MULTIPLE INTEGRALS

5.1 DOUBLE INTEGRALS

The notion of a definite integral can be extended to functions of two or more variables. In this section we will discuss the double integral, which is the extension to functions of two variables.

VOLUME

Recall that the definite integral of a function of one variable

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k \tag{1}$$

arose from the problem of finding areas under curves. [In the rightmost expression in (1), we use the "limit as $n \to +\infty$" to encapsulate the process by which we increase the number of subintervals of $[a, b]$ in such a way that the lengths of the subintervals approach zero.] Integrals of functions of two variables arise from the problem of finding volumes under surfaces.

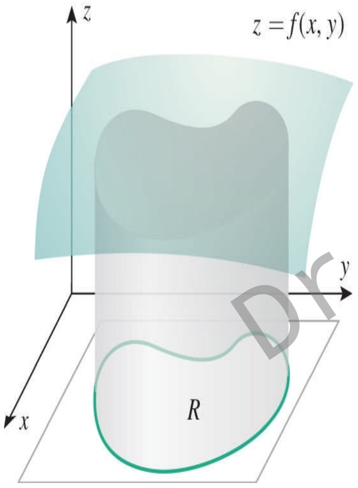


Figure 5.1.1

5.1.1 THE VOLUME PROBLEM Given a function f of two variables that is continuous and nonnegative on a region R in the xy-plane, find the volume of the solid enclosed between the surface $z = f(x, y)$ and the region R (Figure 5.1.1).

Later, we will place more restrictions on the region R, but for now we will just assume that the entire region can be enclosed within some suitably large rectangle with sides parallel to the coordinate axes. This ensures that R does not extend indefinitely in any direction.

The procedure for finding the volume V of the solid in Figure 5.1.1 will be similar to the limiting process used for finding areas, except that now the approximating elements will be rectangular parallelepipeds rather than rectangles. We proceed as follows:

- Using lines parallel to the coordinate axes, divide the rectangle enclosing the region R into subrectangles, and exclude from consideration all those subrectangles that contain any points outside of R. This leaves only rectangles that are subsets of R

DEFINITION OF A DOUBLE INTEGRAL

As in Definition 5.1.2, the notation $n \to +\infty$ in (3) encapsulates a process in which the enclosing rectangle for R is repeatedly subdivided in such a way that both the lengths and the widths of the subrectangles approach zero. Thus, we have extended the notion conveyed by Formula (1) where the definite integral of a one-variable function is expressed as a limit of Riemann sums. By extension, the sums in (3) are also called ***Riemann sums***, and the limit of the Riemann sums is denoted by

$$\iint_R f(x, y)\, dA = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k \tag{4}$$

which is called the ***double integral*** of $f(x, y)$ over R.

If f is continuous and nonnegative on the region R, then the volume formula in (2) can be expressed as

$$V = \iint_R f(x, y)\, dA \tag{5}$$

If f has both positive and negative values on R, then a positive value for the double integral of f over R means that there is more volume above R than below, a negative value for the double integral means that there is more volume below R than above, and a value of zero means that the volume above R is the same as the volume below R.

EVALUATING DOUBLE INTEGRALS

Except in the simplest cases, it is impractical to obtain the value of a double integral from the limit in (4). However, we will now show how to evaluate double integrals by calculating two successive single integrals. For the rest of this section we will limit our discussion to the case where R is a rectangle; in the next section we will consider double integrals over more complicated regions.

The partial derivatives of a function $f(x, y)$ are calculated by holding one of the variables fixed and differentiating with respect to the other variable. Let us consider the reverse of this process, ***partial integration***. The symbols

$$\int_a^b f(x, y)\, dx \quad \text{and} \quad \int_c^d f(x, y)\, dy$$

denote ***partial definite integrals***; the first integral, called the ***partial definite integral with respect to x***, is evaluated by holding y fixed and integrating with respect to x, and the second integral, called the ***partial definite integral with respect to y***, is evaluated by holding x fixed and integrating with respect to y. As the following example shows, the partial definite integral with respect to x is a function of y, and the partial definite integral with respect to y is a function of x.

► Example 1

$$\int_0^1 xy^2\, dx = y^2 \int_0^1 x\, dx = \frac{y^2x^2}{2}\bigg]_{x=0}^{1} = \frac{y^2}{2}$$

$$\int_0^1 xy^2\, dy = x \int_0^1 y^2\, dy = \frac{xy^3}{3}\bigg]_{y=0}^{1} = \frac{x}{3} \blacktriangleleft$$

respect to x. This two-stage integration process is called ***iterated*** (or ***repeated***) ***integration***. We introduce the following notation:

$$V = \iint f(x,y)\,dA$$

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy \qquad (6)$$

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx \qquad (7)$$

These integrals are called ***iterated integrals***.

► Example 2 Evaluate

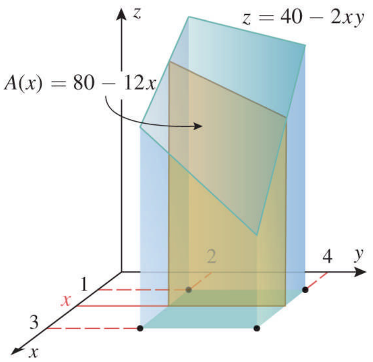
$$\text{(a)} \quad \int_1^3 \int_2^4 (40 - 2xy)\,dy\,dx \qquad \text{(b)} \quad \int_2^4 \int_1^3 (40 - 2xy)\,dx\,dy$$

Solution (a).

$$\begin{aligned}
\int_1^3 \int_2^4 (40 - 2xy)\,dy\,dx &= \int_1^3 \left[\int_2^4 (40 - 2xy)\,dy\right] dx \\
&= \int_1^3 (40y - xy^2)\Big]_{y=2}^{4}\,dx \\
&= \int_1^3 [(160 - 16x) - (80 - 4x)]\,dx \\
&= \int_1^3 (80 - 12x)\,dx \\
&= (80x - 6x^2)\Big]_1^3 = 112
\end{aligned}$$

Solution (b).

$$\begin{aligned}
\int_2^4 \int_1^3 (40 - 2xy)\,dx\,dy &= \int_2^4 \left[\int_1^3 (40 - 2xy)\,dx\right] dy \\
&= \int_2^4 (40x - x^2y)\Big]_{x=1}^{3}\,dy \\
&= \int_2^4 [(120 - 9y) - (40 - y)]\,dy \\
&= \int_2^4 (80 - 8y)\,dy \\
&= (80y - 4y^2)\Big]_2^4 = 112 \quad ◄
\end{aligned}$$



▲ **Figure 5.1.4**

It is no accident that both parts of Example 2 produced the same answer. Consider the solid S bounded above by the surface $z = 40 - 2xy$ and below by the rectangle R defined by $1 \le x \le 3$ and $2 \le y \le 4$. the volume of S is given by

$$V = \int_1^3 A(x)\,dx$$

where $A(x)$ is the area of a vertical cross section of S taken perpendicular to the x-axis (Figure 5.1.4). For a fixed value of x, $1 \le x \le 3$, $z = 40 - 2xy$ is a function of y, so the integral

$$A(x) = \int_2^4 (40 - 2xy)\,dy$$

represents the area under the graph of this function of y. Thus,

$$V = \int_1^3 \left[\int_2^4 (40 - 2xy)\,dy\right] dx = \int_1^3 \int_2^4 (40 - 2xy)\,dy\,dx$$

