Group theory

References

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Chapter one

Binary Operations

Definition 1.1

Let A be a non empty set. A binary operation on a set A is a function from $A \times A$ into A. (i.e.)

$*: A \times A \to A$ is a binary operation iff

1. $a * b \in A, \forall a, b \in A$ (Closure)
2. If $a, b, c, d \in A$ such that $a = c$ and $b = d$, then $a * b = c * d$ (well-define).

Example 1.2

1) The operations $\{+, -, \times\}$ are binary operations on R, Z, Q, C. But " $-$ " is not binary operation on N.

2) The operations $\{+, -\}$ are not binary operations on O **(**odd number).

3) The operation $\div$is abinary operation on $R\backslash\{0\}, Q\backslash\{0\}, C\backslash\{0\}$.

Example 1.3

Let $a * b = a + b + 2, \forall a, b \in Z^+$**.** Is $*$ abinary operation on Z^+?

Solution:

1) Closure: let $a, b \in Z^+$, then $a * b = \overbrace{a + b}^{\in Z^+} + 2 \in Z^+$.
2) well-define: $a, b, c, d \in A$ such that $a = c$ and $b = d$, then

   $a * b = a + b + 2 = c + d + 2 = c * d$

   $\Rightarrow *$ is a binary operation on Z^+.

Example 1.4

Let$a * b = a^b, a, b \in Z$. Show that$*$is abinary operation on Z.

Solution:

1) Closure: if $a = 3$ and $b = -1$. Then $a * b = 3^{-1} = \frac{1}{3} \notin Z \Rightarrow *$ is not a binary operation on Z.

Remark 1.5: Some time we used the symbols $*, \circ, \#, \odot$, ... to denote abinary operation.

Exercises: which of the following are binary operations?

1) $a * b = a + b, \forall a, b \in R\backslash\{0\}$.

2) $a \odot b = \frac{a}{b}, \forall a, b \in Z$.

3) $a \# b = a + b - 3, \forall a, b \in N$.

4) $a \circ b = a + 2b - 5, \forall a, b \in R$.

5) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}, \forall \frac{a}{b}, \frac{c}{d} \in Q\backslash\{0\}$.

Definition 1.6 (Commutative)

A binary operation $*$ on a set$A$ is called a Commutative if and only if $a * b = b * a \forall a, b \in A$.

Definition 1.7 (Associative)

A binary operation $*$ on a set A is called an associative if

(a $*$b) $*$c = a $*$(b $*$c) $\forall$a, b, c $\in$A.

Example 1.8 Let R be a set of real numbers and $*$ be a binary operation on R defined as $a * b = a + b - ab$, then $*$ is commutative and associative.

Solution:

(i) $a * b = a + b - ab = b + a - ba = b * a$

Which implies that $*$ is commutative.

(ii) Let $a, b, c \in R$, then

$$(a * b) * c = (a + b - ab) * c = (a + b - ab) + c - (a + b - ab)c$$

$$= a + b + c - ab - ac - bc + abc \ldots . (1)$$

$$a * (b * c) = a * (b + c - bc)$$

$$= a + (b + c - bc) - a(b + c - bc)$$

$$= a + b + c - bc - ab - ac + abc \ldots (2)$$

$\Rightarrow$ (1)=(2)

$\Rightarrow *$ is an associative.

Exercises: which of the following binary operations is a comm., asso.?

(i) $a * b = a - b, \forall a, b \in Z.$

(ii) $a \odot b = 2ab, \forall a, b \in E.$

(iii) $a \# b = a^3 + b^3 \forall a, b \in R.$

Definition 1.9 (**Mathematical System**)

A Mathematical System or (Mathematical Structure) is a non-empty set of elements with one or more binary operations defined on this set.

Example 1.10

$(R, +), (R, .), (R, -), (R\backslash\{0\}, \div), (R, +, .), (N, +), (E, +, \times)$ are Math. System. But $(N, -), (R, \div), (\mathrm{O}, +, -)$ are not Math. System.

Definition 1.11 (Semi group)

A semi group is a pair $(S, *)$ in which S is an empty set and $*$ is a binary operation on S with associative law.

