftrightarrow$ $\sum b_n$ is convergent

② $\sum a_n$ is divergent $\Leftrightarrow$ $\sum b_n$ is divergent

2- Ratio Test

Let $\sum a_n$ be a positive infinite series,

if $\lim\limits_{n \to \infty} \frac{a_{n+1}}{a_n} = \ell$, ℓ is real number

then:

* If $\ell < 1$, then the series is convergent.
* If $\ell > 1$, then the series is divergent.
* If $\ell = 1$, then the series may be divergent or convergent.

3- nth Root Test

Let $\sum a_n$ be a positive infinite series, if

$\lim\limits_{n \to \infty} \sqrt[n]{a_n} = \ell$, ℓ is real number, then

* If $\ell < 1$, then the series is convergent.
* If $\ell > 1$, then the series is divergent.
* If $\ell = 1$, then the series may be divergent or convergent.

Examples

Determin which of the following Series divergent and which of them Convergent

① $\sum_{n=1}^{\infty} \frac{1}{n!}$ (Use Comparison Test)

$$\sum a_n = \sum_{n=1}^{\infty} \frac{1}{n!}$$

Solve $\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$

$$= 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$$

let $\sum c_n = 1 + \sum_{n=1}^{\infty} \frac{1}{2^n}$

$$= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

Note that, $c_n \geqslant a_n \quad \forall n \in \mathbb{N}$

Also, $\sum c_n = \sum_{n=0}^{\infty} (1)(\frac{1}{2})^n$ is a Geometric series with $a = 1$ and $|r| = \frac{1}{2} < 1$

$\therefore \sum c_n$ is convergent

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n!}$ is convergent

② $\sum a_n = \sum_{n=1}^{\infty} \sin \frac{1}{n}$ (Use the limit Comparsion Test)

Solve

$a_n = \sin \frac{1}{n}$, let $b_n = \frac{1}{n}$

$$\Rightarrow \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{\frac{1}{n} \to 0} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1$$

if $n \to \infty$, then $\frac{1}{n} \to 0$

By using the rule: $\lim_{x \to 0} \frac{\sin x}{x} = 1$

$$\therefore \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{\frac{1}{n} \to 0} \frac{\sin \frac{1}{n}}{\frac{1}{n}} \neq 0$$

Note that $\sum b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is Harmonic series which is divergent.

$\therefore \sum a_n = \sum_{n=1}^{\infty} \sin \frac{1}{n}$ is divergent.

③ $\sum a_n = \sum_{n=1}^{\infty} \frac{1}{n^n}$ (Use the nth Root Test)

Solve $a_n = \frac{1}{n^n} \Rightarrow \sqrt[n]{a_n} = \frac{1}{n}$

$$\lim_{n \to \infty} \sqrt[n]{a_n} = \lim_{n \to \infty} \frac{1}{n} = 0 < 1$$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^n}$ is convergent

④ $\sum a_n = \sum_{n=1}^{\infty} \frac{n!\,n!}{(2n)!}$ (Use the Ratio Test)

Solve

$$a_n = \frac{n!\,n!}{(2n)!} \quad , \quad a_{n+1} = \frac{(n+1)!\,(n+1)!}{(2n+2)!}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!\,(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!\,n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)\cancel{n!}\,(n+1)\cancel{n!}}{(2n+2)(2n+1)\cancel{(2n)!}} \cdot \frac{\cancel{(2n)!}}{\cancel{n!}\,\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)}(n+1)}{2\cancel{(n+1)}(2n+1)}$$

$$= \frac{1}{2} \lim_{n\to\infty} \frac{(n+1)}{(2n+1)}$$

$$= \frac{1}{2} \lim_{n\to\infty} \frac{\frac{n}{n} + \frac{1}{n}}{\frac{2n}{n} + \frac{1}{n}} = \frac{1}{2} \lim_{n\to\infty} \frac{1 + \frac{1}{n}}{2 + \frac{1}{n}}$$

$$= \frac{1}{2} \left(\frac{1+0}{2+0} \right) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{4} < 1$$

$\Rightarrow \quad \sum a_n = \sum_{n=1}^{\infty} \frac{n!\,n!}{(2n)!}$ is Convergent

Exercises

Determin which of the following series convergent and which of them divergent.

