Chapter 4: Numerical Solution of Partial Differential Equations

4.1 Classification of Partial Differential Equations:

A partial differential equation (PDE) is an equation that involves an unknown function (the dependent variable) and some of its partial derivatives with respect to two or more independent variables. The classification of PDEs is important for the numerical solution you choose. Consider the general, second-order, linear partial differential equation in two variables :

$$A(x, y)U_{xx} + 2B(x, y)U_{xy} + C(x, y)U_{yy} = F(x,y ,U_x, U_y, U) \quad (4.1)$$

4.1.1 Elliptic

$$AC > B^2$$

For example, Laplace's equation:

$$U_{xx} + U_{yy} = 0$$

$A = C = 1, B = 0$

4.1.2 Hyperbolic

$$AC < B^2$$

For example the 1-D wave equation:

$$U_{xx} = \frac{1}{c^2} U_{tt}$$

A = 1, C = $\frac{1}{c^2}$, B = 0

4.1.3 Parabolic

$$AC = B^2$$

For example, the heat or diffusion Equation

$$U_t = Uxx$$

$A = 1;B = C = 0$

4.2 Finite Difference Solution of Partial Differential Equations:

4.2.1 Parabolic Equations

Consider the boundary-initial value problem (BIVP):

$$\left.\begin{array}{c} u_{xx} = \frac{1}{c} u_t \ , u = u(x,t), 0 < x < 1 \ , t > 0 \\ u(0,t) = u(1,t) = 0 \quad (boundary\ conditions) \\ u(x,0) = f(x) \ (initial\ condition) \end{array}\right\} \qquad (4.2)$$

Where c is a constant. This problem represents transient heat conduction in a rod with the ends held at zero temperature and an initial temperature profile *f(x)*.

To solve this problem numerically, we discretize x and t such that:

$$x_i = i * h , i = 0,1,2, \dots$$
$$t_j = jk , j = 0,1,2, ..$$

4.2.1.1 Explicit Finite Difference Method

Let u_{ij} be the numerical approximation to $u(x_i , t_j)$. We approximate u_t with the forward finite difference:

$$u_t \approx \frac{u_{i,j+1} - u_{i,j}}{k} \qquad (4.3)$$

and u_{xx} with the central finite difference:

$$u_{xx} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \qquad (4.4)$$

The finite difference approximation to the PDE is then:

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} = \frac{\mathbf{1}}{\boldsymbol{c}} \frac{u_{i,j+1} - u_{i,j}}{k} \qquad (4.5)$$

Define the parameter r as

$$r = \frac{ck}{h^2}$$

in which case Eq. 4.5 becomes:

$$r\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) = \left(u_{i,j+1} - u_{i,j}\right)$$

therefore,

$$u_{i,j+1} = ru_{i+1,j} + (1 - 2r)u_{i,j} + ru_{i-1,j} \qquad (4.6)$$

The domain of the problem and the mesh are illustrated in Fig. 4.1.

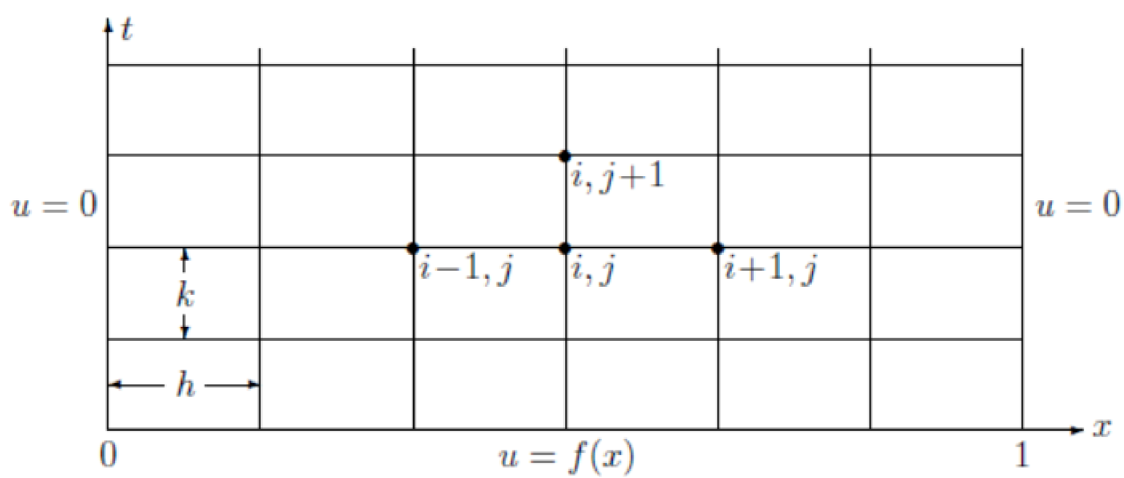

Figure 4.1: Mesh for 1-D Heat Equation.

Eq. 4.6 is a recursive relationship giving u in a given row (time) in terms of three consecutive values of u in the row below (one time step earlier). This equation is referred to as an explicit formula since one unknown value can be found directly in terms of several other known values.

We can write out the matrix system of equations we will solve numerically for the temperature u. Suppose we use 5 grid points x_0 , x_1 , x_2 , x_3 and x_4 .

Now, for *i=1* eq.(4.6) becomes:

$$u_{1,j+1} = ru_{2,j} + (1-2r)u_{1,j} + ru_{0,j}$$

and for *i=2* eq.(4.6) becomes:

$$u_{2,j+1} = ru_{3,j} + (1-2r)u_{2,j} + ru_{1,j}$$

and for *i=3* eq.(4.6) becomes:

$$u_{3,j+1} = ru_{4,j} + (1-2r)u_{3,j} + ru_{2,j}$$

Using boundary condition in eq.(4.2), we get:

$$u_{1,j+1} = ru_{2,j} + (1-2r)u_{1,j}$$
$$u_{2,j+1} = ru_{3,j} + (1-2r)u_{2,j} + ru_{1,j}$$
$$u_{3,j+1} = (1-2r)u_{3,j} + ru_{2,j}$$

Equation above in matrix form becomes:

$$\begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \end{bmatrix} = \begin{bmatrix} 1-2r & r & 0 \\ r & 1-2r & r \\ 0 & r & 1-2r \end{bmatrix} \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \end{bmatrix} \qquad (4.7)$$

where

$$r = \frac{ck}{h^2}$$

Now, for the system of eq's (4.7) substitute j=0,1,2:

for j=0

$$\begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{bmatrix} = \begin{bmatrix} 1-2r & r & 0 \\ r & 1-2r & r \\ 0 & r & 1-2r \end{bmatrix} \begin{bmatrix} u_{1,0} \\ u_{2,0} \\ u_{3,0} \end{bmatrix}$$

where $u_{k,0} = u(x_k, 0) = f(x_k)$ (by using initial condition)

for j=1

$$\begin{bmatrix} u_{1,2} \\ u_{2,2} \\ u_{3,2} \end{bmatrix} = \begin{bmatrix} 1-2r & r & 0 \\ r & 1-2r & r \\ 0 & r & 1-2r \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{bmatrix}$$

for j=2

$$\begin{bmatrix} u_{1,3} \\ u_{2,3} \\ u_{3,3} \end{bmatrix} = \begin{bmatrix} 1-2r & r & 0 \\ r & 1-2r & r \\ 0 & r & 1-2r \end{bmatrix} \begin{bmatrix} u_{1,2} \\ u_{2,2} \\ u_{3,2} \end{bmatrix}$$