2.3 POLAR EQUATIONS OF CONICS

■ A Unified Geometric Description of Conics ■ Polar Equations of Conics

A Unified Geometric Description of Conics

Earlier in this chapter, we defined a parabola in terms of a focus and directrix, but we defined the ellipse and hyperbola in terms of two foci. In this section we give a more unified treatment of all three types of conics in terms of a focus and directrix. If we place one focus at the origin, then a conic section has a simple polar equation.

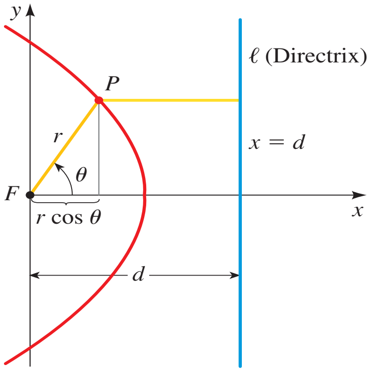


FIGURE 1

EQUIVALENT DESCRIPTION OF CONICS

Let F be a fixed point (the **focus**), ℓ a fixed line (the **directrix**), and let e be a fixed positive number (the **eccentricity**). The set of all points P such that the ratio of the distance from P to F to the distance from P to ℓ is the constant e is a conic. That is, the set of all points P such that

$$\frac{d(P, F)}{d(P, \ell)} = e$$

is a conic. The conic is a parabola if $e = 1$, an ellipse if $e < 1$, or a hyperbola if $e > 1$.

Polar Equations of Conics

we saw that the polar equation of the conic in Figure 1 is $r=e(d-r\cos\theta)$. Solving for r, we get

$$r = \frac{ed}{1 + e\cos\theta}$$

If the directrix is chosen to be to the *left* of the focus $(x = -d)$, then we get the equation $r = ed/(1 - e\cos\theta)$. If the directrix is *parallel* to the polar axis $(y = d$ or $y = -d)$, then we get $\sin\theta$ instead of $\cos\theta$ in the equation. These observations are summarized in the following box and in Figure 2.

POLAR EQUATIONS OF CONICS

A polar equation of the form

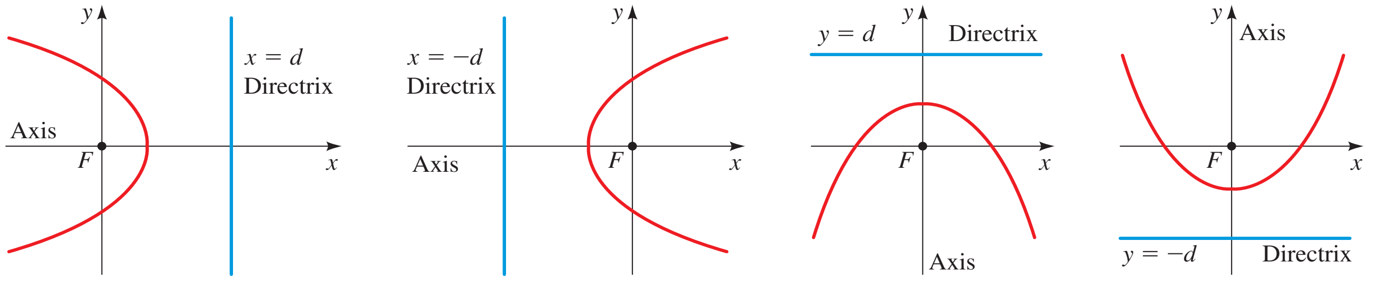
$$r = \frac{ed}{1 \pm e\cos\theta} \qquad \text{or} \qquad r = \frac{ed}{1 \pm e\sin\theta}$$

represents a conic with one focus at the origin and with eccentricity e. The conic is

1. a parabola if $e = 1$,

2. an ellipse if $0 < e < 1$,

3. a hyperbola if $e > 1$.



(a) $r = \dfrac{ed}{1 + e\cos\theta}$ (b) $r = \dfrac{ed}{1 - e\cos\theta}$ (c) $r = \dfrac{ed}{1 + e\sin\theta}$ (d) $r = \dfrac{ed}{1 - e\sin\theta}$

FIGURE 2 The form of the polar equation of a conic indicates the location of the directrix.

To graph the polar equation of a conic, we first determine the location of the directrix from the form of the equation. The four cases that arise are shown in Figure 2. (The figure shows only the parts of the graphs that are close to the focus at the origin. The shape of the rest of the graph depends on whether the equation represents a parabola, an ellipse, or a hyperbola.) The axis of a conic is perpendicular to the directrix—specifically we have the following:

1. For a parabola the axis of symmetry is perpendicular to the directrix.
2. For an ellipse the major axis is perpendicular to the directrix.
3. For a hyperbola the transverse axis is perpendicular to the directrix.

