5.5 TRIPLE INTEGRALS

PROPERTIES OF TRIPLE INTEGRALS

Triple integrals enjoy many properties of single and double integrals:

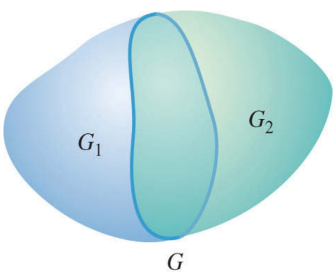
$$\iiint_G cf(x,y,z)\,dV = c\iiint_G f(x,y,z)\,dV \quad (c \text{ a constant})$$

$$\iiint_G [f(x,y,z)+g(x,y,z)]\,dV = \iiint_G f(x,y,z)\,dV + \iiint_G g(x,y,z)\,dV$$

$$\iiint_G [f(x,y,z)-g(x,y,z)]\,dV = \iiint_G f(x,y,z)\,dV - \iiint_G g(x,y,z)\,dV$$

Moreover, if the region G is subdivided into two subregions G_1 and G_2 Figure 5.5.1), then

$$\iiint_G f(x,y,z)\,dV = \iiint_{G_1} f(x,y,z)\,dV + \iiint_{G_2} f(x,y,z)\,dV$$



▲ **Figure 5.5.1**

We omit the proofs.

EVALUATING TRIPLE INTEGRALS OVER RECTANGULAR BOXES

Just as a double integral can be evaluated by two successive single integrations, so a triple integral can be evaluated by three successive integrations.

There are two possible orders of integration for the iterated integrals in Theorem 5.1.3:

$$dx\,dy, \quad dy\,dx$$

Six orders of integration are possible for the iterated integral in Theorem 5.5.1:

$$dx\,dy\,dz, \quad dy\,dz\,dx, \quad dz\,dx\,dy$$
$$dx\,dz\,dy, \quad dz\,dy\,dx, \quad dy\,dx\,dz$$

5.5.1 THEOREM (*Fubini's Theorem)** *Let G be the rectangular box defined by the inequalities*

$$a \le x \le b, \quad c \le y \le d, \quad k \le z \le l$$

If f is continuous on the region G, then

$$\iiint_G f(x,y,z)\,dV = \int_a^b \int_c^d \int_k^l f(x,y,z)\,dz\,dy\,dx \tag{2}$$

Moreover, the iterated integral on the right can be replaced with any of the five other iterated integrals that result by altering the order of integration.

Example 1 Evaluate the triple integral

$$\iiint_G 12xy^2z^3\,dV$$

over the rectangular box G defined by the inequalities $-1 \le x \le 2, 0 \le y \le 3, 0 \le z \le 2$.

Solution. Of the six possible iterated integrals we might use, we will choose the one in (2). Thus, we will first integrate with respect to z, holding x and y fixed, then with respect to y, holding x fixed, and finally with respect to x.

$$\begin{aligned}\iiint_G 12xy^2z^3\,dV &= \int_{-1}^{2}\int_0^3\int_0^2 12xy^2z^3\,dz\,dy\,dx\\ &= \int_{-1}^{2}\int_0^3 \left[3xy^2z^4\right]_{z=0}^{2}\,dy\,dx = \int_{-1}^{2}\int_0^3 48xy^2\,dy\,dx\\ &= \int_{-1}^{2}\left[16xy^3\right]_{y=0}^{3}\,dx = \int_{-1}^{2}432x\,dx\\ &= 216x^2\Big]_{-1}^{2} = 648\end{aligned}$$

EVALUATING TRIPLE INTEGRALS OVER MORE GENERAL REGIONS

Next we will consider how to evaluate triple integrals over solids that are not rectangular boxes. For the moment we will limit our discussion to solids of the type shown in Figure 5.5.2. Specifically, we will assume that the solid G is bounded above by a surface $z = g_2(x, y)$ and below by a surface $z = g_1(x, y)$ and that the projection of the solid on the xy-plane is a type I or type II region R In addition, we will assume that $g_1(x, y)$ and $g_2(x, y)$ are continuous on R and that $g_1(x, y) \le g_2(x, y)$ on R. Geometrically, this means that the surfaces may touch but cannot cross. We call a solid of this type a ***simple xy-solid***.

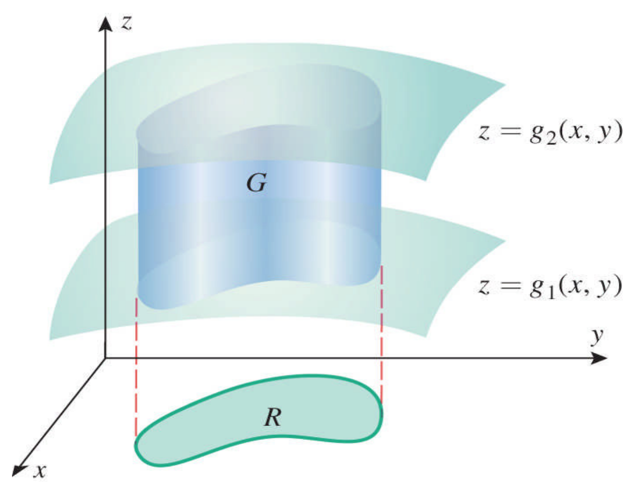


Figure 5.5.2

The following theorem, which we state without proof, will enable us to evaluate triple integrals over simple xy-solids.