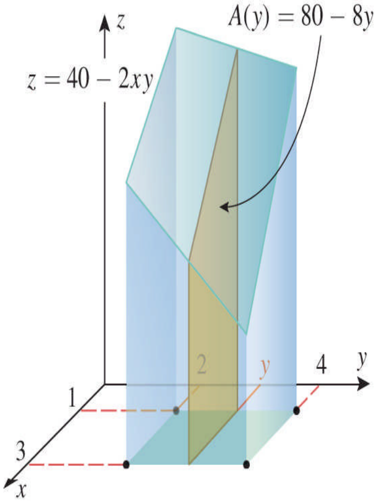


Figure 5.1.5

We will often denote the rectangle

$$\{(x, y) : a \le x \le b, c \le y \le d\}$$

as $[a, b] \times [c, d]$ for simplicity.

is the volume of S. Similarly, by the method of slicing with cross sections of S taken perpendicular to the y-axis, the volume of S is given by

$$V = \int_2^4 A(y)\,dy = \int_2^4 \left[\int_1^3 (40 - 2xy)\,dx\right] dy = \int_2^4 \int_1^3 (40 - 2xy)\,dx\,dy$$

(Figure 5.1.5). Thus, the iterated integrals in parts (a) and (b) of Example 2 both measure the volume of S, which by Formula (5) is the double integral of $z = 40 - 2xy$ over R. That is,

$$\int_1^3 \int_2^4 (40 - 2xy)\,dy\,dx = \iint_R (40 - 2xy)\,dA = \int_2^4 \int_1^3 (40 - 2xy)\,dx\,dy$$

The geometric argument above applies to any continuous function $f(x, y)$ that is nonnegative on a rectangle $R = [a, b] \times [c, d]$, as is the case for $f(x, y) = 40 - 2xy$ on $[1, 3] \times [2, 4]$. The conclusion that the double integral of $f(x, y)$ over R has the same value as either of the two possible iterated integrals is true even when f is negative at some points in R. We state this result in the following theorem and omit a formal proof.

5.1.3 THEOREM (*Fubini's Theorem*) *Let R be the rectangle defined by the inequalities*

$$a \le x \le b, \quad c \le y \le d$$

If $f(x, y)$ is continuous on this rectangle, then

$$\iint_R f(x, y)\,dA = \int_c^d \int_a^b f(x, y)\,dx\,dy = \int_a^b \int_c^d f(x, y)\,dy\,dx$$

Theorem 5.1.3 allows us to evaluate a double integral over a rectangle by converting it to an iterated integral. This can be done in two ways, both of which produce the value of the double integral.

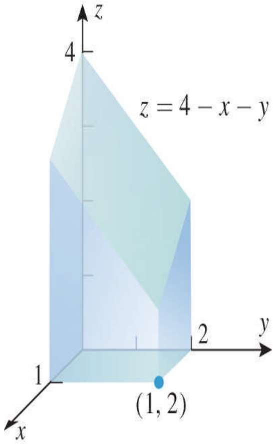


Figure 5.1.6

Example 3 Use a double integral to find the volume of the solid that is bounded above by the plane $z = 4 - x - y$ and below by the rectangle $R = [0, 1] \times [0, 2]$ (Figure 5.1.6).

Solution. The volume is the double integral of $z = 4 - x - y$ over R. Using Theorem 5 .1.3, this can be obtained from either of the iterated integrals

$$\int_0^2 \int_0^1 (4 - x - y)\,dx\,dy \quad \text{or} \quad \int_0^1 \int_0^2 (4 - x - y)\,dy\,dx \tag{8}$$

Using the first of these, we obtain

$$V = \iint_R (4 - x - y)\, dA = \int_0^2 \int_0^1 (4 - x - y)\, dx\, dy$$
$$= \int_0^2 \left[4x - \frac{x^2}{2} - xy \right]_{x=0}^{1} dy = \int_0^2 \left(\frac{7}{2} - y \right) dy$$
$$= \left[\frac{7}{2}y - \frac{y^2}{2} \right]_0^2 = 5$$

You can check this result by evaluating the second integral in (8). ◄

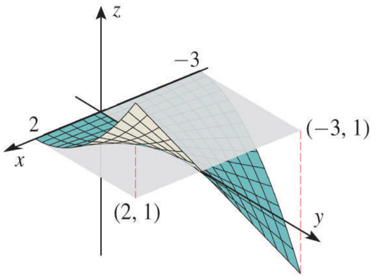
► **Example 4** Evaluate the double integral

$$\iint_R y^2 x\, dA$$

over the rectangle $R = \{(x, y) : -3 \le x \le 2, 0 \le y \le 1\}$.

Solution. In view of Theorem 5.1.3, the value of the double integral can be obtained by evaluating one of two possible iterated double integrals. We choose to integrate first with respect to x and then with respect to y.

$$\iint_R y^2 x\, dA = \int_0^1 \int_{-3}^{2} y^2 x\, dx\, dy = \int_0^1 \left[\frac{1}{2} y^2 x^2 \right]_{x=-3}^{2} dy$$
$$= \int_0^1 \left(-\frac{5}{2} y^2 \right) dy = -\frac{5}{6} y^3 \Big]_0^1 = -\frac{5}{6}$$ ◄



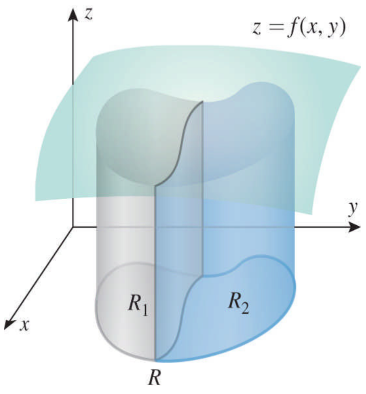
$z = y^2 x$ on $[-3, 2] \times [0, 1]$

Figure 5.1.7

The integral in Example 4 can be interpreted as the net signed volume between the rectangle $[-3, 2] \times [0, 1]$ and the surface $z = y^2 x$. That is, it is the volume below $z = y^2 x$ and above $[0, 2] \times [0, 1]$ minus the volume above $z = y^2 x$ and below $[-3, 0] \times [0, 1]$ (Figure 5.1.7).

PROPERTIES OF DOUBLE INTEGRALS

To distinguish between double integrals of functions of two variables and definite integrals of functions of one variable, we will refer to the latter as ***single integrals***. Because double integrals, like single integrals, are defined as limits, they inherit many of the properties of limits. The following results, which we state without proof, are analogs of those in Theorem 5.5.4.



The volume of the entire solid is the sum of the volumes of the solids above R_1 and R_2.

Figure 5.1.8

$$\iint_R cf(x, y)\, dA = c \iint_R f(x, y)\, dA \quad (c \text{ a constant}) \tag{9}$$

$$\iint_R [f(x, y) + g(x, y)]\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA \tag{10}$$

$$\iint_R [f(x, y) - g(x, y)]\, dA = \iint_R f(x, y)\, dA - \iint_R g(x, y)\, dA \tag{11}$$

Figure 5.1.8 illustrates the result that if $f(x, y)$ is nonnegative on a region R, then subdividing R into two regions R_1 and R_2 has the effect of subdividing the solid between R

and $z = f(x, y)$ into two solids, the sum of whose volumes is the volume of the entire solid. This suggests the following result, which holds even if f has negative values:

$$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA \tag{12}$$

5.2 DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

In this section we will show how to evaluate double integrals over regions other than rectangles.