EXAMPLE 1 Finding a Polar Equation for a Conic

Find a polar equation for the parabola that has its focus at the origin and whose directrix is the line $y = -6$.

SOLUTION Using $e = 1$ and $d = 6$ and using part (d) of Figure 2, we see that the polar equation of the parabola is

$$r = \frac{6}{1 - \sin\theta}$$

To graph a polar conic, it is helpful to plot the points for which $\theta = 0, \pi/2, \pi$, and $3\pi/2$. Using these points and a knowledge of the type of conic (which we obtain from the eccentricity), we can easily get a rough idea of the shape and location of the graph.

EXAMPLE 2 Identifying and Sketching a Conic

A conic is given by the polar equation

$$r = \frac{10}{3 - 2\cos\theta}$$

(a) Show that the conic is an ellipse, and sketch its graph.
(b) Find the center of the ellipse and the lengths of the major and minor axes.

SOLUTION

(a) Dividing the numerator and denominator by 3, we have

$$r = \frac{\frac{10}{3}}{1 - \frac{2}{3}\cos\theta}$$

Since $e = \frac{2}{3} < 1$, the equation represents an ellipse. For a rough graph we plot the points for which $\theta = 0, \pi/2, \pi, 3\pi/2$ (see Figure 3).

θ	r
0	10
$\frac{\pi}{2}$	$\frac{10}{3}$
π	2
$\frac{3\pi}{2}$	$\frac{10}{3}$

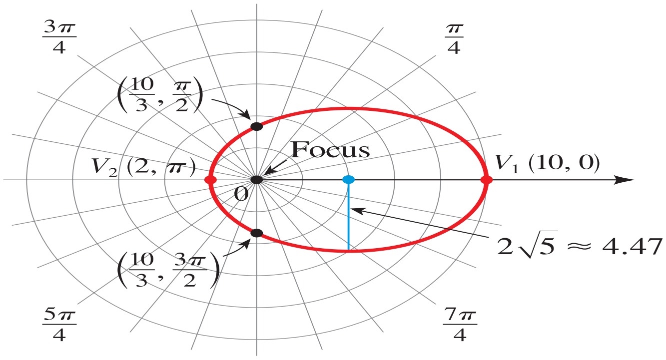


FIGURE 3 $r = \dfrac{10}{3 - 2\cos\theta}$

(b) Comparing the equation to those in Figure 2, we see that the major axis is horizontal. Thus the endpoints of the major axis are $V_1(10, 0)$ and $V_2(2, \pi)$. So the center of the ellipse is at $C(4, 0)$, the midpoint of V_1V_2.

The distance between the vertices V_1 and V_2 is 12; thus the length of the major axis is $2a = 12$, so $a = 6$. To determine the length of the minor axis, we need to find b. we have $c = ae = 6(\frac{2}{3}) = 4$, so

$$b^2 = a^2 - c^2 = 6^2 - 4^2 = 20$$

Thus $b = \sqrt{20} = 2\sqrt{5} \approx 4.47$, and the length of the minor axis is $2b = 4\sqrt{5} \approx 8.94$.

EXAMPLE 3 Identifying and Sketching a Conic

A conic is given by the polar equation

$$r = \frac{12}{2 + 4\sin\theta}$$

(a) Show that the conic is a hyperbola, and sketch its graph.

(b) Find the center of the hyperbola, and sketch the asymptotes.

SOLUTION

(a) Dividing the numerator and denominator by 2, we have

$$r = \frac{6}{1 + 2\sin\theta}$$

Since $e = 2 > 1$, the equation represents a hyperbola. For a rough graph we plot the points for which $\theta = 0, \pi/2, \pi, 3\pi/2$ (see Figure 4).

(b) Comparing the equation to those in Figure 2, we see that the transverse axis is vertical. Thus the endpoints of the transverse axis (the vertices of the hyperbola) are $V_1(2, \pi/2)$ and $V_2(-6, 3\pi/2) = V_2(6, \pi/2)$. So the center of the hyperbola is $C(4, \pi/2)$, the midpoint of V_1V_2.

To sketch the asymptotes, we need to find a and b. The distance between V_1 and V_2 is 4; thus the length of the transverse axis is $2a = 4$, so $a = 2$. To find b, we first find c. we have $c = ae = 2 \cdot 2 = 4$, so

$$b^2 = c^2 - a^2 = 4^2 - 2^2 = 12$$

Thus $b = \sqrt{12} = 2\sqrt{3} \approx 3.46$. Knowing a and b allows us to sketch the central box, from which we obtain the asymptotes shown in Figure 4.

