Chapter2: Numerical Integration

2.1 Introduction to Quadrature:

We now approach the subject of numerical integration. The goal is to approximate the definite integral of f(x) over the interval [a,b] by evaluating f(x) at a finite number of sample points.

Definition(2.1): Suppose that $a=x_0<x_1<\ldots<x_M=b$. A formula of the form:

$$Q[f] = \sum_{k=0}^{M} w_k f(x_k) = w_0 f(x_0) + w_1 f(x_1) + \cdots + w_M f(x_M) \qquad (2.1)$$

With the property that:

$$\int_a^b f(x)dx = Q[f] + E[f] \qquad (2.2)$$

is called a numerical integration or **quadrature** formula. The term *E[f]* is called the **truncation error** for integration. The values $\{x_k\}_{k=0}^{M}$ are called the **quadrature nodes** and $\{w_k\}_{k=0}^{M}$ are called **weights**.

Definition (2.2): The **degree of precision** of a quadrature formula is the positive integer n such that $E[P_i]=0$ for all polynomials $P_i(x)$ of degree $i \leq n$, but for which $E[P_{n+1}]\neq 0$ for some polynomial $P_{n+1}(x)$ of degree n+1.

Theorem(2.1): (closed Newton-cotes Quadrature formula)

Assume that $x_k=x_0+kh$ are equally spaced nodes and $f_k=f(x_k)$. The first four closed Newton-Cotes quadrature formulas are

$$\int_{x_0}^{x_1} f(x)dx \approx \frac{h}{2}(f_0 + f_1) \qquad (2.3)$$ **(the trapezoidal rule)**

$$\int_{x_0}^{x_2} f(x)dx \approx \frac{h}{3}(f_0 + 4f_1 + f_2) \qquad (2.4)$$ **(Simpson rule)**

$$\int_{x_0}^{x_3} f(x)dx \approx \frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3) \qquad (2.5)$$ **(Simpson's $\frac{3}{8}$ rule)**

$$\int_{x_0}^{x_4} f(x)dx \approx \frac{2h}{45}(7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4) \quad (2.6)$$ **(Boole's rule)**

Corollary(2.1): (Newton-Cotes precision)

Assume that *f(x)* is sufficiently differentiable; then *E[f]* for Newton-Cotes quadrature involves an approximate higher derivative. The trapezoidal rule has degree of precision n=1. If $f \in C^2[a,b]$, then:

$$\int_{x_0}^{x_1} f(x)dx = \frac{h}{2}(f_0 + f_1) - \frac{h^3}{12}f^{(2)}(c) \qquad (2.7)$$

Simpson's rule has degree of precision $n=3$. If $f \in C^4[a,b]$, then:

$$\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}(f_0 + 4f_1 + f_2) - \frac{h^5}{90}f^{(4)}(c) \qquad (2.8)$$

Simpson's $\frac{3}{8}$ rule has degree of precision n=3. If $f \in C^4[a,b]$, then:

$$\int_{x_0}^{x_3} f(x)dx = \frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3) - \frac{3h^5}{80}f^{(4)}(c) \qquad (2.9)$$

Boole's rule has degree of precision n=5. If $f \in C^6[a,b]$, then:

$$\int_{x_0}^{x_4} f(x)dx = \frac{2h}{45}(7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4) - \frac{8h^7}{945}f^{(6)}(c) \qquad (2.10)$$

Proof of Theorem(2.1): Start with the Lagrange polynomial $P_M(x)$ based on $x_0, x_1, \dots, x_M$ that can be used to approximate *f(x)*:

$$f(x) \approx P_M(x) = \sum_{k=0}^{M} f(x_k) \prod_{\substack{j=0 \\ j\neq k}}^{M} \frac{(x-x_j)}{(x_k-x_j)} \qquad (2.11)$$

An approximate for the integral is obtained by replacing the integrand *f(x)* with the polynomial $P_M(x)$. This is the general method for obtaining a Newton-Cotes integration formula:

$$\int_{x_0}^{x_M} f(x)dx \approx \int_{x_0}^{x_M} P_M(x)dx = \int_{x_0}^{x_M}\left(\sum_{k=0}^{M} f_k \prod_{\substack{j=0\\ j\neq k}}^{M} \frac{(x-x_j)}{(x_k-x_j)}\right) \qquad (2.12)$$