① $\sum a_n = \sum_{n=1}^{\infty} \frac{2n}{3n^2-4n+1}$ (Use Limit ~~Compassion~~ Compasion Test)

② $\sum a_n = \sum_{n=1}^{\infty} \frac{3^n}{4^n\sqrt{n}}$ (Use Ratio Test)

③ $\sum a_n = \sum_{n=1}^{\infty} \frac{2^n}{n^2}$ (Use nth Root Test)

④ $\sum a_n = \sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$ (Use Comparsion Test)

6.4 Power Series

Definition (6.4.1)

Let $\sum a_n$ be an infinite series (positive, negative, or alternating). $\sum a_n$ is said to be absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Definition (6.4.2)

Let $\sum_{n=1}^{\infty} (-1)^n a_n$ be an alternating series.

$\sum_{n=1}^{\infty} (-1)^n a_n$ is said to be conditionally convergent if:

1. $a_n > 0 \quad \forall n$.
2. $a_{n+1} \leq a_n \quad \forall n$.
3. $\lim_{n \to \infty} a_n = 0$.

Tylor Series

Definition (6.4.3)

Let f be a smooth function and let a be an element of the domain of f. By an extended Taylor polynomail of f at $x=a$, we mean an equation of the form:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

If $n \to \infty$, then:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

is said to be Taylor Series of f at $x=a$.

If $a=0$, then

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

is said to be Taylor polynomail at $x=0$.

If $n \to \infty$ and $x = 0$, then

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \text{---------}$$

is said to be Makluren Series of f at $x=0$.

Examples

(1) Find Taylor polynomial and Makluren Series generated by $f(x) = e^x$ at $x=0$.

Solve Taylor polynomial of f at $x=0$ is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \text{----} + \frac{f^{(n)}(0)}{n!}x^n$$

$$\Rightarrow e^x = e^0 + e^0 x + \frac{e^0}{2!}x^2 + \frac{e^0}{3!}x^3 + \text{----} + \frac{e^0}{n!}x^n$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \text{----} + \frac{x^n}{3!}$$

If $n \to \infty$, then

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \text{----}$$

This is a Makluren Series of e^x at $x=0$

② Find Taylor series generated by

$f(x) = \cos x$ at $x = 2\pi$

solve Taylor series of f at $x=a$ is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

For $f(x) = \cos x$ and $a = 2\pi$, we have

$$f(2\pi) = \cos 2\pi = 1$$

$$f'(x) = -\sin x \Rightarrow f'(2\pi) = -\sin 2\pi = 0$$

$$f''(x) = -\cos x \Rightarrow f''(2\pi) = -\cos 2\pi = -1$$

$$f^{(3)}(x) = \sin x \Rightarrow f^{(3)}(2\pi) = \sin 2\pi = 0$$

$$\Rightarrow \cos x = f(2\pi) + f'(2\pi)(x-2\pi) + \frac{f''(2\pi)}{2!}(x-2\pi)^2 + \cdots$$

$$\cos x = 1 + (0)(x-2\pi) + \frac{-1}{2!}(x-2\pi)^2 + (0)(x-2\pi)^3 + \cdots$$

$$\cos x = 1 + 0 - \frac{1}{2!}(x-2\pi)^2 + 0 + \frac{1}{4!}(x-2\pi)^4 + 0 - \cdots$$

$$\cos x = 1 - \frac{1}{2!}(x-2\pi)^2 + \frac{1}{4!}(x-2\pi)^4 - \frac{1}{6!}(x-2\pi)^6 + \cdots$$

Convergence of Power Series

Theorem (6.4.5)

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series

* If $\sum a_n x^n$ is convergent at $x = c$, then it is convergent for each x where $|x| < c$, $c \neq 0$.

* If $\sum a_n x^n$ is divergent at $x = b$, then it is divergent for each x where $|x| > b$, $b \neq 0$.

Example Find the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$.

Solve

We test the convergence of the series of absolute values using the ratio test, as follows:

6

$$\sum_{n=1}^{\infty} \left| \frac{x^n}{n} \right| \quad , \quad a_n = \left| \frac{x^n}{n} \right| \quad , \quad a_{n+1} = \left| \frac{x^{n+1}}{n+1} \right|$$

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right|$$

$$\Rightarrow \lim_{n \to \infty} \frac{nx}{n+1} = |x| .$$

$\Rightarrow \sum_{n=1}^{\infty} \left| \frac{x^n}{n} \right|$ is convergent if $|x| < 1$

$$\Rightarrow \quad -1 < x < 1$$

- if $x = -1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is Conditionally convergent.

if $x = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{(1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ is a Harmonic Series which is divergent.

Hence the interval of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n}$ is $[-1, 1) = \{x : -1 \leq x < 1\}$.