θ	r
0	6
$\frac{\pi}{2}$	2
π	6
$\frac{3\pi}{2}$	-6

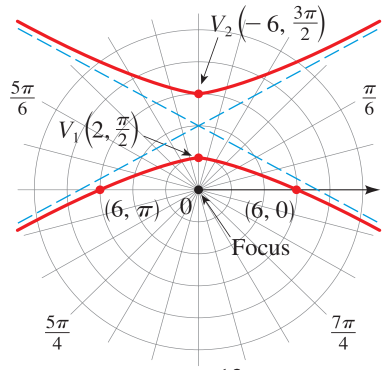


FIGURE 4 $r = \dfrac{12}{2 + 4\sin\theta}$

When we rotate conic sections, it is much more convenient to use polar equations than Cartesian equations. We use the fact that the graph of $r = f(\theta - \alpha)$ is the graph of $r = f(\theta)$ rotated counterclockwise about the origin through an angle α

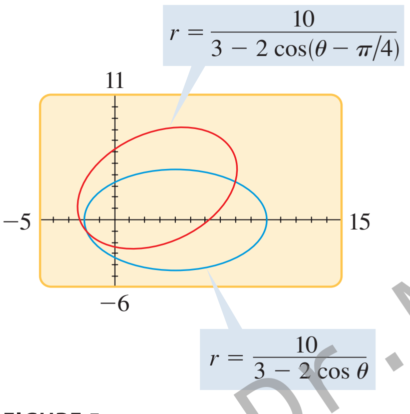


FIGURE 5

EXAMPLE 4 Rotating an Ellipse

Suppose the ellipse of Example 2 is rotated through an angle $\pi/4$ about the origin. Find a polar equation for the resulting ellipse, and draw its graph.

SOLUTION We get the equation of the rotated ellipse by replacing θ with $\theta - \pi/4$ in the equation given in Example 2. So the new equation is

$$r = \frac{10}{3 - 2\cos(\theta - \pi/4)}$$

We use this equation to graph the rotated ellipse in Figure 5. Notice that the ellipse has been rotated about the focus at the origin.

In Figure 6 we use a computer to sketch a number of conics to demonstrate the effect of varying the eccentricity e. Notice that when e is close to 0, the ellipse is nearly circular, and it becomes more elongated as e increases. When $e = 1$, of course, the conic is a parabola. As e increases beyond 1, the conic is an ever steeper hyperbola.

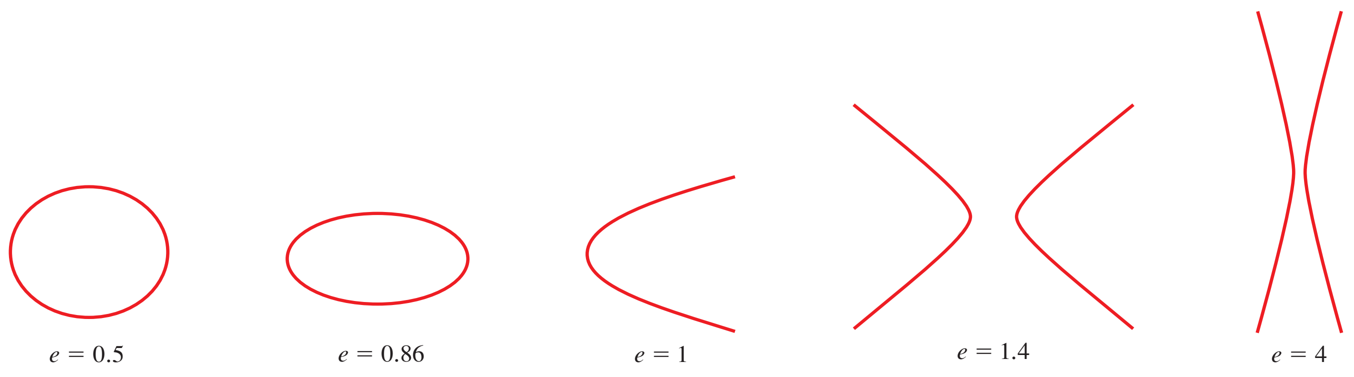


FIGURE 6

2.4 TANGENT LINES, ARC LENGTH, AND AREA FOR POLAR CURVES

In this section we will derive the formulas required to find slopes, tangent lines, and arc lengths of polar curves. We will then show how to find areas of regions that are bounded by polar curves.

TANGENT LINES TO POLAR CURVES

Our first objective in this section is to find a method for obtaining slopes of tangent lines to polar curves of the form $r = f(\theta)$ in which r is a differentiable function of θ. We showed in the last section that a curve of this form can be expressed parametrically in terms of the parameter θ by substituting $f(\theta)$ for r in the equations $x = r\cos\theta$ and $y = r\sin\theta$. This yields

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta$$

from which we obtain

$$\begin{aligned} \frac{dx}{d\theta} &= -f(\theta)\sin\theta + f'(\theta)\cos\theta = -r\sin\theta + \frac{dr}{d\theta}\cos\theta \\ \frac{dy}{d\theta} &= f(\theta)\cos\theta + f'(\theta)\sin\theta = r\cos\theta + \frac{dr}{d\theta}\sin\theta \end{aligned} \qquad (1)$$

Thus, if $dx/d\theta$ and $dy/d\theta$ are continuous and if $dx/d\theta \neq 0$, then y is a differentiable function of x.