The details for the general proof of the theorem are tedious. We shall give a Simpson's rule, which is the case M=2. This case involves the approximation polynomial

$$P_2(x) = f_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \qquad (2.13)$$

Since f_0, f_1 and f_2 are constant with respect to integration, the relations in (2.12) lead to:

$$\int_{x_0}^{x_2} f(x)dx \approx \int_{x_0}^{x_2} f_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}dx + \int_{x_0}^{x_2} f_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}dx + \int_{x_0}^{x_2} f_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}dx \qquad (2.14)$$

We introduce the change of variable $x=x_0+th$ with $dx=hdt$ to assist with the evaluation of the integrals in (2.14). The new limits of integration are from $t=0$ to $t=2$. The equal spacing of the nodes $x_k=x_0+kh$ leads to $x_k-x_j=(k-j)h$ and $x-x_k=(t-k)h$, which are used to simplify (2.14), and get:

$$\int_{x_0}^{x_2} f(x)dx \approx f_0\int_0^2 \frac{h(t-1)h(t-2)}{(-h)(-2h)}hdt + f_1\int_0^2 \frac{h(t-0)h(t-2)}{(h)(-h)}hdt + f_2\int_0^2 \frac{h(t-0)h(t-1)}{(2h)(h)}hdt$$

$$= f_0\frac{h}{2}\int_0^2 (t^2-3t+2)dt + f_1 h\int_0^2 (t^2-2t)dt + f_2\frac{h}{2}\int_0^2 (t^2-t)dt$$

$$= f_0\frac{h}{2}\left(\frac{t^3}{3}-\frac{3t^2}{2}+2t\right)\Big|_{t=0}^{t=2} - f_1 h\left(\frac{t^3}{3}-\frac{2t^2}{2}\right)\Big|_{t=0}^{t=2} + f_2\frac{h}{2}\left(\frac{t^3}{3}-\frac{t^2}{2}\right)\Big|_{t=0}^{t=2}$$

$$= f_0 \frac{h}{2}\left(\frac{2}{3}\right) - f_1 h\left(\frac{-4}{3}\right) + f_2 \frac{h}{2}\left(\frac{2}{3}\right)$$

$$= \frac{h}{3}(f_0 + 4f_1 + f_2)$$

and the proof is complete.

Example(2.1): Consider the function f(x)=1+e^{-x}sin(4x), the equally spaced quadrature nodes x_0 =0, x_1 =0.5, x_2 =1, x_3=1.5, x_4 =2 and the corresponding function values f_0 =1, f_1=1.55152, f_2=0.72159, f_3=0.93765 and f_4=1.13390. Apply the various quadrature formulas (2.3) through (2.6).

The step size is h=0.5, and the computations are:

$$\int_0^{0.5} f(x)dx \approx \frac{0.5}{2}(1 + 1.55152) = 0.63788$$

$$\int_0^{1} f(x)dx \approx \frac{0.5}{3}(1 + 4(1.55152) + 0.72159) = 1.32128$$

$$\int_0^{1.5} f(x)dx \approx \frac{3(0.5)}{8}(1 + 3(1.55152) + 3(0.72159) + 0.93765) = 1.64193$$

$$\int_0^{2} f(x)dx \approx \frac{2(0.5)}{45}\big(7(1) + 32(1.55152) + 12(0.72159) + 32(0.93765) + 7(1.1339)\big)$$

$$= 2.29444$$

Examples (2.2): Consider the integration of the function f(x)=1+e^{-x}sin(4x) over the fixed interval [a,b]=[0,1]. Apply the various formulas (2.3) through (2.6).