ITERATED INTEGRALS WITH NONCONSTANT LIMITS OF INTEGRATION

Later in this section we will see that double integrals over nonrectangular regions can often be evaluated as iterated integrals of the following types:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy \right] dx \tag{1}$$

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x, y)\, dx \right] dy \tag{2}$$

Note that in (1) and (2) the limits of integration in the outer integral are constants. This is consistent with the fact that the value of each iterated integral is a number that represents a net signed volume.

We begin with an example that illustrates how to evaluate such integrals.

Example 1 Evaluate

$$\text{(a)} \int_0^1 \int_{-x}^{x^2} y^2 x\, dy\, dx \qquad \text{(b)} \int_0^{\pi/3} \int_0^{\cos y} x \sin y\, dx\, dy$$

Solution (a).

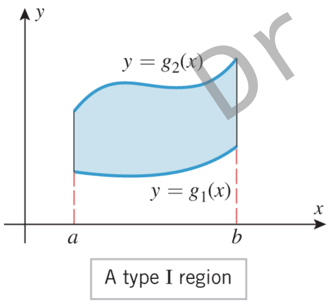
$$\int_0^1 \int_{-x}^{x^2} y^2 x\, dy\, dx = \int_0^1 \left[\int_{-x}^{x^2} y^2 x\, dy \right] dx = \int_0^1 \left. \frac{y^3 x}{3} \right]_{y=-x}^{x^2} dx$$

$$= \int_0^1 \left[\frac{x^7}{3} + \frac{x^4}{3} \right] dx = \left(\frac{x^8}{24} + \frac{x^5}{15} \right) \Bigg]_0^1 = \frac{13}{120}$$

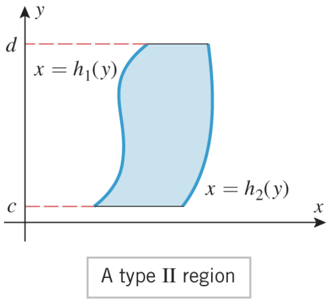
Solution (b).

$$\int_0^{\pi/3} \int_0^{\cos y} x \sin y\, dx\, dy = \int_0^{\pi/3} \left[\int_0^{\cos y} x \sin y\, dx \right] dy = \int_0^{\pi/3} \left. \frac{x^2}{2} \sin y \right]_{x=0}^{\cos y} dy$$

$$= \int_0^{\pi/3} \left[\frac{1}{2} \cos^2 y \sin y \right] dy = -\frac{1}{6} \cos^3 y \Bigg]_0^{\pi/3} = \frac{7}{48} \blacktriangleleft$$



(a)



(b)

DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

Plane regions can be extremely complex, and the theory of double integrals over very general regions is a topic for advanced courses in mathematics. We will limit our study of double integrals to two basic types of regions, which we will call *type I* and *type II*; they are defined as follows.

5.2.1 DEFINITION

(a) A ***type I region*** is bounded on the left and right by vertical lines $x = a$ and $x = b$ and is bounded below and above by continuous curves $y = g_1(x)$ and $y = g_2(x)$, where g

(b) A ***type II region*** is bounded below and above by horizontal lines $y = c$ and $y = d$ and is bounded on the left and right by continuous curves $x = h_1(y)$ and $x = h_2(y)$ satisfying $h_1(y) \le h_2(y)$ for $c \le y \le d$ (Figure 5.2.1b).

The following theorem will enable us to evaluate double integrals over type I and type II regions using iterated integrals.

5.2.2 THEOREM

(a) If R is a type I region on which $f(x, y)$ is continuous, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx \tag{3}$$

(b) If R is a type II region on which $f(x, y)$ is continuous, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy \tag{4}$$

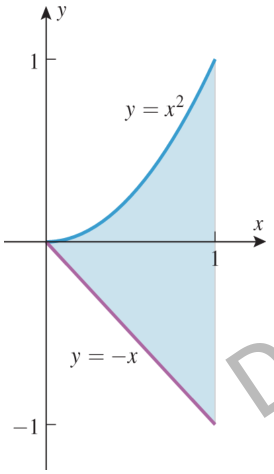


Figure 5.2.2

Example 2 Each of the iterated integrals in Example 1 is equal to a double integral over a region R. Identify the region R in each case.

Solution. Using Theorem 5.2.2, the integral in Example 1(a) is the double integral of the function $f(x, y) = y^2x$ over the type I region R bounded on the left and right by the vertical lines $x = 0$ and $x = 1$ and bounded below and above by the curves $y = -x$ and $y = x^2$ (Figure 5.2.2). The integral in Example 1(b) is the double integral of the function $f(x, y) = x \sin y$ over the type II region R bounded below and above by the horizontal lines $y = 0$ and $y = \pi/3$ and bounded on the left and right by the curves $x = 0$ and $x = \cos y$ (Figure 5.2.3). ◀

We will not prove Theorem 5.2.2, but for the case where $f(x, y)$ is nonnegative on the region R, it can be made plausible by a geometric argument that is similar to that given for Theorem 5.1.3. Since $f(x, y)$ is nonnegative, the double integral can be interpreted as the volume of the solid S that is bounded above by the surface $z = f(x, y)$ and below by the region R, so it suffices to show that the iterated integrals also represent this volume. Consider the iterated integral in (3), for example. For a fixed value of x, the function $f(x, y)$ is a function of y, and hence the integral

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy$$

represents the area under the graph of this function of y between $y = g_1(x)$ and $y = g_2(x)$. This area, shown in yellow in Figure 5.2.4, is the cross-sectional area at x of the solid S, and hence by the method of slicing, the volume V of the solid S is

$$V = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

which shows that in (3) the iterated integral is equal to the double integral. Similarly, the iterated integral in (4) is equal to the corresponding double integral.

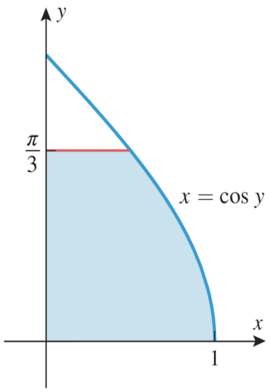


Figure 5.2.3

SETTING UP LIMITS OF INTEGRATION FOR EVALUATING DOUBLE INTEGRALS

To apply Theorem 5.2.2, it is helpful to start with a two-dimensional sketch of the region R. [It is not necessary to graph $f(x, y)$.] For a type I region, the limits of integration in Formula (3) can be obtained as follows:

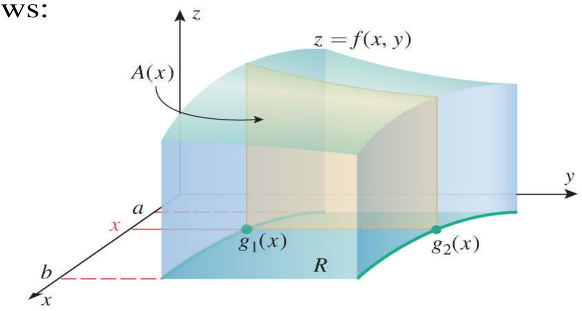


Figure 5.2.4

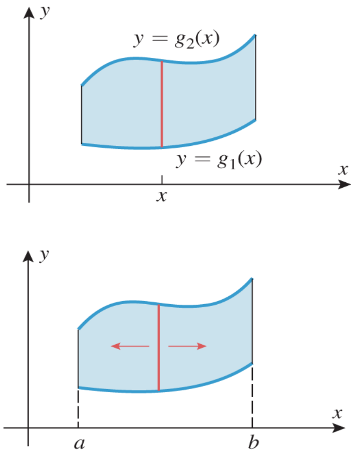


Figure 5.2.5

Determining Limits of Integration: Type I Region

Step 1. Since x is held fixed for the first integration, we draw a vertical line through the region R at an arbitrary fixed value x (Figure 5.2.5). This line crosses the boundary of R twice. The lower point of intersection is on the curve $y = g_1(x)$ and the higher point is on the curve $y = g_2(x)$. These two intersections determine the lower and upper y-limits of integration in Formula (3).