(i.e.) $(S, *)$ is semi group$\Leftrightarrow$ (1) $S \neq \emptyset$,

(2) $*$ is a binary operation,

(3) $\forall a, b, c \in S$, (a $*$b) $*$c = a $*$(b $*$c).

Example 1.12

(1) $(Z, +), (Z, \times), (N, +), (N, \times), (E, +), (E, \times)$ are semi groups.

(2) $(\mathrm{O}, +), (Z, -), (\mathrm{E}, -), (R\backslash\{0\}, \div)$ are not semi groups.

Definition 1.13 (**The identity element**)

Let $(S, *)$ be a Mathematical System and $e \in S$. Then e is called an identity element if $a * e = e * a = a, \forall a \in S.$

Definition 1.14 (**The inverse element**)

Let $(S,*)$ be a Mathematical System and $a,b \in S$. Then b is called an inverse of a if $a * b = b * a = e$.

Definition 1.15 (**The Group**)

The pair $(G,*)$ is a group iff $(G,*)$ is a semi group with identity in which each element of G has an inverse.

Definition 1.16 (**The Group**)

A group $(G,*)$ is a non-empty set G and a binary operation $*$, such that the following axioms are satisfied:

(1) The binary operation $*$ is associative.

(i.e.) $(a * b) * c = a * (b * c),\ \forall a, b, c \in G$

(2) There is an element e in G such that

$a * e = e * a = a, \forall a \in G.$

This element e is an identity element for $*$ on G.

(3)for each a in G, there is an element b in G such that

$a * b = b * a = e.$

The element b is an inverse of a and denoted by a^{-1}.

Remark 1.17

Every group is a semi group but the converse is not true as in the following example shows.

$(N,+)$ is a semigroup but not group because $\nexists a^{-1} \in N, \forall a \in N$.

Definition 1.18 (**Commutative group**)

A group $(G,*)$ is called a Commutative group iff $a * b = b * a, \forall a, b \in G$.

Example 1.19

i. $(Z,+),(E,+),(Q,+),(N,\times),(C,+)$ are commutative groups .

ii. $(Z^+,+)$ is not a group because there is no identity element for $+$ in Z^+.

iii. $(Z^+,\times)$ is not a group because there is an identity element 1 but no inverse of 5.

iv. $(G=\{1,0,-1,2\},+)$ is not group since $+$ is not a binary operation on G, $1+2=3\notin G$.

v. $(G=\{1,-1\},\times)$ is comm. Group.

vi. $(R\backslash\{0\},\times)$, $(Q\backslash\{0\},\times)$,$(C\backslash\{0\},\times)$ are comm. Groups.

Example 1.20

Let $G=\{a,b,c,d\}$ be a set. Define a binary operation $*$ on G by the following table.

$*$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

Is $(G,*)$ a commutative group?

Solution:

(1) Closure is true.

(2) Asso.

(a $*$b) $*$c = a $*$(b $*$c) ?

b$*$ c=a $*$ d

d = d

b$*$(a$*$c)=b$*$c=d=(b$*$a)$*$c

c$*$(a$*$b)=c$*$b=d=(c$*$a)$*$b

d$*$(a$*$c)=d$*$c=b=(d$*$a)$*$c ….$\rightarrow$

$\Rightarrow*$ is asso.

(3) The identity: To prove $\exists e \in G \ \ s.t. a * e = e * a = a, \forall a \in G.$

$a * a = a, b * a = b, c * a = c, d * a = d.$

$\Rightarrow e = a$ is an identity element of G.

(4) The inverse:

$a * a = a \Rightarrow a^{-1} = a$

$b * d = a \Rightarrow b^{-1} = d$

$c * c = a \Rightarrow c^{-1} = c$

$a * a = a \Rightarrow a^{-1} = a$

$d * b = a \Rightarrow d^{-1} = b$

(5) Comm.

$a * b = b * a$?

$b = b$

$a * c = c * a = c$

$a * d = d * a = d$

$b * c = c * b = d$

$b * d = d * b = a$

$c * d = d * c = b$

$\Rightarrow *$ is a comm.