For the trapezoidal rule, h=1 and

$$\int_0^1 f(x)dx \approx \frac{1}{2}\big(f(0)+f(1)\big) = \frac{1}{2}(1+0.72159) = 0.86079$$

For Simpson's rule, h=1/2, and we get:

$$\int_0^1 f(x)dx \approx \frac{1/2}{3}\left(f(0)+4f\left(\frac{1}{2}\right)+f(1) = \frac{1}{6}(1+4(1.55152)+0.72159) = 1.32128\right.$$

For Simpson's $\frac{3}{8}$ rule, h=1/3, and we obtain:

$$\int_0^1 f(x)dx \approx \frac{3\left(\frac{1}{3}\right)}{8}\left(f(0)+3f\left(\frac{1}{3}\right)+3f\left(\frac{2}{3}\right)+f(1)\right)$$

$$= \frac{1}{8}(1+3(1.69642)+3(1.23447)+0.72159) = 1.31440$$

For Boole's rule, h=1/4, and the result is:

$$\int_0^1 f(x)dx \approx \frac{2\left(\frac{1}{4}\right)}{45}\left(7f(0)+32f\left(\frac{1}{4}\right)+12f\left(\frac{1}{2}\right)+32f\left(\frac{3}{4}\right)+7f(1)\right)$$

$$= \frac{1}{90}\big(7(1)+32(1.65534)+12(1.55152)+32(1.06666)+7(0.72159)\big)$$

=1.30859

The true value of the definite integral is:

$$\int_0^1 f(x)dx = 1.308\,250\,604$$

To make a fair comparison of quadrature methods, we must use the same number of function evaluations in each method. Our final example is concerned with comparing integration over a fixed interval [a,b] using exactly five function evaluation $f_k=f(x_k)$, for

$k=0,1,\ldots,4$ for each method. When the trapezoidal rule is applied on the four subintervals $[x_0,x_1]$, $[x_1,x_2]$, $[x_2,x_3]$ and $[x_3,x_4]$, it is called a **composite trapezoidal rule**:

$$\int_{x_0}^{x_4} f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \int_{x_2}^{x_3} f(x)dx + \int_{x_3}^{x_4} f(x)dx$$

$$\approx \frac{h}{2}(f_0+f_1) + \frac{h}{2}(f_1+f_2) + \frac{h}{2}(f_2+f_3) + \frac{h}{2}(f_3+f_4)$$

$$= \frac{h}{2}(f_0 + 2f_1 + 2f_2 + 2f_3 + f_4) \qquad (2.15)$$

Simpson's rule can also be used in this manner. When Simpson's rule is applied on the two subintervals $[x_0,x_2]$ and $[x_2,x_4]$, it is called a **composite Simpson's rule**:

$$\int_{x_0}^{x_4} f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx$$

$$\approx \frac{h}{3}(f_0 + 4f_1 + f_2) + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

$$= \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + f_4) \qquad (2.16)$$

Example(2.3): Consider the integration of the function $f(x)=1+e^{-x}\sin(4x)$ over $[a,b]=[0,1]$. Use exactly five function evaluations and compare the results from the composite trapezoidal rule and composite Simpson's rule.

The uniform step size is $h=1/4$. The composite trapezoidal rule (2.15) produces:

$$\int_0^1 f(x)dx \approx \frac{1/4}{2}\left(f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right) + f(1)\right)$$

$$= \frac{1}{8}(1 + 2(1.65534) + 2(1.55152) + 2(1.06666) + 0.72159)$$

$$=1.28358$$

Using the composite Simpson's rule (2.16), we get:

$$\int_0^1 f(x)dx \approx \frac{1/4}{3}\left(f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1)\right)$$

$$= \frac{1}{12}(1 + 4(1.65534) + 2(1.55152) + 4(1.06666) + 0.72159)$$

=1.30938

<u>Example(2.4)</u>: Determine the degree of precision of Simpson's $\frac{3}{8}$ rule.

