Geometric Transformations

Two-Dimensional Transformations

A graphic system should allow the programer to define pictures that include a variety of transformations. For example, he should be able to magnify a picture so that detail appears more clearly, or reduce it so that more of the picture is visible.

Transformation: is a single mathematical entity and as such can be denoted by a single name or symbol (translation, rotation, or scaling). Each of these transformations is used to generate a new point ($\bar{x}$, $\bar{y}$) from the coordinates of a point (x,y) in the original picture description. If the original definition includes a line, it suffices to apply the transformation to the endpoints of the line and display the line between the two transformed endpoints.

1- Translation:

The form of translation transformation is:

$$\bar{x} = x + T_x , \quad \bar{y} = y + T_y$$

Translation transformation can be represented in a uniform way by a 3×3 matrix as shown below:

$$[\bar{x} \quad \bar{y} \quad 1] = [x \quad y \quad 1] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ Tx & Ty & 1 \end{bmatrix}$$

<u>**Example:**</u>

Consider triangle defined by its three vertices (20,0), (60,0), (40,100) being translated 100 units to the right and 10 units up. What are the new vertices?

$T_x=100$, $T_y=10$

The new vertices are:

(20,0)$\rightarrow$(20+100,0+10) $\rightarrow$(120,10)

(60,0) $\rightarrow$(60+100,0+10)$\rightarrow$(160,10)

(40,100) $\rightarrow$(40+100,100+10) $\rightarrow$(140,110)

or:

$$[\bar{x} \quad \bar{y} \quad 1] = [20 \quad 0 \quad 1]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 100 & 10 & 1 \end{bmatrix} = [120 \quad 10 \quad 1]$$

$$[\bar{x} \quad \bar{y} \quad 1] = [60 \quad 0 \quad 1]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 100 & 10 & 1 \end{bmatrix} = [160 \quad 10 \quad 1]$$

$$[\bar{x} \quad \bar{y} \quad 1] = [40 \quad 100 \quad 1]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 100 & 10 & 1 \end{bmatrix} = [140 \quad 110 \quad 1]$$

H.W.: Draw the vertices of the above example before and after the translation?

2- Rotation:

To rotate a point (x,y) through a clockwise angle θ about the origin of the coordinate system, we write:

$$\bar{x} = x\cos\theta + y\sin\theta \;, \quad \bar{y} = -x\sin\theta + y\cos\theta$$

Rotation transformation can be represented in a uniform way by a 3$\times$3 matrix as shown below:

$$[\bar{x} \quad \bar{y} \quad 1] = [x \quad y \quad 1]\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example:

Consider triangle defined by its three vertices (20,0), (60,0), (40,100) being rotated 45^0 clockwise about the origin. What are the new vertices?

The new vertices are:

(20,0) $\rightarrow$ $\bar{x}$ **=20cos45+0 sin45=14.14**

$\rightarrow$ $\bar{y}$ **=-20sin45+ 0 cos45=-14.14**

$\rightarrow$ **(14.14, -14.14)**

(60,0) $\rightarrow \bar{x}$ **=60cos45+0 sin45=42.43**

$\rightarrow$ $\bar{y}$ **=-60sin45+ 0 cos45=-42.43**

$\rightarrow$ **(42.43, -42.43)**

(40,100) $\rightarrow \bar{x}$ **=40cos45+100 sin45=98.99**

$\rightarrow$ $\bar{y}$ **=-40sin45+ 100 cos45=42.43**

$\rightarrow$ **(98.99, 42.43)**

or:

$$[\bar{x} \quad \bar{y} \quad 1] = [20 \quad 0 \quad 1]\begin{bmatrix}\cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1\end{bmatrix} = [14.14 \quad -14.14 \quad 1]$$

$$[\bar{x} \quad \bar{y} \quad 1] = [60 \quad 0 \quad 1]\begin{bmatrix}\cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1\end{bmatrix} = [42.43 \quad -42.43 \quad 1]$$

$$[\bar{x} \quad \bar{y} \quad 1] = [40 \quad 100 \quad 1]\begin{bmatrix}\cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1\end{bmatrix} = [98.99 \quad 42.43 \quad 1]$$

H.W.: Draw the vertices of the above example before and after the rotation?

3- Scaling:

The form of the scaling transformation is:

$$\bar{x} = xS_x , \quad \bar{y} = yS_y$$

The scaling transformation can be used for a variety of purposes. If the picture is to be enlarged to twice it's original size we might choose S_x=S_y=2. Notice that the elargement is relative to the origin of the coordinate system.

scaling transformation can be represented in a uniform way by a 3×3 matrix as shown below:

$$[\bar{x} \quad \bar{y} \quad 1] = [x \quad y \quad 1]\begin{bmatrix} Sx & 0 & 0 \\ 0 & Sy & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example:

Consider triangle defined by it's three vertices (20,0), (60,0), (40,100) being twice enlarged.. What are the new vertices?