Step 2. Imagine moving the line drawn in Step 1 first to the left and then to the right (Figure 5.2.5). The leftmost position where the line intersects the region R is $x = a$, and the rightmost position where the line intersects the region R is $x = b$. This yields the limits for the x-integration in Formula (3).

Example 3 Evaluate

$$\iint_R xy\, dA$$

over the region R enclosed between $y = \frac{1}{2}x$, $y = \sqrt{x}$, $x = 2$, and $x = 4$.

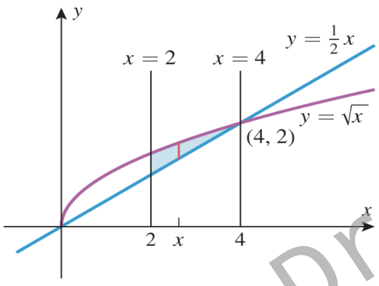


Figure 5.2.6

Solution. We view R as a type I region. The region R and a vertical line corresponding to a fixed x are shown in Figure 4.2.6. This line meets the region R at the lower boundary $y = \frac{1}{2}x$ and the upper boundary $y = \sqrt{x}$. These are the y-limits of integration. Moving this line first left and then right yields the x-limits of integration, $x = 2$ and $x = 4$. Thus,

$$\iint_R xy\, dA = \int_2^4 \int_{x/2}^{\sqrt{x}} xy\, dy\, dx = \int_2^4 \left[\frac{xy^2}{2}\right]_{y=x/2}^{\sqrt{x}} dx = \int_2^4 \left(\frac{x^2}{2} - \frac{x^3}{8}\right) dx$$

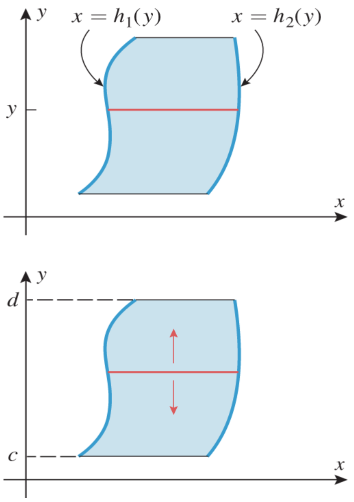
$$= \left[\frac{x^3}{6} - \frac{x^4}{32}\right]_2^4 = \left(\frac{64}{6} - \frac{256}{32}\right) - \left(\frac{8}{6} - \frac{16}{32}\right) = \frac{11}{6} \blacktriangleleft$$

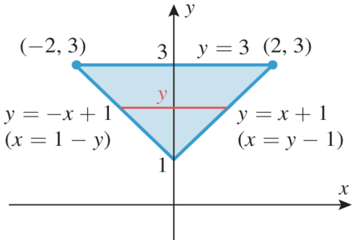
If R is a type II region, then the limits of integration in Formula (4) can be obtained as follows:

Determining Limits of Integration: Type II Region

Step 1. Since y is held fixed for the first integration, we draw a horizontal line through the region R at a fixed value y (Figure 5.2.7). This line crosses the boundary of R twice. The leftmost point of intersection is on the curve $x = h_1(y)$ and the rightmost point is on the curve $x = h_2(y)$. These intersections determine the x-limits of integration in (4).

Step 2. Imagine moving the line drawn in Step 1 first down and then up (Figure 5.2.7). The lowest position where the line intersects the region R is $y = c$, and the highest position where the line intersects the region R is $y = d$. This yields the y-limits of integration in (4).





▲ **Figure 5.2.8**

▶ **Example 4** Evaluate

$$\iint_R (2x - y^2)\, dA$$

over the triangular region R enclosed between the lines $y = -x + 1$, $y = x + 1$, and $y = 3$.

Solution. We view R as a type II region. The region R and a horizontal line corresponding to a fixed y are shown in Figure 5.2.8. This line meets the region R at its left-hand boundary $x = 1 - y$ and its right-hand boundary $x = y - 1$. These are the x-limits of integration. Moving this line first down and then up yields the y-limits, $y = 1$ and $y = 3$. Thus,

$$\iint_R (2x - y^2)\, dA = \int_1^3 \int_{1-y}^{y-1} (2x - y^2)\, dx\, dy = \int_1^3 \left[x^2 - y^2 x\right]_{x=1-y}^{y-1} dy$$

$$= \int_1^3 [(1 - 2y + 2y^2 - y^3) - (1 - 2y + y^3)]\, dy$$

$$= \int_1^3 (2y^2 - 2y^3)\, dy = \left[\frac{2y^3}{3} - \frac{y^4}{2}\right]_1^3 = -\frac{68}{3} ◀$$

To integrate over a type II region, the left- and right-hand boundaries must be expressed in the form $x = h_1(y)$ and $x = h_2(y)$. This is why we rewrote the boundary equations

$$y = -x + 1 \quad \text{and} \quad y = x + 1$$

as

$$x = 1 - y \quad \text{and} \quad x = y - 1$$

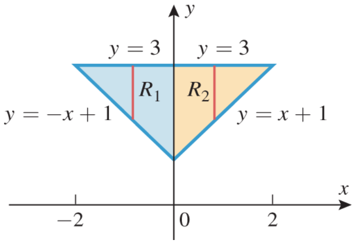
in Example 4.

In Example 4 we could have treated R as a type I region, but with an added complication. Viewed as a type I region, the upper boundary of R is the line $y = 3$ (Figure 5.2.9) and the lower boundary consists of two parts, the line $y = -x + 1$ to the left of the y-axis and the line $y = x + 1$ to the right of the y-axis. To carry out the integration it is necessary to decompose the region R into two parts, R_1 and R_2, as shown in Figure 5.2.9, and write

$$\iint_R (2x - y^2)\, dA = \iint_{R_1} (2x - y^2)\, dA + \iint_{R_2} (2x - y^2)\, dA$$

$$= \int_{-2}^{0} \int_{-x+1}^{3} (2x - y^2)\, dy\, dx + \int_0^2 \int_{x+1}^{3} (2x - y^2)\, dy\, dx$$

This will yield the same result that was obtained in Example 4. (Verify.)