5.5.2 THEOREM *Let G be a simple xy-solid with upper surface $z = g_2(x, y)$ and lower surface $z = g_1(x, y)$, and let R be the projection of G on the xy-plane. If $f(x, y, z)$ is continuous on G, then*

$$\iiint_G f(x,y,z)\,dV = \iint_R \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\right] dA \tag{3}$$

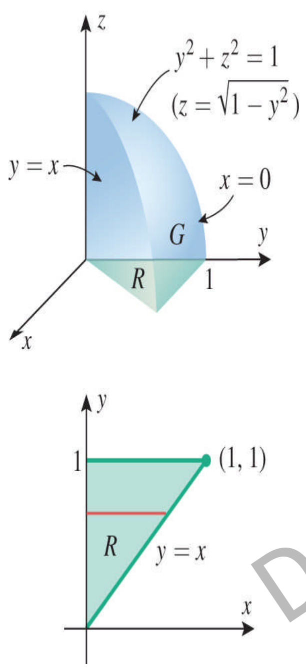


Figure 5.5.3

Example 2 Let G be the wedge in the first octant that is cut from the cylindrical solid $y^2 + z^2 \le 1$ by the planes $y = x$ and $x = 0$. Evaluate

$$\iiint_G z\,dV$$

Solution. The solid G and its projection R on the xy-plane are shown in Figure 5.5.3. The upper surface of the solid is formed by the cylinder and the lower surface by the xy-plane. Since the portion of the cylinder $y^2 + z^2 = 1$ that lies above the xy-plane has the equation $z = \sqrt{1 - y^2}$, and the xy-plane has the equation $z = 0$, it follows from (3) that

$$\iiint_G z\,dV = \iint_R \left[\int_0^{\sqrt{1-y^2}} z\,dz\right] dA \tag{4}$$

For the double integral over R, the x- and y-integrations can be performed in either order, since R is both a type I and type II region. We will integrate with respect to x first. With this choice, (4) yields

$$\iiint_G z\,dV = \int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} z\,dz\,dx\,dy = \int_0^1 \int_0^y \frac{1}{2} z^2 \bigg]_{z=0}^{\sqrt{1-y^2}} dx\,dy$$

$$= \int_0^1 \int_0^y \frac{1}{2}(1 - y^2)\,dx\,dy = \frac{1}{2}\int_0^1 (1 - y^2)x \bigg]_{x=0}^{y} dy$$

$$= \frac{1}{2}\int_0^1 (y - y^3)\,dy = \frac{1}{2}\left[\frac{1}{2}y^2 - \frac{1}{4}y^4\right]_0^1 = \frac{1}{8}$$

VOLUME CALCULATED AS A TRIPLE INTEGRAL

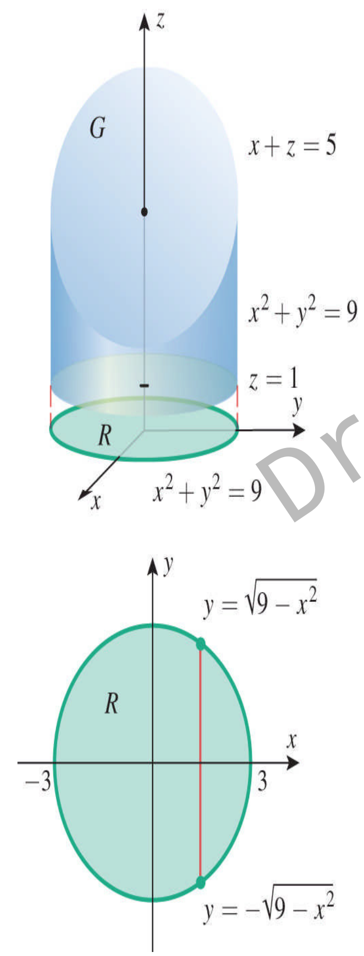
Triple integrals have many physical interpretations, some of which we will consider in the special case where $f(x, y, z) = 1$, Formula (1) yields

$$\iiint_G dV = \lim_{n \to +\infty} \sum_{k=1}^{n} \Delta V_k$$

which Figure 5.5.1 suggests is the volume of G; t hat is,

$$\text{volume of } G = \iiint_G dV \tag{5}$$

▶ **Example 3** Use a triple integral to find the volume of the solid within the cylinder $x^2 + y^2 = 9$ and between the planes $z = 1$ and $x + z = 5$.