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta + \sin\theta\dfrac{dr}{d\theta}}{-r\sin\theta + \cos\theta\dfrac{dr}{d\theta}} \qquad (2)$$

EXAMPLE 1 Find the slope of the tangent line to the circle $r = 4\cos\theta$ at the point where $\theta = \pi/4$.

Solution. From (2) with $r = 4\cos\theta$, so that $dr/d\theta = -4\sin\theta$, we obtain

$$\frac{dy}{dx} = \frac{4\cos^2\theta - 4\sin^2\theta}{-8\sin\theta\cos\theta} = -\frac{\cos^2\theta - \sin^2\theta}{2\sin\theta\cos\theta}$$

Using the double-angle formulas for sine and cosine,

$$\frac{dy}{dx} = -\frac{\cos 2\theta}{\sin 2\theta} = -\cot 2\theta$$

Thus, at the point where $\theta = \pi/4$ the slope of the tangent line is

$$m = \left.\frac{dy}{dx}\right|_{\theta=\pi/4} = -\cot\frac{\pi}{2} = 0$$

which implies that the circle has a horizontal tangent line at the point where $\theta = \pi/4$ (Figure 1). ◀

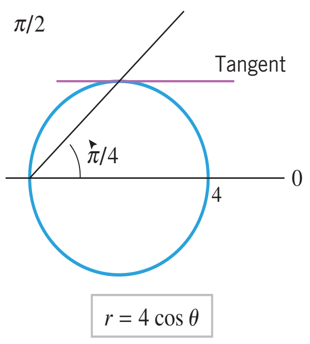


Figure 1

EXAMPLE 2 Find the points on the cardioid $r = 1 - \cos\theta$ at which there is a horizontal tangent line, a vertical tangent line, or a singular point.

Solution. A horizontal tangent line will occur where $dy/d\theta = 0$ and $dx/d\theta \neq 0$, a vertical tangent line where $dy/d\theta \neq 0$ and $dx/d\theta = 0$, and a singular point where $dy/d\theta = 0$ and $dx/d\theta = 0$. We could find these derivatives from the formulas in (1). However, an alternative approach is to go back to basic principles and express the cardioid parametrically by substituting $r = 1 - \cos\theta$ in the conversion formulas $x = r\cos\theta$ and $y = r\sin\theta$. This yields

$$x = (1 - \cos\theta)\cos\theta, \quad y = (1 - \cos\theta)\sin\theta \qquad (0 \le \theta \le 2\pi)$$

Differentiating these equations with respect to θ and then simplifying yields (verify)

$$\frac{dx}{d\theta} = \sin\theta(2\cos\theta - 1), \quad \frac{dy}{d\theta} = (1 - \cos\theta)(1 + 2\cos\theta)$$

Thus, $dx/d\theta = 0$ if $\sin\theta = 0$ or $\cos\theta = \frac{1}{2}$, and $dy/d\theta = 0$ if $\cos\theta = 1$ or $\cos\theta = -\frac{1}{2}$. We leave it for you to solve these equations and show that the solutions of $dx/d\theta = 0$ on the interval $0 \le \theta \le 2\pi$ are

$$\frac{dx}{d\theta} = 0: \quad \theta = 0, \ \frac{\pi}{3}, \ \pi, \ \frac{5\pi}{3}, \ 2\pi$$

and the solutions of $dy/d\theta = 0$ on the interval $0 \le \theta \le 2\pi$ are

$$\frac{dy}{d\theta} = 0: \quad \theta = 0, \ \frac{2\pi}{3}, \ \frac{4\pi}{3}, \ 2\pi$$

Thus, horizontal tangent lines occur at $\theta = 2\pi/3$ and $\theta = 4\pi/3$; vertical tangent lines occur at $\theta = \pi/3$, π, and $5\pi/3$; and singular points occur at $\theta = 0$ and $\theta = 2\pi$ (Figure .2). Note, however, that $r = 0$ at both singular points, so there is really only one singular point on the cardioid—the pole. ◀