▲ **Figure 5.2.9**

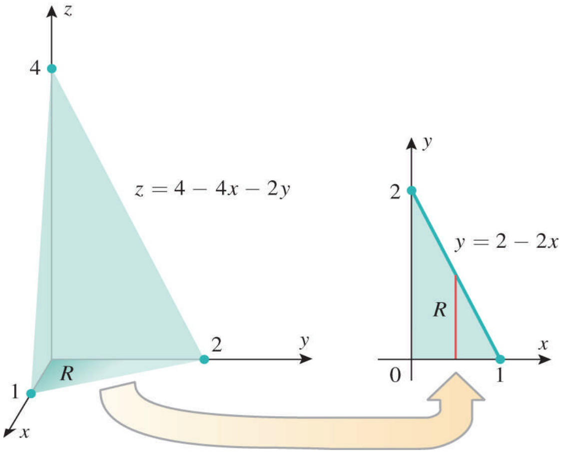
▶ **Example 5** Use a double integral to find the volume of the tetrahedron bounded by the coordinate planes and the plane $z = 4 - 4x - 2y$.

Solution. The tetrahedron in question is bounded above by the plane

$$z = 4 - 4x - 2y \tag{5}$$

and below by the triangular region R shown in Figure 5.2.10. Thus, the volume is given by

$$V = \iint_R (4 - 4x - 2y)\, dA$$



▶ **Figure 5.2.10**

area calculated as a double integral

although double integrals arose in the context of calculating volumes, they can also be used to calculate areas. to see why this is so, recall that a right cylinder is a solid that is generated when a plane region is translated along a line that is perpendicular to the region.

▶ **Example 6** Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

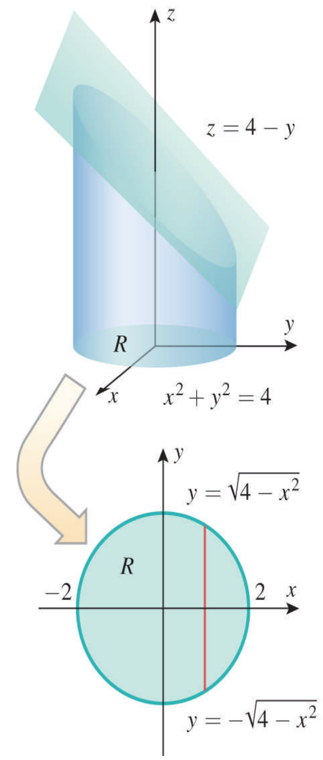
Solution. The solid shown in Figure 5.2.11 is bounded above by the plane $z = 4 - y$ and below by the region R within the circle $x^2 + y^2 = 4$. The volume is given by

$$V = \iint_R (4 - y)\, dA$$

Treating R as a type I region we obtain

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - y)\, dy\, dx = \int_{-2}^{2} \left[4y - \frac{1}{2}y^2\right]_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} 8\sqrt{4 - x^2}\, dx = 8(2\pi) = 16\pi \quad ◀$$



▲ Figure 5.2.11

REVERSING THE ORDER OF INTEGRATION

Sometimes the evaluation of an iterated integral can be simplified by reversing the order of integration. The next example illustrates how this is done.

▶ **Example 7** Since there is no elementary antiderivative of e^{x^2}, the integral

$$\int_0^2 \int_{y/2}^1 e^{x^2}\, dx\, dy$$

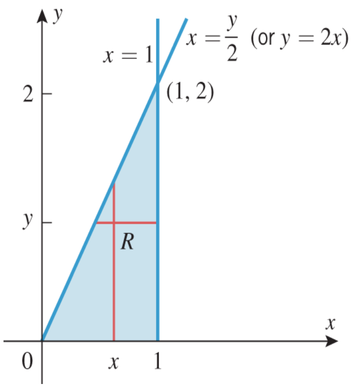
cannot be evaluated by performing the x-integration first. Evaluate this integral by expressing it as an equivalent iterated integral with the order of integration reversed.

Solution. For the inside integration, y is fixed and x varies from the line $x = y/2$ to the line $x = 1$ (Figure 5.2.12). For the outside integration, y varies from 0 to 2, so the given iterated integral is equal to a double integral over the triangular region R in Figure 5.2.12.

To reverse the order of integration, we treat R as a type I region, which enables us to write the given integral as

$$\int_0^2 \int_{y/2}^1 e^{x^2}\, dx\, dy = \iint_R e^{x^2}\, dA = \int_0^1 \int_0^{2x} e^{x^2}\, dy\, dx = \int_0^1 \left[e^{x^2} y\right]_{y=0}^{2x} dx$$

$$= \int_0^1 2xe^{x^2}\, dx = e^{x^2}\Big]_0^1 = e - 1 \quad ◀$$

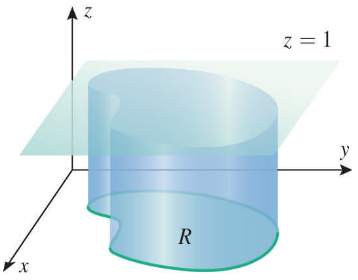


▲ Figure 5.2.12

AREA CALCULATED AS A DOUBLE INTEGRAL

Although double integrals arose in the context of calculating volumes, they can also be used to calculate areas. To see why this is so, recall that a *right cylinder* is a solid that is generated when a plane region is translated along a line that is perpendicular to the region. In Formula (2) of Section 6.2 we stated that the volume V of a right cylinder with cross-sectional area A and height h is

$$V = A \cdot h \tag{6}$$



Cylinder with base R and height 1

Figure 5.2.13

Formula (7) can be confusing because it equates an area and a volume; the formula is intended to equate only the *numerical values* of the area and volume and not the units, which must, of course, be different.

Now suppose that we are interested in finding the area A of a region R in the xy-plane. If we translate the region R upward 1 unit, then the resulting solid will be a right cylinder that has cross-sectional area A, base R, and the plane $z = 1$ as its top (Figure 5.2.13). Thus, it follows from (6) that

$$\iint_R 1\, dA = (\text{area of } R) \cdot 1$$

which we can rewrite as

$$\text{area of } R = \iint_R 1\, dA = \iint_R dA \tag{7}$$

Example 8 Use a double integral to find the area of the region R enclosed between the parabola $y = \frac{1}{2}x^2$ and the line $y = 2x$.

Solution. The region R may be treated equally well as type I (Figure 5.2.14*a*) or type II (Figure 5.2.14*b*). Treating R as type I yields

$$\begin{aligned}\text{area of } R = \iint_R dA &= \int_0^4 \int_{x^2/2}^{2x} dy\, dx = \int_0^4 [y]_{y=x^2/2}^{2x}\, dx \\ &= \int_0^4 \left(2x - \frac{1}{2}x^2\right) dx = \left[x^2 - \frac{x^3}{6}\right]_0^4 = \frac{16}{3}\end{aligned}$$

Treating R as type II yields

$$\begin{aligned}\text{area of } R = \iint_R dA &= \int_0^8 \int_{y/2}^{\sqrt{2y}} dx\, dy = \int_0^8 [x]_{x=y/2}^{\sqrt{2y}}\, dy \\ &= \int_0^8 \left(\sqrt{2y} - \frac{1}{2}y\right) dy = \left[\frac{2\sqrt{2}}{3}y^{3/2} - \frac{y^2}{4}\right]_0^8 = \frac{16}{3}\end{aligned}$$ ◀

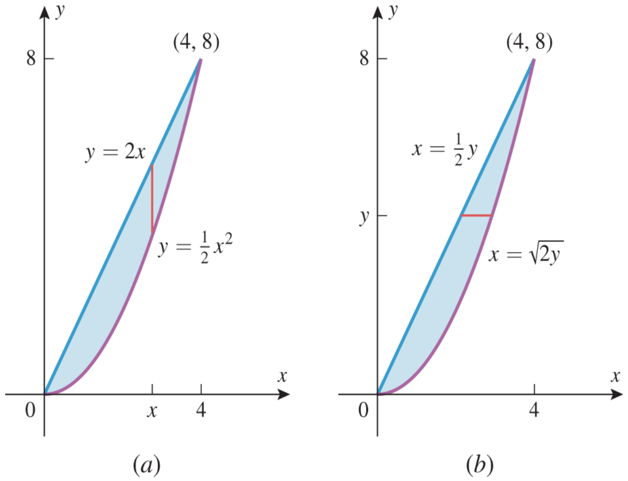


Figure 5.2.14

5.3 DOUBLE INTEGRALS IN POLAR COORDINATES

In this section we will study double integrals in which the integrand and the region of integration are expressed in polar coordinates. Such integrals are important for two reasons: first, they arise naturally in many applications, and second, many double integrals in rectangular coordinates can be evaluated more easily if they are converted to polar coordinates.