▲ Figure 5.5.4

Solution. The solid G and its projection R on the xy-plane are shown in Figure 5.5.4 . The lower surface of the solid is the plane $z = 1$ and the upper surface is the plane $x + z = 5$ or, equivalently, $z = 5 - x$. Thus, from (3) and (5)

$$\text{volume of } G = \iiint_G dV = \iint_R \left[\int_1^{5-x} dz\right] dA \tag{6}$$

For the double integral over R, we will integrate with respect to y first. Thus, (6) yields

$$\text{volume of } G = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_1^{5-x} dz\,dy\,dx = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} z\Big]_{z=1}^{5-x} dy\,dx$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4 - x)\,dy\,dx = \int_{-3}^{3} (8 - 2x)\sqrt{9 - x^2}\,dx$$

$$= 8\int_{-3}^{3} \sqrt{9 - x^2}\,dx - \int_{-3}^{3} 2x\sqrt{9 - x^2}\,dx$$

For the first integral, see Formula (3) of Section 7.4.

$$= 8\left(\frac{9}{2}\pi\right) - \int_{-3}^{3} 2x\sqrt{9 - x^2}\,dx$$

The second integral is 0 because the integrand is an odd function.

$$= 8\left(\frac{9}{2}\pi\right) - 0 = 36\pi$$ ◀

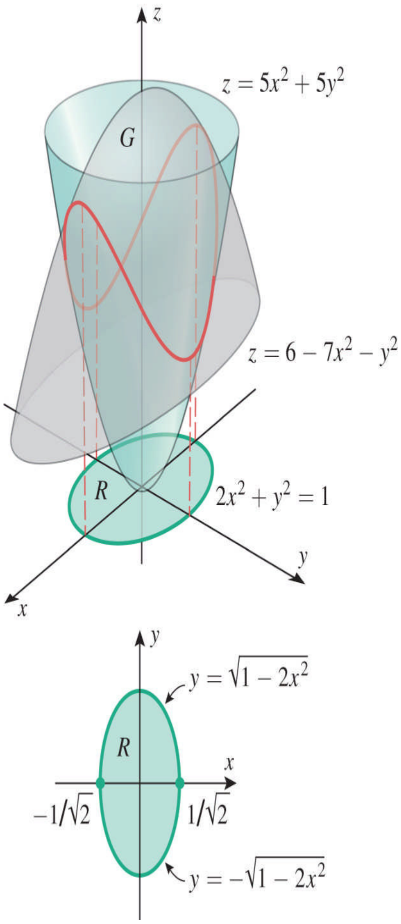


Figure 5.5.5

Example 4 Find the volume of the solid enclosed between the paraboloids

$$z = 5x^2 + 5y^2 \quad \text{and} \quad z = 6 - 7x^2 - y^2$$

Solution. The solid G and its projection R on the xy-plane are shown in Figure 5.5.5. The projection R is obtained by solving the given equations simultaneously to determine where the paraboloids intersect. We obtain

$$5x^2 + 5y^2 = 6 - 7x^2 - y^2$$

or

$$2x^2 + y^2 = 1 \tag{7}$$

which tells us that the paraboloids intersect in a curve on the elliptic cylinder given by (7). The projection of this intersection on the xy-plane is an ellipse with this same equation. Therefore,

$$\text{volume of } G = \iiint_G dV = \iint_R \left[\int_{5x^2+5y^2}^{6-7x^2-y^2} dz\right] dA$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} \int_{5x^2+5y^2}^{6-7x^2-y^2} dz\,dy\,dx$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} (6 - 12x^2 - 6y^2)\,dy\,dx$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left[6(1-2x^2)y - 2y^3\right]_{y=-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} dx$$

$$= 8\int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1-2x^2)^{3/2}\,dx = \frac{8}{\sqrt{2}}\int_{-\pi/2}^{\pi/2} \cos^4\theta\,d\theta = \frac{3\pi}{\sqrt{2}}$$

Let $x = \frac{1}{\sqrt{2}}\sin\theta$.

Use the Wallis cosine formula in Exercise 70 of Section 7.3.

Example 5 In Example 2 we evaluated

$$\iiint_G z\,dV$$

over the wedge in Figure 5.5.4 by integrating first with respect to z. Evaluate this integral by integrating first with respect to x.

Solution. The solid is bounded in the back by the plane $x = 0$ and in the front by the plane $x = y$, so

$$\iiint_G z\,dV = \iint_R \left[\int_0^y z\,dx\right] dA$$

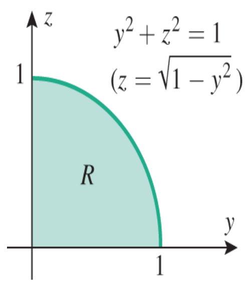


Figure 5.5.6

where R is the projection of G on the yz-plane (Figure 5.5.6). The integration over R can be performed first with respect to z and then y or vice versa. Performing the z-integration first yields

$$\iiint_G z\,dV = \int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^y z\,dx\,dz\,dy = \int_0^1 \int_0^{\sqrt{1-y^2}} zx\Big]_{x=0}^{y} dz\,dy$$

$$= \int_0^1 \int_0^{\sqrt{1-y^2}} zy\,dz\,dy = \int_0^1 \frac{1}{2}z^2 y\Big]_{z=0}^{\sqrt{1-y^2}} dy = \int_0^1 \frac{1}{2}(1-y^2)y\,dy = \frac{1}{8}$$