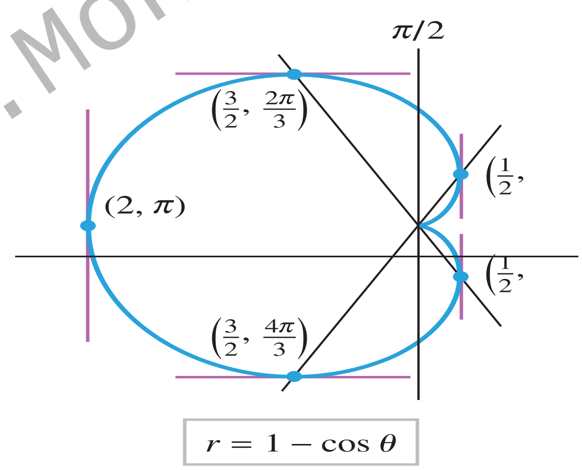


Figure 2

TANGENT LINES TO POLAR CURVES AT THE ORIGIN

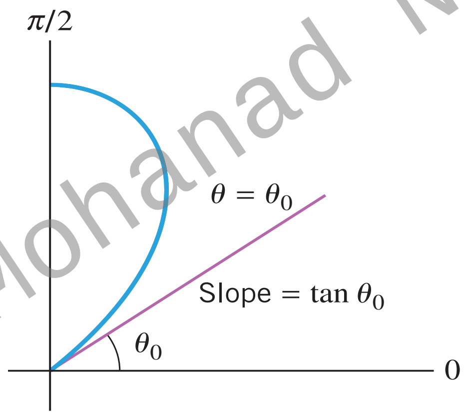
Formula (2) reveals some useful information about the behavior of a polar curve $r = f(\theta)$ that passes through the origin. If we assume that $r = 0$ and $dr/d\theta \neq 0$ when $\theta = \theta_0$, then it follows from Formula (2) that the slope of the tangent line to the curve at $\theta = \theta_0$ is

$$\frac{dy}{dx} = \frac{0 + \sin\theta_0 \dfrac{dr}{d\theta}}{0 + \cos\theta_0 \dfrac{dr}{d\theta}} = \frac{\sin\theta_0}{\cos\theta_0} = \tan\theta_0$$

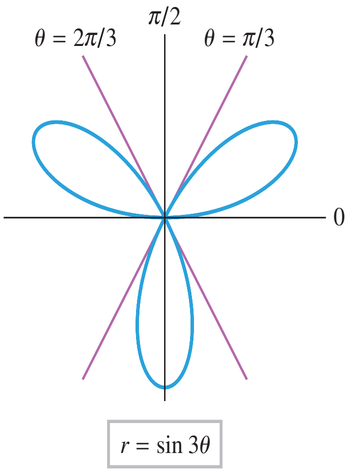
(Figure 3). However, $\tan\theta_0$ is also the slope of the line $\theta = \theta_0$, so we can conclude that this line is tangent to the curve at the origin. Thus, we have established the following result.

THEOREM *If the polar curve $r = f(\theta)$ passes through the origin at $\theta = \theta_0$, and if $dr/d\theta \neq 0$ at $\theta = \theta_0$, then the line $\theta = \theta_0$ is tangent to the curve at the origin.*

This theorem tells us that equations of the tangent lines at the origin to the curve $r = f(\theta)$ can be obtained by solving the equation $f(\theta) = 0$. It is important to keep in mind, however, that $r = f(\theta)$ may be zero for more than one value of θ, so there may be more than one tangent line at the origin. This is illustrated in the next example.



▲ **Figure 3**



▲ **Figure 4**

EXAMPLE 3 The three-petal rose $r = \sin 3\theta$ in Figure 4 has three tangent lines at the origin, which can be found by solving the equation

$$\sin 3\theta = 0$$

The complete rose is traced once as θ varies over the interval $0 \leq \theta < \pi$, so we need only look for solutions in this interval. We leave

$$\theta = 0, \quad \theta = \frac{\pi}{3}, \quad \text{and} \quad \theta = \frac{2\pi}{3}$$

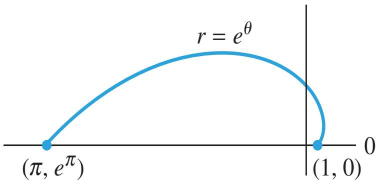
Since $dr/d\theta = 3\cos 3\theta \neq 0$ for these values of θ, these three lines are tangent to the rose at the origin, which is consistent with the figure. ◄

ARC LENGTH OF A POLAR CURVE

A formula for the arc length of a polar curve $r = f(\theta)$ can be derived by expressing the curve in parametric form and applying for the arc length of aparametric curve. We leave it as an exercise to show the following.

ARC LENGTH FORMULA FOR POLAR CURVES If no segment of the polar curve $r = f(\theta)$ is traced more than once as θ increases from α to β, and if $dr/d\theta$ is continuous for $\alpha \le \theta \le \beta$, then the arc length L from $\theta = \alpha$ to $\theta = \beta$ is

$$L = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad (3)$$



▲ **Figure 5**

EXAMPLE 4 Find the arc length of the spiral $r = e^\theta$ in Figure 5 between $\theta = 0$ and $\theta = \pi$.

Solution.