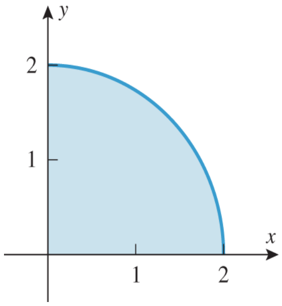


Figure 5.3.1

SIMPLE POLAR REGIONS

Some double integrals are easier to evaluate if the region of integration is expressed in polar coordinates. This is usually true if the region is bounded by a cardioid, a rose curve, a spiral, or, more generally, by any curve whose equation is simpler in polar coordinates than in rectangular coordinates. For example, the quarter-disk in Figure 5.3.1 is described in rectangular coordinates by

$$0 \le y \le \sqrt{4-x^2}, \quad 0 \le x \le 2$$

However, in polar coordinates the region is described more simply by

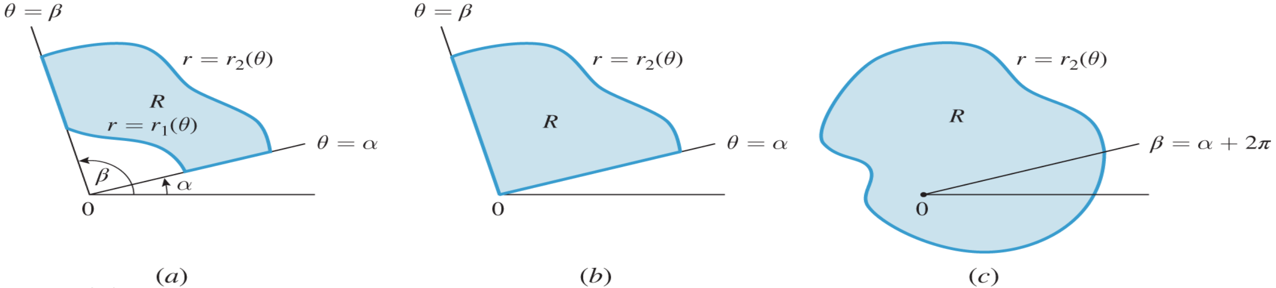
$$0 \le r \le 2, \quad 0 \le \theta \le \pi/2$$

Moreover, double integrals whose integrands involve $x^2 + y^2$ also tend to be easier to evaluate in polar coordinates because this sum simplifies to r^2 when the conversion formulas $x = r\cos\theta$ and $y = r\sin\theta$ are applied.

Figure 5.3.2*a* shows a region R in a polar coordinate system that is enclosed between two rays, $\theta = \alpha$ and $\theta = \beta$, and two polar curves, $r = r_1(\theta)$ and $r = r_2(\theta)$. If, as shown in the figure, the functions $r_1(\theta)$ and $r_2(\theta)$ are continuous and their graphs do not cross, then the region R is called a *simple polar region*. If $r_1(\theta)$ is identically zero, then the boundary $r = r_1(\theta)$ reduces to a point (the origin), and the region has the general shape shown in Figure 5.3.2*b*. If, in addition, $\beta = \alpha + 2\pi$, then the rays coincide, and the region has the general shape shown in Figure 5.3.2*c*. The following definition expresses these geometric ideas algebraically.

5.3.1 DEFINITION A ***simple polar region*** in a polar coordinate system is a region that is enclosed between two rays, $\theta = \alpha$ and $\theta = \beta$, and two continuous polar curves, $r = r_1(\theta)$ and $r = r_2(\theta)$, where the equations of the rays and the polar curves satisfy the following conditions:

(i) $\alpha \le \beta$ (ii) $\beta - \alpha \le 2\pi$ (iii) $0 \le r_1(\theta) \le r_2(\theta)$



(*a*) (*b*) (*c*)

Figure 5.3.2 Simple polar regions

DOUBLE INTEGRALS IN POLAR COORDINATES

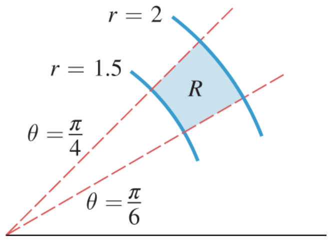


Figure 5.3.3

5.3.2 THE VOLUME PROBLEM IN POLAR COORDINATES Given a function $f(r,\theta)$ that is continuous and nonnegative on a simple polar region R, find the volume of the solid that is enclosed between the region R and the surface whose equation in cylindrical coordinates is $z = f(r,\theta)$ (Figure 5.3.4).

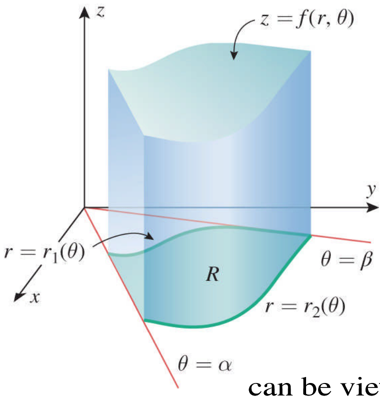


Figure 5.3.4

To motivate a formula for the volume V of the solid in Figure 5.3.4, we will use a limit process similar to that used to obtain Formula (2) of Section 5.1, except that here we will use circular arcs and rays to subdivide the region R into polar rectangles. As shown in Figure 5.3.5, we will exclude from consideration all polar rectangles that contain any points outside of R, leaving only polar rectangles that are subsets of R. Assume that there are n such polar rectangles, and denote the area of the kth polar rectangle by ΔA_k. Let (r_k^*, θ_k^*) be any point in this polar rectangle. As shown in Figure 5.3.6, the product $f(r_k^*, \theta_k^*)\Delta A_k$ is the volume of a solid with base area ΔA_k and height $f(r_k^*, \theta_k^*)$, so the sum

$$\sum_{k=1}^{n} f(r_k^*, \theta_k^*)\Delta A_k$$

can be viewed as an approximation to the volume V of the entire solid.