Therefore $(G,*)$ is a comm. Group and called Klein 4-group.

Example 1.21

Let $G = \{1, -1, i, -i\}$ be a set and "." be abinary operation.

Is $(G, .)$ a group ?

Solution:

.	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

1- Closure is true.

2- Asso. Law is true

3- 1 is an identity element.

4- $1^{-1} = 1$, $-1^{-1} = -1, i^{-1} = -i, -i^{-1} = i$

5- Comm .is true

$\therefore$ (G , .) is a comm.group.

Example 1.22

Let G = Z , $a * b = a + b + 2$, show that (G , $*$) is a comm . group.

Solution:

1- Closure : let a, b $\in$Z , Then

$a * b = a + b + 2 \in Z \rightarrow$Closure is true

2- asso. Low : Let a, b, c $\in$Z , then

$a *(b * c) = a * (b + c + 2)$= a + (b + c + 2) + 2

= a + b + c + 4 ……..(1)

$(a * b) * c = (a + b + 2) * c$= (a + b + 2) + C + 2

= a + b + c + 4 …….(2)

$\therefore$ (1) = (2) $\Rightarrow *$ is asso .

3- Identity : let e $\in$**Z** $\ni$ $a * e = e * a = a$, then

$a * e = a + e + 2 = a \Longrightarrow e = -2$

$e * a = e + a + 2 = a \Longrightarrow e = -2$

$\therefore$ -2 is an identity element of G.

4- Inverse : let a , b $\in$ Z $\ni$ $a * b = b * a = e$

$a * b = a + b + 2 = -2 \Longrightarrow b = -a - 4$

$b * a = b + a + 2 = -2 \Longrightarrow b = -a - 4$

$\therefore a^{-1} = -(a+4) \in Z$

$\therefore (G, *)$ is a group.

5- Comm. To prove $a * b = b * a \ \forall a, b \in Z$

$a * b = a + b + 2 = b + a + 2 = b * a$

$\therefore (G, *)$ is a comm. Group.

Example 1.25

Let (G , *) be an arbitrary group .The set of the function from G in to G with the composition (F_G, o) is forms a group , where

$F_G = \{ f_a : a \in G \}$, $f_a: G \to G$ s.t .

$f_a(x) = a * x$, $x \in G$, prove that

Proof : (1) Closure: let $f_a, f_b \in F_G$, $a,b \in G$

$(f_a \text{ o } f_b)(x) = f_a(f_b(x)) = f_a(b * x)$

$= a * (b * x)$

$= (a * b) * x$, since G is a group .

$= f_{a*b}(x) \in F_G$, since $a*b \in G$

(2) asso : Let $f_a, f_b, f_c \in F_G$, $a,b,c \in G$

$(f_a \text{ o } f_b) \text{ o } f_c = f_{a*b} \text{ o } f_c = f_{(a*b)*c}$

Since * is asso. on G

$= f_{a*(b*c)} = f_a \text{ o } f_{b*c} = f_a \text{ o } (f_a \text{ o } f_c)$

(3) identity : f_e is an identity of FG, since

$f_a \text{ o } f_e = f_{a*e} = f_{e*a} = f_e \text{ o } f_a = f_a$

(4) inverse : The inverse of f_a in F_G is f_a^{-1}, since

$F_a \text{ o } f_a^{-1} = f_{a*a^{-1}} = f_{a^{-1}*a} = f_a^{-1} \text{o } f_a = f_e$

Also, if G is a Comm . group, then (F_G, o) is a comm. group .

(Exercises): Determine the systems $(G , *)$ described abelian (comm..) group

1) $G= Z , a * b = a+b+3$
2) $G = R \times R = \{ (a, b) : a , b \in R\}$ s.t
   $(a, b) * (c,d) = (a+b, b+d + 2bd)$.
3) $(G = \{f_1 , f_2, f_3, f_4, f_5, f_6 \}, o)$, where
   $f_1(x) = x$, $f_2 (x) = \frac{1}{x}$, $f_3 (x) =1-x$, $f_4(x) = \frac{x-1}{x}$, $f_5(x) = \frac{x}{x-1}$, $f_6(x) = \frac{1}{1-x}$
4) $G = \{ (a,b) : a, b \in R , a \neq 0 , b \neq 0 \}$ s.t.
   $(a,b) * (c,d) = (ac,bd)$
5) $(G = \{ an : n \in z \} , +)$
6) $G = Q^+ , a * b = \frac{ab}{2}$.