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^\pi \sqrt{(e^\theta)^2 + (e^\theta)^2}\, d\theta$$

$$= \int_0^\pi \sqrt{2}\, e^\theta\, d\theta = \sqrt{2}\, e^\theta \Big]_0^\pi = \sqrt{2}(e^\pi - 1) \approx 31.3 \blacktriangleleft$$

EXAMPLE 5 Find the total arc length of the cardioid $r = 1 + \cos\theta$.

Solution. The cardioid is traced out once as θ varies from $\theta = 0$ to $\theta = 2\pi$. Thus,

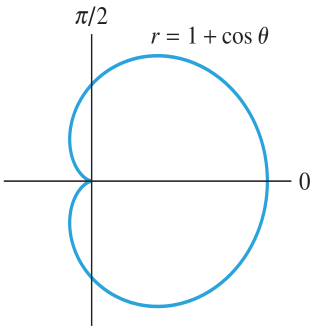
$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{(1 + \cos\theta)^2 + (-\sin\theta)^2}\, d\theta$$

$$= \sqrt{2} \int_0^{2\pi} \sqrt{1 + \cos\theta}\, d\theta$$

$$= 2 \int_0^{2\pi} \sqrt{\cos^2 \tfrac{1}{2}\theta}\, d\theta$$

Identity (45) of Appendix B

$$= 2 \int_0^{2\pi} \left|\cos \tfrac{1}{2}\theta\right| d\theta$$



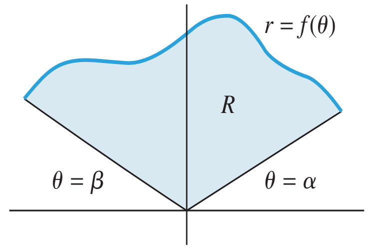
▲ **Figure 6**

Since $\cos \frac{1}{2}\theta$ changes sign at π, we must split the last integral into the sum of two integrals: the integral from 0 to π plus the integral from π to 2π. However, the integral from π to 2π is equal to the integral from 0 to π, since the cardioid is symmetric about the polar axis (Figure 6). Thus,

$$L = 2 \int_0^{2\pi} \left|\cos \tfrac{1}{2}\theta\right| d\theta = 4 \int_0^\pi \cos \tfrac{1}{2}\theta\, d\theta = 8 \sin \tfrac{1}{2}\theta \Big]_0^\pi = 8 \blacktriangleleft$$

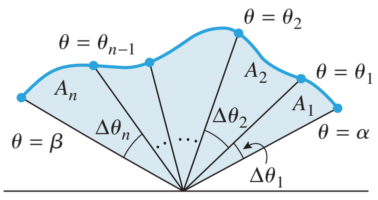
AREA IN POLAR COORDINATES

We begin our investigation of area in polar coordinates with a simple case.



▲ **Figure 7**

> **AREA PROBLEM IN POLAR COORDINATES** Suppose that α and β are angles that satisfy the condition
>
> $$\alpha < \beta \le \alpha + 2\pi$$
>
> and suppose that $f(\theta)$ is continuous and nonnegative for $\alpha \le \theta \le \beta$. Find the area of the region R enclosed by the polar curve $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$ (Figure 7).



▲ **Figure 8**

In rectangular coordinates we obtained areas under curves by dividing the region into an increasing number of vertical strips, approximating the strips by rectangles, and taking a limit. In polar coordinates rectangles are clumsy to work with, and it is better to partition the region into ***wedges*** by using rays

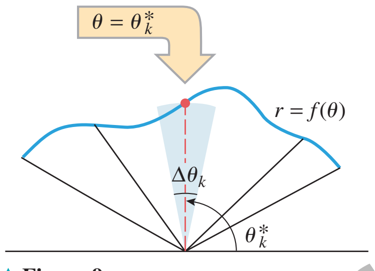
$$\theta = \theta_1,\ \theta = \theta_2, \ldots, \theta = \theta_{n-1}$$

such that

$$\alpha < \theta_1 < \theta_2 < \cdots < \theta_{n-1} < \beta$$

(Figure 8). As shown in that figure, the rays divide the region R into n wedges with areas $A_1, A_2, \ldots, A_n$ and central angles $\Delta\theta_1, \Delta\theta_2, \ldots, \Delta\theta_n$. The area of the entire region can be written as

$$A = A_1 + A_2 + \cdots + A_n = \sum_{k=1}^{n} A_k \qquad (4)$$



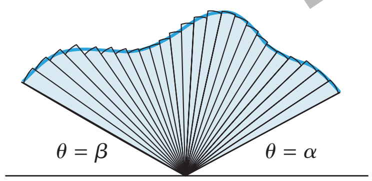
▲ **Figure 9**

If $\Delta\theta_k$ is small, then we can approximate the area A_k of the kth wedge by the area of a sector with central angle $\Delta\theta_k$ and radius $f(\theta_k^*)$, where $\theta = \theta_k^*$ is any ray that lies in the kth wedge (Figure 9). Thus, from (4) and Formula (5) of Appendix B for the area of a sector, we obtain

$$A = \sum_{k=1}^{n} A_k \approx \sum_{k=1}^{n} \tfrac{1}{2}[f(\theta_k^*)]^2 \Delta\theta_k \qquad (5)$$

