Chapter 5: Numerical Solution of Integral Equations

5.1 Classification of Integral Equations:

An integral equation is an equation in which the unknown function u(x) appears under an integral sign. The most general linear integral equation in u(x) can be presented as:

$$h(x)u(x) = f(x) + \int_a^{b(x)} k(x,t)u(t)dt \qquad (5.1)$$

where k(x,t) is a function of two variables called the **kernel** of the integral equation.

This equation is called a ***Volterra integral equation*** when b(x)=x,

$$h(x)u(x) = f(x) + \int_a^{x} k(x,t)u(t)dt \qquad (5.2)$$

when h(x)=0 it is called a ***Volterra equation of the first kind***,

$$-f(x) = \int_a^{x} k(x,t)u(t)dt \qquad (5.3)$$

and is called a ***Volterra equation of the second kind*** when h(x)=1,

$$u(x) = f(x) + \int_a^{x} k(x,t)u(t)dt \qquad \ldots(5.4)$$

The integral equation (5.1) is called a ***Fredholm integral equation*** when b(x)=b, where b constant,

$$h(x)u(x) = f(x) + \int_a^{b} k(x,t)u(t)dt \qquad \ldots(5.5)$$

It is also called a ***Fredholm equation of the first and second kinds*** when h(x)=0 and h(x)=1, respectively:

$$-f(x) = \int_a^{b} k(x,t)u(t)dt \qquad \ldots(5.6)$$

$$u(x) = f(x) + \int_a^{b} k(x,t)u(t)dt \qquad \ldots(5.7)$$

5.2 Numerical Solution of Volterra Integral Equations:

Let us consider the Volterra equation of the second kind:

$$u(x) = f(x) + \int_a^{x} k(x,t)u(t)dt$$

we will subdivide the interval of integration (a,x) into n equal subintervals of width $h=(x_n-a)/n$, $n\geq 1$, where x_n is the end point we choose for x, we shall set $t_0=a$ and $t_j=a+jh$. Note that the particular value $u(x_0)=f(a)$, so if we use the trapezoidal rule with n subintervals to approximate the integral in the Volterra integral equation of the second kind (5.4), we have:

$$\int_a^x \boldsymbol{k(x,t)u(t)dt} \approx \frac{h}{2}\begin{bmatrix} \boldsymbol{k(x,t_0)u(t_0)+2k(x,t_1)u(t_1)+\cdots+2k(x,t_{n-1})u(t_{n-1})} \\ \boldsymbol{+k(x,t_n)u(t_n)} \end{bmatrix} \tag{5.8}$$

and the integral equation (5.4) is then approximated by the sum:

$$u(x)=f(x)+\frac{h}{2}\left[k(x,t_0)u(t_0)+2\sum_{j=1}^{n-1}k(x,t_j)u(t_j)+k(x,t_n)u(t_n)\right] \tag{5.9}$$

If we consider n+1 sample values of $u(x)$, $u(x_i), i=0,1,\ldots,n$, equation (5.9) will become a set of n+1 equations in $u(x_i)$ (or u_i)[note that $u(x_0)=f(x_0)$ since the integral in (5.4) vanishes for $x=x_0=a$].

$$\left.\begin{array}{c} u_0=f_0 \\ u_i=f_i+\frac{h}{2}\left[k_{i0}u_0+2\sum_{j=1}^{m-1}k_{i,j}u_j+k_{i,m}u_m\right], \\ i=1,2,\ldots,n, k_{ij}=k(x_i,t_j),\ \ j\leq i \end{array}\right\} \tag{5.10}$$

which are n+1 equations in u_i, the approximation to the solution $u(x)$ of (5.4) at $x_i=a+ih$ for $i=0,1,\ldots,n$.

Example 5.1: Use trapezoidal method to find an approximate values to the solution for the following Volterra integral equation $u(x)=x-\int_0^x(x-t)u(t)dt$ at x=0,1,2,3,and 4.

Here, f(x)=x, k(x,t)=t-x for $t\leq x=4$ and is zero for $t>x=4$, and a=0 with u(0)=0. We also have n=4 and hence h=(4-0)/4=1. So using (5.10) to obtain:

$u_0=f_0=0$

$u_1=f_1+\frac{h}{2}[k_{10}u_0+k_{11}u_1]=1+\frac{1}{2}[(0-1)(0)+(1-1)u_1]=1$

$u_2=f_2+\frac{h}{2}[k_{20}u_0+2k_{21}u_1+k_{22}u_2]$

$=2+\frac{1}{2}[(0-2)(0)+2(1-2)(1)+(2-2)u_2]=1$

$u_3=f_3+\frac{h}{2}[k_{30}u_0+2k_{31}u_1+2k_{32}u_2+k_{33}u_3]=3+\frac{1}{2}[(0-3)(0)+2(1-3)(1)+2(2-3)(1)+(3-3)u_3]=3+\frac{1}{2}[-4-2]=0$