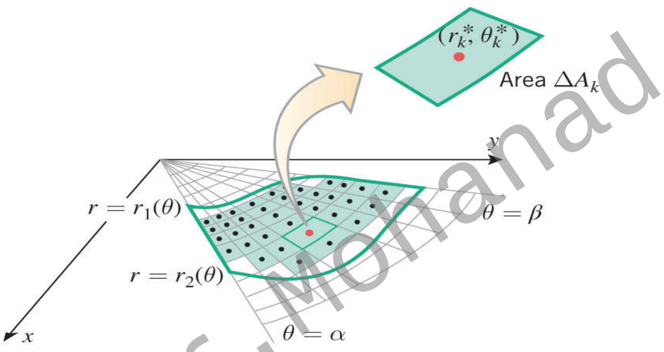


Figure 5.3.5

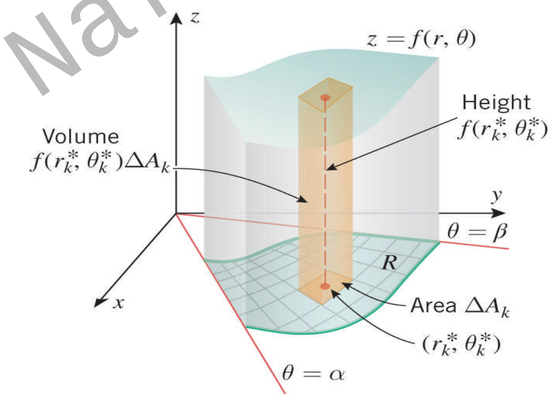


Figure 5.3.6

If we now increase the number of subdivisions in such a way that the dimensions of the polar rectangles approach zero, then it seems plausible that the errors in the approximations approach zero, and the exact volume of the solid is

$$V = \lim_{n \to +\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*)\Delta A_k \tag{1}$$

If $f(r,\theta)$ is continuous on R and has both positive and negative values, then the limit

$$\lim_{n \to +\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*)\Delta A_k \tag{2}$$

represents the net signed volume between the region R and the surface $z = f(r,\theta)$ (as with double integrals in rectangular coordinates). The sums in (2) are called ***polar Riemann sums***, and the limit of the polar Riemann sums is denoted by

$$\iint_R f(r,\theta)\,dA = \lim_{n \to +\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*)\Delta A_k \tag{3}$$

which is called the ***polar double integral*** of $f(r,\theta)$ over R. If $f(r,\theta)$ is continuous and nonnegative on R, then the volume formula (1) can be expressed as

$$V = \iint_R f(r,\theta)\,dA \tag{4}$$

EVALUATING POLAR DOUBLE INTEGRALS

In Sections 5.1 and 5.2 we evaluated double integrals in rectangular coordinates by expressing them as iterated integrals. Polar double integrals are evaluated the same way. To motivate the formula that expresses a double polar integral as an iterated integral, we will assume that $f(r,\theta)$ is nonnegative so that we can interpret (3) as a volume. To begin, let us choose the arbitrary point (r_k^*, θ_k^*) in (3) to be at the "center" of the kth polar rectangle as shown in Figure 5.3.7. Suppose also that this polar rectangle has a central angle $\Delta\theta_k$ and a "radial thickness" Δr_k. Thus, the inner radius of this polar rectangle is $r_k^* - \frac{1}{2}\Delta r_k$ and the outer radius is $r_k^* + \frac{1}{2}\Delta r_k$. Treating the area ΔA_k of this polar rectangle as the difference in area of two sectors, we obtain

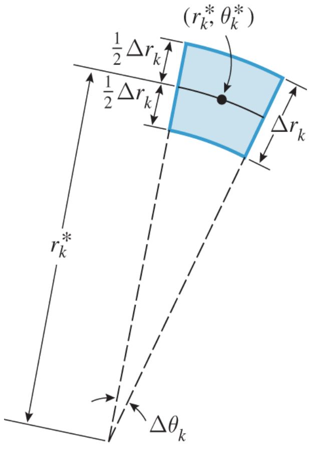


Figure 5.3.7

$$\Delta A_k = \tfrac{1}{2}\left(r_k^* + \tfrac{1}{2}\Delta r_k\right)^2 \Delta\theta_k - \tfrac{1}{2}\left(r_k^* - \tfrac{1}{2}\Delta r_k\right)^2 \Delta\theta_k$$

which simplifies to

$$\Delta A_k = r_k^* \Delta r_k \Delta\theta_k \tag{5}$$

Thus, from (3) and (4)

$$V = \iint_R f(r,\theta)\,dA = \lim_{n\to+\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*) r_k^* \Delta r_k \Delta\theta_k$$

which suggests that the volume V can be expressed as the iterated integral

$$V = \iint_R f(r,\theta)\,dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta) r\,dr\,d\theta \tag{6}$$

in which the limits of integration are chosen to cover the region R; that is, with θ fixed between α and β, the value of r varies from $r_1(\theta)$ to $r_2(\theta)$ (Figure 5.3.8).

Although we assumed that $f(r,\theta)$ to be nonnegative in deriving Formula (6), it can be proved that the relationship between the polar double integral and the iterated integral in this formula also holds if f has negative values. Accepting this to be so, we obtain the following theorem, which we state without formal proof.

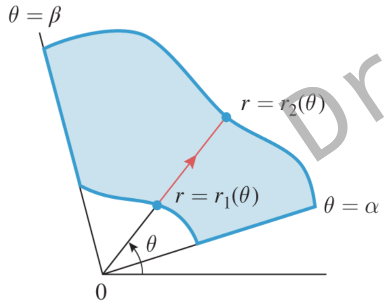


Figure 5.3.8

5.3.3 THEOREM *If R is a simple polar region whose boundaries are the rays $\theta = \alpha$ and $\theta = \beta$ and the curves $r = r_1(\theta)$ and $r = r_2(\theta)$ shown in Figure 5.3.8, and if $f(r,\theta)$ is continuous on R, then*

$$\iint_R f(r,\theta)\,dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta) r\,dr\,d\theta \tag{7}$$

To apply this theorem you will need to be able to find the rays and the curves that form the boundary of the region R, since these determine the limits of integration in the iterated integral. This can be done as follows:

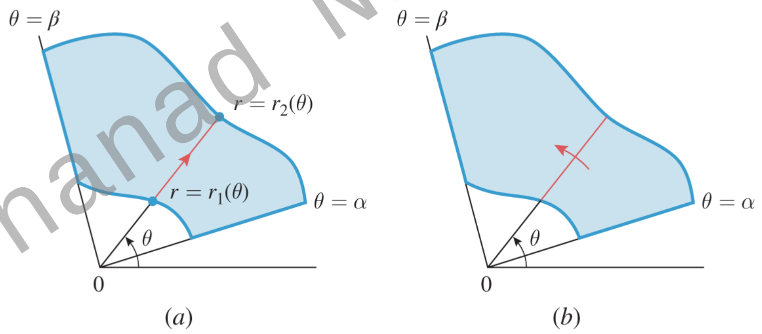
Determining Limits of Integration for a Polar Double Integral: Simple Polar Region

Step 1. Since θ is held fixed for the first integration, draw a radial line from the origin through the region R at a fixed angle θ (Figure 5.3.9a). This line crosses the boundary of R at most twice. The innermost point of intersection is on the inner boundary curve $r = r_1(\theta)$ and the outermost point is on the outer boundary curve $r = r_2(\theta)$. These intersections determine the r-limits of integration in (7).

Step 2. Imagine rotating the radial line from Step 1 about the origin, thus sweeping out the region R. The least angle at which the radial line intersects the region R is $\theta = \alpha$ and the greatest angle is $\theta = \beta$ (Figure 5.3.9b). This determines the θ-limits of integration.