If we now increase n in such a way that $\max \Delta\theta_k \to 0$, then the sectors will become better and better approximations of the wedges and it is reasonable to expect that (5) will approach the exact value of the area A (Figure 10); that is,

$$A = \lim_{\max \Delta\theta_k \to 0} \sum_{k=1}^{n} \tfrac{1}{2}[f(\theta_k^*)]^2 \Delta\theta_k = \int_{\alpha}^{\beta} \tfrac{1}{2}[f(\theta)]^2\, d\theta$$



▲ **Figure 10**

Note that the discussion above can easily be adapted to the case where $f(\theta)$ is nonpositive for $\alpha \le \theta \le \beta$. We summarize this result below.

> **AREA IN POLAR COORDINATES** If α and β are angles that satisfy the condition
>
> $$\alpha < \beta \le \alpha + 2\pi$$
>
> and if $f(\theta)$ is continuous and either nonnegative or nonpositive for $\alpha \le \theta \le \beta$, then the area A of the region R enclosed by the polar curve $r = f(\theta)$ $(\alpha \le \theta \le \beta)$ and the lines $\theta = \alpha$ and $\theta = \beta$ is
>
> $$A = \int_{\alpha}^{\beta} \tfrac{1}{2}[f(\theta)]^2\, d\theta = \int_{\alpha}^{\beta} \tfrac{1}{2} r^2\, d\theta \qquad (6)$$

The hardest part of applying (6) is determining the limits of integration. This can be done as follows:

Area in Polar Coordinates: Limits of Integration

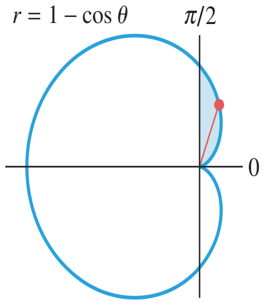
Step 1. Sketch the region R whose area is to be determined.

Step 2. Draw an arbitrary "radial line" from the pole to the boundary curve $r = f(\theta)$.

Step 3. Ask, "Over what interval of values must θ vary in order for the radial line to sweep out the region R?"

Step 4. Your answer in Step 3 will determine the lower and upper limits of integration.

EXAMPLE 6 Find the area of the region in the first quadrant that is within the cardioid $r = 1 - \cos\theta$.



The shaded region is swept out by the radial line as θ varies from 0 to $\pi/2$.

Figure 11

Solution. The region and a typical radial line are shown in Figure 11. For the radial line to sweep out the region, θ must vary from 0 to $\pi/2$. Thus, from (6) with $\alpha = 0$ and $\beta = \pi/2$, we obtain

$$A = \int_0^{\pi/2} \frac{1}{2}r^2\,d\theta = \frac{1}{2}\int_0^{\pi/2}(1-\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^{\pi/2}(1-2\cos\theta+\cos^2\theta)\,d\theta$$

With the help of the identity $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$, this can be rewritten as

$$A = \frac{1}{2}\int_0^{\pi/2}\left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta = \frac{1}{2}\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} = \frac{3}{8}\pi - 1$$

EXAMPLE 7 Find the entire area within the cardioid of Example 6.

Solution. For the radial line to sweep out the entire cardioid, θ must vary from 0 to 2π. Thus, from (6) with $\alpha = 0$ and $\beta = 2\pi$,

$$A = \int_0^{2\pi}\frac{1}{2}r^2\,d\theta = \frac{1}{2}\int_0^{2\pi}(1-\cos\theta)^2\,d\theta$$

If we proceed as in Example 6, this reduces to

$$A = \frac{1}{2}\int_0^{2\pi}\left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta = \frac{3\pi}{2}$$

Alternative Solution. Since the cardioid is symmetric about the x-axis, we can calculate the portion of the area above the x-axis and double the result. In the portion of the cardioid above the x-axis, θ ranges from 0 to π, so that

$$A = 2\int_0^{\pi}\frac{1}{2}r^2\,d\theta = \int_0^{\pi}(1-\cos\theta)^2\,d\theta = \frac{3\pi}{2}$$

USING SYMMETRY

Although Formula (6) is applicable if $r = f(\theta)$ is negative, area computations can sometimes be simplified by using symmetry to restrict the limits of integration to intervals where $r \geq 0$. This is illustrated in the next example.

EXAMPLE 8 Find the area of the region enclosed by the rose curve $r = \cos 2\theta$.