Some properties of Groups:

Theorem (1) : If G is a group with a binary operation $*$, then the left and right cancellation laws hold in G , that is:

1) $a * b = a * c$ implies $b = c$
2) $b * a = c* a$ implies $b = c$

For all $a, b, c \in G$.

Proof :

1) suppose $a*b = a*c$

$\exists\ a^{-1} \in G$ s . t . $a^{-1} * (a * b) = a^{-1} * (a * c)$

$(a^{-1} * a) * b = (a^{-1} * a) * c$

$e * b = e * c$

$\therefore b = c$

2) H.W

Theorem(2): In a group (G , $*$), there is only one element e in G such that $e * a = a * e = a$, $\forall\, a \in G$.

Proof:

Suppose that G has two identity elements e and e' that mean $\forall a \in G$. Then

$a * e = e * a = a$ and $a * e' = e' * a = a$

Since each e and e' belong to G , so

$e * e' = e' * e = e$ (e عنصر و e' عنصر محايد)

and

$e' * e = e * e' = e'$ (e' عنصر و e عنصر محايد)

It follows that $e' = e$.

Theorem(3): In a group (G , $*$), the inverse element of each element in G is unique.

Proof :

Let $a \in G$ and a has two inverse x and x' . Such that $a * x = x * a = e$
$a * x' = x' * a = e$

$$\Longrightarrow x = x * e = x * (a * x')$$
$$= (x * a) * x'$$
$$= e * x'$$
$$= x'$$

$\therefore\ x = x' \Longrightarrow$ the inverse is an unique element.

Theorem(4): If (G , $*$) is group , then

1- $e^{-1} = e$
2- $(a^{-1})^{-1} = a \quad \forall\ a \in G$
3- $(a * b)^{-1} = b^{-1} * a^{-1} \forall \quad a , b \in G$

Proof :-

1- Let $e^{-1} = x$

e is the identity element of G $\Longrightarrow x * e = e * x = x$ ---- (1)

x is the inverse of $e \Longrightarrow e * x = x * e = e$ ------- (2)

from (1) and(2) $\Longrightarrow x = e \Rightarrow e^{-1} = e$.

2- $(a^{-1})^{-1} = (a^{-1})^{-1} * e$

$= (a^{-1})^{-1} * (a^{-1} * a)$

$= ((a^{-1})^{-1} * a^{-1}) * a$

$= e * a = a.$

3) $(a * b)^{-1} = b^{-1} * a^{-1}, \quad \forall a, b \in G$

Proof :

Since $(a * b) \in G \Longrightarrow (a * b)^{-1} \in G$

$(a * b) * (a * b)^{-1} = (a * b)^{-1} * (a * b) = e$ (def . of inverse)

$(a * b) * (a * b)^{-1} = e$

$a^{-1} * (a * b) * (a * b)^{-1} = a^{-1} * e$

$(a^{-1} * a) * b * (a * b)^{-1} = a^{-1}$

$e * b * (a * b)^{-1} = a^{-1}$

$b^{-1} * b * (a * b)^{-1} = b^{-1} * a^{-1}$

$e * (a * b)^{-1} = b^{-1} * a^{-1}$

$\therefore (a * b)^{-1} = b^{-1} * a^{-1}$

Theorem(5) : Let $(G, *)$ be a group . Then

i- $(a * b)^{-1} = a^{-1} * b^{-1} \Leftrightarrow$ G is comm. group.

Proof :

$(\Longrightarrow)$ Let $(G, *)$ be a group and $(a * b)^{-1} = a^{-1} * b^{-1}$. To prove G is comm.