$u_4=f_4+\frac{h}{2}[k_{40}u_0+2k_{41}u_1+2k_{42}u_2+2k_{43}u_3+k_{44}u_4]=4+\frac{1}{2}[(0-4)(0)+2(1-4)(1)+2(2-4)(1)+2(3-4)(0)+(4-4)u_4]=4+\frac{1}{2}[-6-4]=-1$

x_k	0	1	2	3	4
u_k	0	1	1	0	-1

5.3 Numerical Solution of Fredholm Integral Equations:

Let us consider the Fredholm equation of the second kind:

$$u(x)=f(x)+\int_a^b k(x,t)u(t)dt \tag{5.11}$$

we will subdivide the interval of integration (a,b) into n equal subintervals of width h=(b-a)/n, n≥1, we shall set $t_0=a, t_n=b$ and $t_j=a+jh$. Note that the particular value , so if we use the trapezoidal rule with n subintervals to approximate the integral in the Fredholm integral equation of the second kind (5.11), we have:

$$\int_a^b \boldsymbol{k(x,t)u(t)dt} \approx \frac{\boldsymbol{h}}{\boldsymbol{2}}\begin{bmatrix}\boldsymbol{k(x,t_0)u(t_0)+2k(x,t_1)u(t_1)+\cdots+2k(x,t_{n-1})u(t_{n-1})} \\ \boldsymbol{+k(x,t_n)u(t_n)}\end{bmatrix} \tag{5.12}$$

and the integral equation (5.11) is then approximated by the sum:

$$u(x)=f(x)+\frac{h}{2}\left[k(x,t_0)u(t_0)+2\sum_{j=1}^{n-1}k(x,t_j)u(t_j)+k(x,t_n)u(t_n)\right] \tag{5.13}$$

If we consider n+1 sample values of *u(x)*, $u(x_i)$, $i=0,1,\ldots,n$, equation (5.13) will become a set of n+1 equations in $u(x_i)$ (or u_i).

$$\left.\begin{aligned} u_i&=f_i+\frac{h}{2}\left[k_{i0}u_0+2\sum_{j=1}^{m-1}k_{i,j}u_j+k_{i,m}u_m\right], \\ &i=1,2,\ldots,n, k_{ij}=k(x_i,t_j),\ \ j\le i\end{aligned}\right\} \tag{5.14}$$

which are n+1 equations in u_i, the approximation to the solution *u(x)* of (5.11) at $x_i=a+ih$ for $i=0,1,\ldots,n$.

Example 5.2: Use trapezoidal method to find an approximate values to the solution for the integral equation u(x)=$x^2+\frac{1}{4}-\frac{1}{3}x+\int_0^1(x-t)u(t)dt$ with h=0.25 notice that the real solution is u(x)=x^2

We have f(x)= $x^2+\frac{1}{4}-\frac{1}{3}x$ and k(x,t)=x-t.

Since h=0.25, we have $x_0=t_0=0, x_1=t_1=0.25, x_2=t_2=0.5$, $x_3=t_3=0.75$ and $x_4=t_4=1$

for i=0,1,2,3 and 4, we have:

$$u_0=f_0+\frac{h}{2}[k_{00}u_0+2k_{01}u_1+2k_{02}u_2+2k_{03}u_3+k_{04}u_4]$$

$$u_1=f_1+\frac{h}{2}[k_{10}u_0+2k_{11}u_1+2k_{12}u_2+2k_{13}u_3+k_{14}u_4]$$

$$u_2=f_2+\frac{h}{2}[k_{20}u_0+2k_{21}u_1+2k_{22}u_2+2k_{23}u_3+k_{24}u_4]$$

$$u_3=f_3+\frac{h}{2}[k_{30}u_0+2k_{31}u_1+2k_{32}u_2+2k_{33}u_3+k_{34}u_4]$$

$$u_4=f_4+\frac{h}{2}[k_{40}u_0+2k_{41}u_1+2k_{42}u_2+2k_{43}u_3+k_{44}u_4]$$

therefore, we hence:

$$u_0=0.25+\frac{0.25}{2}[(0-0)u_0+2(0-0.25)u_1+2(0-0.5)u_2+2(0-0.75)u_3+(0-1)u_4]$$

$$u_1=0.22917+\frac{0.25}{2}[(0.25-0)u_0+2(0.25-0.25)u_1+2(0.25-0.5)u_2+2(0.25-0.75)u_3+(0.25-1)u_4]$$

$$u_2=0.33333+\frac{0.25}{2}[(0.5-0)u_0+2(0.5-0.25)u_1+2(0.5-0.5)u_2+2(0.5-0.75)u_3+(0.5-1)u_4]$$

$$u_3 = 0.5625 + \frac{0.25}{2}[(0.75-0)u_0 + 2(0.75-0.25)u_1 + 2(0.75-0.5)u_2 + 2(0.75-0.75)u_3 + (0.75-1)u_4]$$

$$u_4 = 0.91667 + \frac{0.25}{2}[(1-0)u_0 + 2(1-0.25)u_1 + 2(1-0.5)u_2 + 2(1-0.75)u_3 + (1-1)u_4]$$

then,

$8u_0 + 0.5u_1 + u_2 + 1.5u_3 + u_4 = 2$

$-0.25u_0+8u_1+0.5u_2+u_3+0.75u_4=1.8333$

$-0.5u_0-0.5u_1+8u_2+0.5u_3+0.5u_4=2.6667$

$-0.75u_0-u_1-0.5u_2+8u_3+0.25u_4=4.5$

$-u_0-1.5u_1-u_2-0.5u_3+8u_4=7.3333$

solving this system, we get:

u=[-0.010417 0.052083 0.23958 0.55208 0.98958]^T

x_k	u_k	$u(x_k)$
0	-0.010417	0
0.25	0.052083	0.0625
0.5	0.23958	0.25
0.75	0.55208	0.5625
1	0.98958	1

Exercise:

1. Use trapezoidal method to find an approximate values to the solution for the integral equation u(x)=$x - \frac{x^3}{3} + \int_0^x tu(t)dt$, x∈[0,1], with h=0.25.(note that u(x)=x)
2. Use trapezoidal method to find an approximate values to the solution for the integral equation u(x)=$e^x - xe^1 + x + \int_0^1 xu(t)dt$, with h=0.5 (note that u(x)=e^x).