Solution. the area in the first quadrant that is swept out for $0 \leq \theta \leq \pi/4$, is one-eighth of the total area inside the rose. Thus, form Formula (6)

$$A = 8\int_0^{\pi/4} \frac{1}{2} r^2\, d\theta = 4\int_0^{\pi/4} \cos^2 2\theta\, d\theta$$

$$= 4\int_0^{\pi/4} \frac{1}{2}(1+\cos 4\theta)\, d\theta = 2\int_0^{\pi/4} (1+\cos 4\theta)\, d\theta$$

$$= 2\theta + \frac{1}{2}\sin 4\theta \Big]_0^{\pi/4} = \frac{\pi}{2}$$

Sometimes the most natural way to satisfy the restriction $\alpha < \beta \leq \alpha + 2\pi$ required by Formula (6) is to use a negative value for α. For example, suppose that we are interested in finding the area of the shaded region in Figure 12*a* .The first step would be to determine the intersections of the cardioid $r = 4 + 4\cos\theta$ and the circle $r = 6$, since this information is needed for the limits of integration. To find the points of intersection, we can equate the two expressions for r. This yields

$$4 + 4\cos\theta = 6 \quad \text{or} \quad \cos\theta = \frac{1}{2}$$

which is satisfied by the positive angles

$$\theta = \frac{\pi}{3} \quad \text{and} \quad \theta = \frac{5\pi}{3}$$

However, there is a problem here because the radial lines to the circle and cardioid do not sweep through the shaded region shown in Figure 12.*b* as θ varies over the interval $\pi/3 \leq \theta \leq 5\pi/3$. There are two ways to circumvent this problem—one is to take advantage of the symmetry by integrating over the interval $0 \leq \theta \leq \pi/3$ and doubling the result, and the second is to use a negative lower limit of integration and integrate over the interval $-\pi/3 \leq \theta \leq \pi/3$ (Figure 12*c*). The two methods are illustrated in the next example.

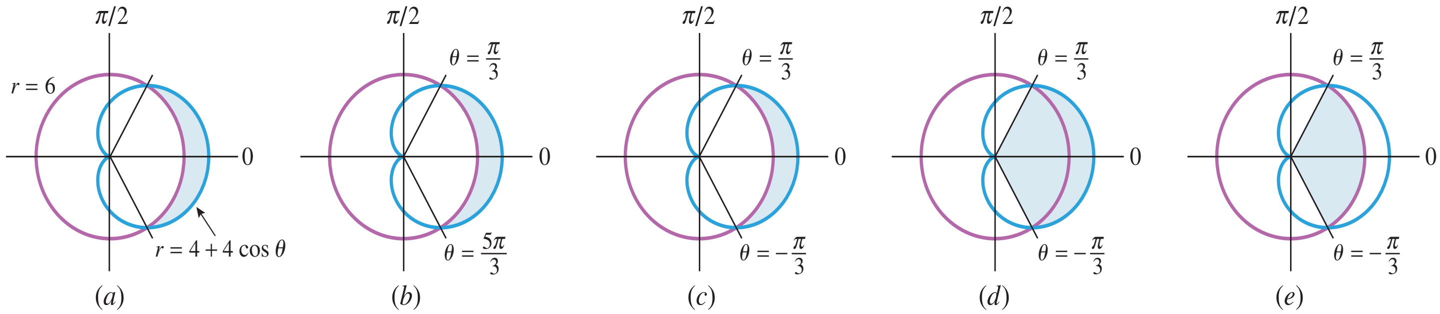


Figure 12

EXAMPLE 9 Find the area of the region that is inside of the cardioid $r = 4 + 4\cos\theta$ and outside of the circle $r = 6$.

Solution Using a Negative Angle. The area of the region can be obtained by subtracting the areas in Figures 12*d* and 12*e*:

$$A = \int_{-\pi/3}^{\pi/3} \frac{1}{2}(4 + 4\cos\theta)^2\, d\theta - \int_{-\pi/3}^{\pi/3} \frac{1}{2}(6)^2\, d\theta$$

Area inside cardioid minus area inside circle.

$$= \int_{-\pi/3}^{\pi/3} \frac{1}{2}[(4 + 4\cos\theta)^2 - 36]\, d\theta = \int_{-\pi/3}^{\pi/3} (16\cos\theta + 8\cos^2\theta - 10)\, d\theta$$

$$= \left[16\sin\theta + (4\theta + 2\sin 2\theta) - 10\,\theta\right]_{-\pi/3}^{\pi/3} = 18\sqrt{3} - 4\pi$$

Solution Using Symmetry. Using symmetry, we can calculate the area above the polar axis and double it. This yields (verify)

$$A = 2\int_{0}^{\pi/3} \frac{1}{2}[(4 + 4\cos\theta)^2 - 36]\, d\theta = 2(9\sqrt{3} - 2\pi) = 18\sqrt{3} - 4\pi$$

which agrees with the preceding result. ◄