It will suffice to apply Simpson's $\frac{3}{8}$ rule over the interval [0,3] with the five test functions f(x)=1, x, x^2, x^3, and x^4. For the first four functions. Simpson's $\frac{3}{8}$ rule is exact.

$$\int_0^3 1dx = \frac{3}{8}(1 + 3(1) + 3(1) + 1) = 3$$

$$\int_0^3 xdx = \frac{3}{8}(0 + 3(1) + 3(2) + 3) = \frac{9}{2}$$

$$\int_0^3 x^2dx = \frac{3}{8}(0 + 3(1) + 3(4) + 9) = 9$$

$$\int_0^3 x^3dx = \frac{3}{8}(0 + 3(1) + 3(8) + 27) = \frac{81}{4}$$

the function f(x)=x^4 is the lowest power of x for which the rule is not exact.

$$\int_0^3 x^4dx = \frac{3}{8}(0 + 3(1) + 3(16) + 81) = \frac{99}{2}$$

Therefore, the degree of precision of Simpson's $\frac{3}{8}$ rule is n=3.

Exercises:

1. Consider a general interval [a,b]. Show that Simpson's rule produces exact results for the function $f(x)=x^2$ and $f(x)=x^3$, that is

   a. $\int_a^b x^2 dx = \frac{b^3}{3} - \frac{a^3}{3}$ b. $\int_a^b x^3 dx = \frac{b^4}{4} - \frac{a^4}{4}$

2. Integrate the Lagrange interpolation polynomial

   $$P_1(x) = f_0 \frac{(x-x_1)}{(x_0-x_1)} + f_1 \frac{(x-x_0)}{(x_1-x_0)}$$

   over the interval $[x_0,x_1]$ and establish the trapezoidal rule.

3. Determine the degree of precision of the trapezoidal rule.

2.2 Other Ways to Derive Integration Formulas Using Newton Forward Polynomial:

During the integration we will need to change the variable of integration from x to t since our polynomials are expressed in terms of t. Observe that dx=hdt.

$$\int_{x_0}^{x_1} f(x)dx = h \int_{t=0}^{t=1} \left[f_0 + t\Delta f_0 + \frac{t(t-1)}{2!}\Delta^2 f_{0+} \frac{t(t-1)(t-2)}{3!}\Delta^3 f_0 + \cdots \right] dt$$

$$= h \int_0^1 \left[f_0 + t\Delta f_0 + \frac{t^2-t}{2}\Delta^2 f_0 + \frac{t^3-3t^2+2t}{6}\Delta^3 f_0 + \cdots \right] dt$$

$$= h \left[f_0 t + \frac{t^2}{2}\Delta f_0 + \left(\frac{t^3}{6} - \frac{t^2}{4}\right)\Delta^2 f_0 + \left(\frac{t^4}{24} - \frac{t^3}{6} + \frac{t^2}{6}\right)\Delta^3 f_0 + \cdots \right]_{t=0}^{t=1}$$

$$= h \left[f_0 + \frac{1}{2}\Delta f_0 - \frac{1}{12}\Delta^2 f_0 + \frac{1}{24}\Delta^3 f_0 + \cdots \right]$$

using first two terms only, we get:

$$\int_{x_0}^{x_1} f(x)dx = h\left[f_0 + \frac{1}{2}\Delta f_0\right] = h\left[f_0 + \frac{1}{2}(f_1 - f_0)\right] = \frac{h}{2}[f_0 + f_1]$$

Exercise:

Derive Simpson's formula using Newton Forward polynomial.

2.3 Composite Trapezoidal and Simpson's Rule:

Theorem(2.2): (Composite Trapezoidal Rule)

Suppose that the interval [a,b] is subdivided into subinterval $[x_k, x_{k+1}]$ of width h=(b-a)/M by using equally spaced nodes x_k=a+kh, for k=0,1,…,M. The **composite trapezoidal rule for M subintervals** can be expressed in:

$$\int_a^b f(x)dx \approx T(f,h) = \frac{h}{2}[f_0 + 2(f_1 + \cdots + f_{M-1}) + f_M]$$

$$= \frac{h}{2}[f(a) + f(b)] + h\sum_{k=1}^{M-1} f(x_k) \qquad (2.17)$$

Proof: Apply the trapezoidal rule over each subinterval $[x_{k-1}, x_k]$. Use the additive property of the integral for subintervals:

$$\int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{M-1}}^{x_M} f(x)dx$$

$$= \frac{h}{2}[f_0 + f_1] + \frac{h}{2}[f_1 + f_2] + \cdots + \frac{h}{2}[f_{M-1} + f_M]$$

$$= \frac{h}{2}[f_0 + 2(f_1 + f_2 + \cdots + f_{M-1}) + f_M].$$

Example(2.5): Consider $f(x) = 2 + \sin(2\sqrt{x})$. Use the composite trapezoidal rule with 11 sample points to compute an approximation to the integral of f(x) taken over [1,6].