► Example 1 Evaluate

$$\iint_R \sin\theta \, dA$$



(a) (b)

► Figure 5.3.9

where R is the region in the first quadrant that is outside the circle $r = 2$ and inside the cardioid $r = 2(1 + \cos\theta)$.

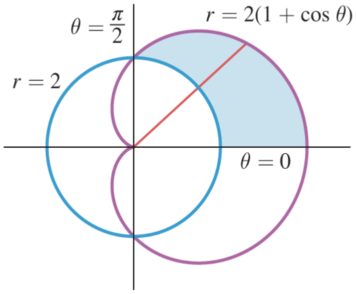
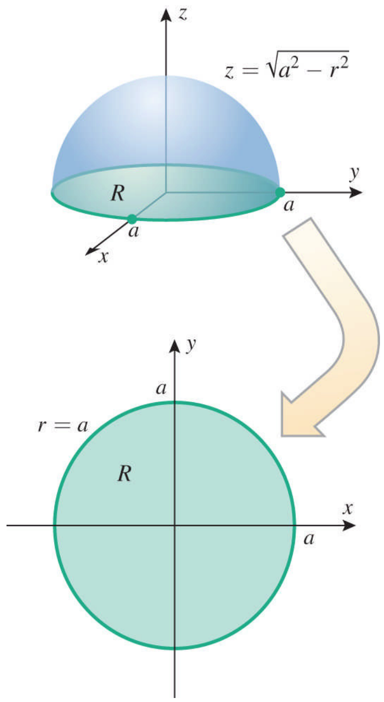


Figure 5.3.10

Solution. The region R is sketched in Figure 5.3.10. Following the two steps outlined above we obtain

$$\begin{aligned}
\iint_R \sin\theta \, dA &= \int_0^{\pi/2} \int_2^{2(1+\cos\theta)} (\sin\theta) r \, dr \, d\theta \\
&= \int_0^{\pi/2} \left[\frac{1}{2} r^2 \sin\theta \right]_{r=2}^{2(1+\cos\theta)} d\theta \\
&= 2 \int_0^{\pi/2} [(1+\cos\theta)^2 \sin\theta - \sin\theta] \, d\theta \\
&= 2 \left[-\frac{1}{3}(1+\cos\theta)^3 + \cos\theta \right]_0^{\pi/2} \\
&= 2\left[-\frac{1}{3} - \left(-\frac{5}{3} \right) \right] = \frac{8}{3} \blacktriangleleft
\end{aligned}$$



▲ **Figure 5.3.11**

▶ **Example 2** The sphere of radius a centered at the origin is expressed in rectangular coordinates as $x^2 + y^2 + z^2 = a^2$, and hence its equation in cylindrical coordinates is $r^2 + z^2 = a^2$. Use this equation and a polar double integral to find the volume of the sphere.

Solution. In cylindrical coordinates the upper hemisphere is given by the equation

$$z = \sqrt{a^2 - r^2}$$

so the volume enclosed by the entire sphere is

$$V = 2 \iint_R \sqrt{a^2 - r^2}\, dA$$

where R is the circular region shown in Figure 5.3.11. Thus,

$$V = 2 \iint_R \sqrt{a^2 - r^2}\, dA = \int_0^{2\pi} \int_0^{a} \sqrt{a^2 - r^2}(2r)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[-\frac{2}{3}(a^2 - r^2)^{3/2} \right]_{r=0}^{a} d\theta = \int_0^{2\pi} \frac{2}{3} a^3\, d\theta$$

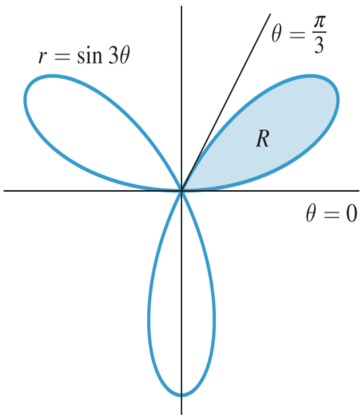
$$= \left[\frac{2}{3} a^3 \theta \right]_0^{2\pi} = \frac{4}{3}\pi a^3 ◀$$

FINDING AREAS USING POLAR DOUBLE INTEGRALS

Recall from Formula (7) of Section 5.2 that the area of a region R in the xy-plane can be expressed as

$$\text{area of } R = \iint_R 1\, dA = \iint_R dA \tag{8}$$

The argument used to derive this result can also be used to show that the formula applies to polar double integrals over regions in polar coordinates.



▲ **Figure 5.3.12**

▶ **Example 3** Use a polar double integral to find the area enclosed by the three-petaled rose $r = \sin 3\theta$.

Solution. The rose is sketched in Figure 5.3.12. We will use Formula (8) to calculate the area of the petal R in the first quadrant and multiply by 3.

$$A = 3 \iint_R dA = 3 \int_0^{\pi/3} \int_0^{\sin 3\theta} r\, dr\, d\theta$$

$$= \frac{3}{2} \int_0^{\pi/3} \sin^2 3\theta\, d\theta = \frac{3}{4} \int_0^{\pi/3} (1 - \cos 6\theta)\, d\theta$$

$$= \frac{3}{4} \left[\theta - \frac{\sin 6\theta}{6} \right]_0^{\pi/3} = \frac{1}{4}\pi ◀$$

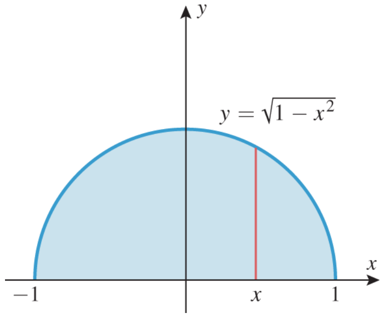
CONVERTING DOUBLE INTEGRALS FROM RECTANGULAR TO POLAR COORDINATES

Sometimes a double integral that is difficult to evaluate in rectangular coordinates can be evaluated more easily in polar coordinates by making the substitution $x = r\cos\theta$, $y = r\sin\theta$ and expressing the region of integration in polar form; that is, we rewrite the double integral in rectangular coordinates as

$$\iint_R f(x,y)\,dA = \iint_R f(r\cos\theta, r\sin\theta)\,dA = \iint_{\substack{\text{appropriate}\\ \text{limits}}} f(r\cos\theta, r\sin\theta)r\,dr\,d\theta \tag{9}$$

▶ **Example 4** Use polar coordinates to evaluate $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\,dy\,dx$.

Solution. In this problem we are starting with an iterated integral in rectangular coordinates rather than a double integral, so before we can make the conversion to polar coordinates we will have to identify the region of integration. Observe that for fixed x the y-integration runs from $y = 0$ to $y = \sqrt{1-x^2}$, which tells us that the lower boundary of the region is the x-axis and the upper boundary is a semicircle of radius 1 centered at the origin. From the x-integration we see that x varies from -1 to 1, so we conclude that the region of integration is as shown in Figure 4.3.13. In polar coordinates, this is the region swept out as r varies between 0 and 1 and θ varies between 0 and π. Thus,



$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\,dy\,dx = \iint_R (x^2+y^2)^{3/2}\,dA$$

$$= \int_0^{\pi}\int_0^1 (r^3)r\,dr\,d\theta = \int_0^{\pi} \frac{1}{5}\,d\theta = \frac{\pi}{5} \blacktriangleleft$$