$S_x = S_y = 2$

The new vertices are:

$(20,0) \rightarrow (20\times2, 0\times2) \rightarrow (40,0)$

$(60,0) \rightarrow (60\times2, 0\times2) \rightarrow (120,0)$

$(40,100) \rightarrow (40\times2, 100\times2) \rightarrow (80,200)$

or:

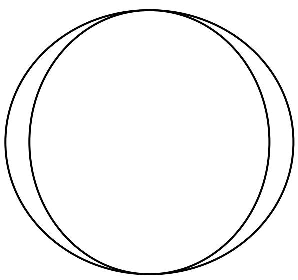
$$[\bar{x} \quad \bar{y} \quad 1] = [20 \quad 0 \quad 1]\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [40 \quad 0 \quad 1]$$

$$[\bar{x} \quad \bar{y} \quad 1] = [60 \quad 0 \quad 1]\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [120 \quad 0 \quad 1]$$

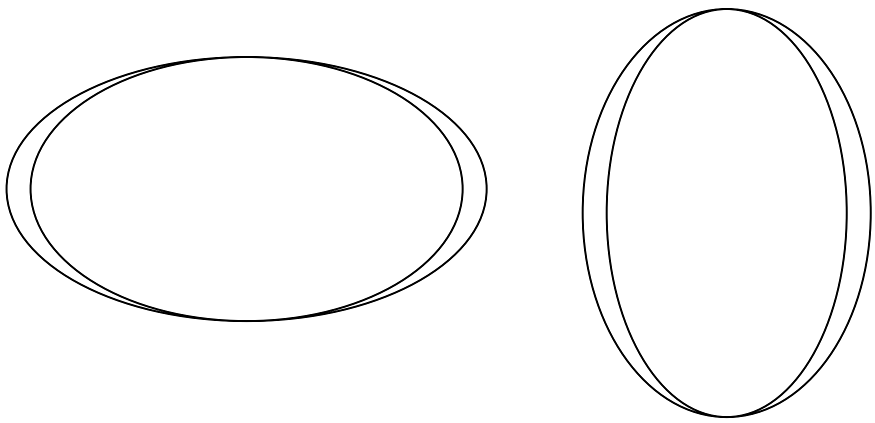
$$[\bar{x} \quad \bar{y} \quad 1] = [40 \quad 100 \quad 1]\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [80 \quad 200 \quad 1]$$

H.W.: Draw the vertices of the above example before and after the scaling?

Note: If S_x and S_y are not equal, they have the effect of distorting pictures by elongating or shrinking them along the directions parallel to the coordinate axes. For instance the foolowing figure:

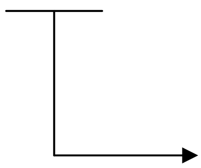


Can be distorted as shown in figures below:



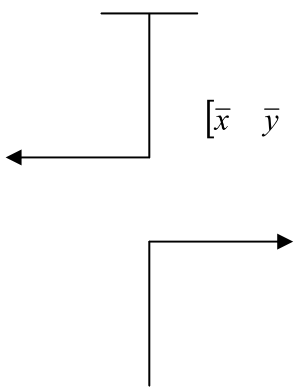
4- Reflection (Mirroring):

Reflection transformation can be considered as a special case of scaling, because the mirror image of an object can be generated using negative values of S_x or S_y. Mirror images of figure below:



Can be generated as shown below:

$\bar{x} = x(-S_x)$, $\bar{y} = yS_y$



$$[\bar{x} \quad \bar{y} \quad 1] = [x \quad y \quad 1]\begin{bmatrix} -Sx & 0 & 0 \\ 0 & Sy & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\bar{x}$ =xS_x , $\bar{y}$ =y(-S_y)

$$[\bar{x} \quad \bar{y} \quad 1] = [x \quad y \quad 1]\begin{bmatrix} Sx & 0 & 0 \\ 0 & -Sy & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\bar{x}$ =x(-S_x) , $\bar{y}$ =y(-S_y)

$$[\bar{x} \quad \bar{y} \quad 1] = [x \quad y \quad 1]\begin{bmatrix} -Sx & 0 & 0 \\ 0 & -Sy & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

H.W.: Write complete program in order to perform all types of transformations on a line between (0,1) and (5,6)?

Example:

The triangle with vertices ((20,0),(60,0),(40,100)) is rotated through 90^0, and then translated with T_x=-80,T_y=0. What is the new vertices of the resulted figure?

Rotation:

$\bar{x}$ **=x cosθ + y sinθ = x cos90 + y sin90= y**

$\bar{y}$ **= - x sinθ + y cosθ = - sin90 + y sin90= -x**

Translation:

$\bar{\bar{x}}$ **=** $\bar{x}$ **+ T_x = y – 80**

$\bar{\bar{y}}$ **=** $\bar{y}$ **+ T_y = -x**

then:

(20,0): $\bar{\bar{x}}$ **= y – 80 =0-80=-80,** $\bar{\bar{y}}$ **= -x = -20 $\rightarrow$ (-80,-20)**

(60,0): $\bar{\bar{x}}$ **= y – 80 =0-80=-80,** $\bar{\bar{y}}$ **= -x = -60 $\rightarrow$ (-80,-60)**

(40,100): $\bar{\bar{x}}$ = y – 80 =100-80=20, $\bar{\bar{y}}$ = -x = -40 → (20,-40)

H.W.: Draw the vertices of the above example before and after the scaling?

Example:

Perform a (45^0) rotation of triangle A(0,0), B(1,1), C(5,2) about the origin using matrix representation?

Representation of triangle: $\begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

Rotation matrix= R_{45} = $\begin{bmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **=** $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then:

$$[\bar{A} \quad \bar{B} \quad \bar{C}] = R_{45} \times [A \quad B \quad C] = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{3\sqrt{2}}{2} \\ 0 & \sqrt{2} & \frac{7\sqrt{2}}{2} \\ 1 & 1 & 1 \end{bmatrix}$$

Thus: $\bar{A} = (0,0),\ \bar{B} = (0,\sqrt{2}), \bar{C} = (\frac{3\sqrt{2}}{2}, \frac{7\sqrt{2}}{2})$

H.W.: Draw the vertices of the above example before and after the scaling?