To generate 11 sample points, we use M=10 and h=(6-1)/10=1/2.

x	*1*	*1.5*	*2*	*2.5*	*3*	*3.5*	*4*	*4.5*	*5*	*5.5*	*6*
f(x)	*2.909297*	*2.638157*	*2.308071*	*1.979316*	*1.683052*	*1.4353041*	*1.243197*	*1.108317*	*1.028722*	*1.000241*	*1.017357*

$\int_1^6 f(x)dx = \frac{\frac{1}{2}}{2}[f(1) + 2\big(f(1.5) + f(2) + f(2.5) + f(3) + f(3.5) + f(4) + f(4.5) + f(5) + f(5.5)\big) + f(6)]$=8.193854.

Theorem(2.3): (Composite Simpson Rule)

Suppose that [a,b] is subdivided into 2M subintervals $[x_k, x_{k+1}]$ of equal width with h=(b-a)/(2M) by using $x_k=a+kh$ for k=0,1,…,2M. The **composite Simpson rule for 2M subintervals** can be expressed in:

$$\int_a^b f(x)dx \approx S(f,h) = \frac{h}{3}[f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 2f_{2M-2} + 4f_{2M-1} + f_{2M}]$$

$$= \frac{h}{3}[f(a) + f(b)] + \frac{2h}{3}\sum_{k=1}^{M-1} f(x_{2k}) + \frac{4h}{3}\sum_{k=1}^{M} f(x_{2k-1}) \quad (2.18)$$

proof: (EXC)

Example(2.6): Consider $f(x) = 2 + \sin(2\sqrt{x})$. Use the composite Simpson rule with 11 sample points to compute an approximation to the integral of f(x) taken over [1,6].

$\int_a^b f(x)dx = \frac{1/2}{3}[f(1) + f(6)] + \frac{1}{3}[f(2) + f(3) + f(4) + f(5)] + \frac{2}{3}[f(1.5) + f(2.5) + f(3.5) + f(4.5) + f(5.5)]$=8.1830155

Error Analysis:

Corollary(2.2): (Trapezoidal Rule: Error Analysis)

Suppose that [a,b] is subdivided into M subintervals $[x_k, x_{k+1}]$ of width h=(b-a)/M. The composite trapezoidal rule:

$$T(f,h) = \frac{h}{2}[f(a) + f(b)] + h\sum_{k=1}^{M-1} f(x_k) \quad (2.19)$$

is an approximation to the integral:

$$\int_a^b f(x)dx = T(f,h) + E_T(f,h) \quad (2.20)$$

Furthermore, if $f \in C^2[a,b]$, there exists a value c with a<c<b so that the error term $E_T(f,h)$ has the form:

$$E_T(f,h) = \frac{-(b-a)f^{(2)}(c)h^2}{12} = O(h^2) \qquad (2.21)$$

Proof: We first determine the error term when the rule is applied over $[x_0, x_1]$. Integrating the Lagrange polynomial $P_1(x)$ and its remainder yields:

$$\int_{x_0}^{x_1} f(x)dx = \int_{x_0}^{x_1} P_1(x)dx + \int_{x_0}^{x_1} \frac{(x-x_0)(x-x_1)f^{(2)}(c(x))}{2!}dx \qquad (2.22)$$

The term $(x-x_0)(x-x_1)$ does not change sign on $[x_0, x_1]$, and $f^{(2)}(c(x))$ is continuous. Hence the second Mean value Theorem for integrals implies that there exists a value c_1 so that:

$$\int_{x_0}^{x_1} f(x)dx = \frac{h}{2}[f_0 + f_1] + f^{(2)}(c_1)\int_{x_0}^{x_1} \frac{(x-x_0)(x-x_1)}{2!}dx \qquad (2.23)$$

Use the change of variable $x=x_0+ht$ in the integral on the right side of (2.23)

$$\int_{x_0}^{x_1} f(x)dx = \frac{h}{2}[f_0 + f_1] + \frac{f^{(2)}(c_1)}{2}\int_0^1 h(t-0)h(t-1)hdt$$

$$= \frac{h}{2}[f_0 + f_1] + \frac{f^{(2)}(c_1)h^3}{2}\int_0^1 (t^2 - t)dt$$

$$= \frac{h}{2}[f_0 + f_1] - \frac{f^{(2)}(c_1)h^3}{12} \qquad (2.24)$$

Now we are ready to add up the error terms for all of the intervals $[x_k, x_{k+1}]$:

$$\int_a^b f(x)dx = \sum_{k=1}^{M}\int_{x_{k-1}}^{x_k} f(x)dx = \sum_{k=1}^{M}\frac{h}{2}[f(x_{k-1}) + f(x_k)] - \frac{h^3}{12}\sum_{k=1}^{M} f^{(2)}(c_k) \quad (2.25)$$

The first sum is the composite trapezoidal rule T(f,h). In the second term, one factor of h is replaced with its equivalent h=(b-a)/M, and the result is:

$$\int_a^b f(x)dx = T(f,h) - \frac{(b-a)h^2}{12}\left(\frac{1}{M}\sum_{k=1}^{M} f^{(2)}(c_k)\right)$$

The term in parentheses can be recognized as an average of values for the second derivative and hence is replaced by $f^{(2)}(c)$. Therefore, we have established that:

$$\int_a^b f(x)dx = T(f,h) - \frac{(b-a)f^{(2)}(c)h^2}{12}$$

and the proof is complete.

__Corollary(2.3)__: (Simpson's rule: Error analysis)

Suppose that [a,b] is subdivided into 2M subintervals $[x_k, x_{k+1}]$ of equal width h=(b-a)/(2M). The composite Simpson rule

$$S(f,h) = \frac{h}{3}(f(a)+f(b)) + \frac{2h}{3}\sum_{k=1}^{M-1} f(x_{2k}) + \frac{4h}{3}\sum_{k=1}^{M} f(x_{2k-1}) \qquad (2.26)$$

is an approximation to the integral:

$$\int_a^b f(x)dx = S(f,h) + E_S(f,h) \qquad (2.27)$$

Furthermore, if $f \in C^4[a,b]$, there exists a value c with a<c<b so that the error term $E_S(f,h)$ has the form:

$$E_S(f,h) = \frac{-(b-a)f^{(4)}(c)h^4}{180} = O(h^4) \qquad (2.28)$$

__Example(2.7)__: Consider $f(x) = \frac{1}{x}$. Investigate the error when the composite trapezoidal rule is used over [1,6] and the number of subintervals is 10.

h=(6-1)/10=0.5, since:

$$E_T(f,h) = \frac{-(b-a)f^{(2)}(c)h^2}{12} = O(h^2)$$

we first compute $f'(x) = \frac{-1}{x^2}$ and $f''(x) = \frac{2}{x^3}$,therefore:

$$f''(1) = 2, f''(2) = \frac{1}{4}, f''(6) = \frac{2}{6^3} = 0.009\,259$$

and hence f''(c)=2 and $E_T(f,h)=\frac{-(6-1)(2)(0.5)^2}{12}=\frac{-2.5}{12}=-0.208\,333$

Example(2.8): Find the number M and the step size h so that the error $E_S(f,h)$ for the Simpson's rule is less than 5×10^{-9} for the approximation $\int_2^7 dx/x \approx S(f,h)$.

$$f(x)=\frac{1}{x}\xrightarrow{yields}f'(x)=\frac{-1}{x^2}\xrightarrow{yields}f''(x)=\frac{2}{x^3}\xrightarrow{yields}f^{(3)}(x)=\frac{-6}{x^4}\xrightarrow{yields}f^{(4)}(x)=\frac{24}{x^5}$$

the maximum value of $|f^{(4)}(x)|$ taken over [2,7] occurs at the end point x=2 and $f^{(4)}(2)=3/4$, then:

$$|E_S(f,h)|=\frac{|-(b-a)f^{(4)}(c)h^4|}{180}\le\frac{(7-2)\frac{3}{4}h^4}{180}=\frac{h^4}{48}$$

The step size h and number M satisfy the relation h=5/(2M), and this is used in the above equation to get the relation

$$|E_S(f,h)|\le\frac{625}{768M^4}\le5\times10^{-9}$$

$$\xrightarrow{yields}\frac{125}{768}\times10^9\le M^4\xrightarrow{yields}112.95\le M$$

since M must be integer, we chose M=113

and the corresponding step size h=5/226=0.022123

Exercises:

1. Approximate the integral $\int_{-1}^{1}\frac{dx}{1+x^2}$using the composite trapezoidal rule with M=10.
2. The length of the curve y=f(x) over the interval $a\le x\le b$ is L=$\int_a^b\sqrt{1+(f'(x)^2}$ approximate the length of the function $f(x)=x^3$ over [0,1] using composite Simpsons rule with M=5.

3. Verify that the trapezoidal rule (M=1, h=1) is exact for polynomials of degree≤1 of the form $f(x)=c_1x+c_0$ over [0,1].
4. Determine the number M and the interval width h so that the composite trapezoidal rule for M subintervals can be used to compute the integral $\int_0^2 xe^{-x}dx$ with an accuracy of 5×10^{-9} .

2.4 Romberg Integration:

The discussion here is based upon the trapezium rule. Let the integration domain [a,b] be divided by three equispaced nodes $x_0=a$, $x_1=(a+b)/2$ and $x_2=b$ at interval of size h. Two successive trapezium estimates using one and two subintervals respectively are:

$$T_1 = \frac{2h}{2}[f(x_0) + f(x_1)] \ \ and \ \ T_2 = \frac{h}{2}[f(x_0) + 2f(x_1) + f(x_2)]$$

On including the truncation error for this estimate we can write:

$$I = T_1 - \frac{(2h)^2}{12} f''(x_0) - G(2h)^4 - \cdots$$

$$I = T_2 - \frac{h^2}{12} f''(x_0) - Gh^4 - \cdots$$

where G is independent of the step size h. Four times the second estimate minus the first estimate gives:

$$I = \frac{1}{3}[4T_2 - T_1] + 4Gh^4 + O(h^6) \qquad (2.29)$$

Taken as an estimate to I, the values $(4T_2\text{-}T_1)/3$ has leading error of $O(h^4)$. Expand this estimate:

$$I \approx \frac{1}{3}[4T_2 - T_1] = \frac{1}{3}\left[4\left\{\frac{h}{2}(f_0 + 2f_1 + f_2)\right\} - \frac{2h}{2}(f_0 + f_2)\right]$$

$$= \frac{h}{3}[f_0 + 4f_1 + f_2]$$

Shows it to be the Simpson estimate S_2 using two sub-intervals of size h=(b-a)/2.

This process can be carried out for any two trapezium estimates T_N and T_{2N} to give the more accuracy Simpson's estimate S_{2N}.

Trapezoidal	Simpson	
T_1		
T_2	S_2	
T_4	S_4	**In general S_{2N}=1/3{4T_{2N}-T_N}**
T_8	S_8	

In the same way we get:

$$I \approx \frac{1}{15}[16S_4 - S_2] + O(h^6) \qquad (2.30)$$

known as Boole's rule.

Trapezoidal	Simpson	Boole's	
T_1			
T_2	S_2		
T_4	S_4	B_4	**In general S_{2N}=1/3{4T_{2N}-T_N}**
T_8	S_8	B_8	**In general B_{4N}=1/15{16S_{4N}-S_{2N}}**

Example(2.9): Estimate the value of $\int_0^1 e^{sinx}dx$ using Romberg integration

N	Trapezium k=1	Simpson k=2	Boole k=3	k=4
1	1.659 888			
2	1.637 517	1.630 060		
4	1.633 211	1.631 776	1.631 891	
8	1.632 201	1.631 864	1.631 869	1.631 869

Exercises:

1. Use Romberg integration to estimate $\int_0^2 x^2 e^{-x^2} dx$ as accurately as possible, working to four decimal places.