Let $a, b \in G$. To prove $a * b = b * a, \forall a, b \in G$

$a * b = ((a * b)^{-1})^{-1}$ $\quad$ (by $(a^{-1}) = a$)

$= (b^{-1} * a^{-1})^{-1}$ $\quad$ (by Th.4)

$= (b^{-1})^{-1} * (a^{-1})^{-1}$

$= b * a$ $\quad$ (by $(a^{-1})^{-1} = a$)

$\therefore$ G is comm. gp.

$(\Longleftarrow)$ Let $(G, *)$ is a comm . gp. To prove $(a * b)^{-1} = a^{-1} * b^{-1}$

$(a * b)^{-1} = b^{-1} * a^{-1}$ (by Th.4)

$= a^{-1} * b^{-1}$ (by comm.)

ii) if $a = a^{-1}$ then G is a comm. gp. (Is the converse true?)

proof :

Let $a = a^{-1}$ T. P. $a * b = b * a$, $\forall a, b \in G$

Let $a, b \in G$ and $a * b \in G \Longrightarrow (a * b) = (a * b)^{-1}$

$= b^{-1} * a^{-1}$ (by Th.4)

$= b * a$

$\therefore$ G is a comm. Group.

The converse of this part is not true.

(i-e.) if $(G, *)$ is a comm . $\nRightarrow$ $a = a^{-1}$

For example:

Let (G = { 1 , -1 , i , -i } , .) be a comm. group,

Let a = i $\Longrightarrow$ $a^{-1} = -i$

$\therefore a \neq a^{-1}$

Give another example (H. W.)

Theorem (6): In a group (G , $*$) , the equations $a * x = b$ and $y * a = b$ have a unique solution.

Definition.(**The integral powers of *a***)

Let (G , $*$) be a group . The integral powers of *a*, $a \in G$ is defined by :

1- $a^n = \underbrace{a * a \ldots * a}_{n-times}$

2- $a^0 = e$

3- $a^{-n} = (a^{-1})^n, n \in Z^+$

4- $a^{n+1} = a^n * a \ , n \in Z^+$.

For example :

(1) In (R , +) ,

$3^0 = 0$,

$3^3 = 3 + 3 + 3 = 9$,

$3^{-2} = (3^{-1})^2 = (-3) + (-3)$

$= -6$.

(2) In (R , .) ,

$2^0 = 1$,

$2^3 = 2 \times 2 \times 2 = 8$,

$2^{-4} = (2^{-1})^4 = (\frac{1}{2})^4$

$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{16}$

(3) In (G = { 1, -1 , i , -i}, .) ,

$i^o = 1$, $i^2 = i \times i = -1$, $i^{-2} = (i^{-1})^2 = (-i)^2$

$= - i \times -i$

$= -1$

Theorem:

Let (G , $*$) be agroup and $a \in G$, $m , n \in Z$, then :

1- $a^n * a^m = a^{n+m}$ $\forall n , m \in Z$ (H. W.)

2- $(a^n)^m = a^{n\,m}$ $\forall n , m \in Z^+$

3- $a^{-n} = (a^n)^{-1}$ $\forall n \in Z^+$

4- $(a * b)^n = a^n * b^n$ $\forall n \in Z \Leftrightarrow$ G is a comm. group.

Definition: ((The **order of a group**))

The number of elements of a group G is called the order of G and is denoted by | G | or o (G).

G is called a finite group if $|G| < \infty$ and infinite group otherwise

Definition: (The **order of an element**)

The order of an element *a* , $a \in G$ is the least positive integer *n* such that $a^n = e$, where *e* is the identity element of G . We denoted to order *a* by | *a* | or o (*a*) .

(i.e.) $|a| = n$ if $a^n = e$, $n \in Z^+$

Example (1): (Z , +) is an infinite group

Example (2): the trivial group G = { 0 }

| G | = 1 , G is the only group of order 1.

Example (3): find the order of G and the order of each element of (G, .) . Such that G = { 1, -1 , i, -i}.

Ans.

| G | = 4 and

| *a* | : *a* = 1 , then | *a* | = | 1 | = 1 (since *e* = 1)

If *a* = -1, then | -1 | : $(-1)^2 = 1 \Longrightarrow$ | -1 | = 2

If *a* = i , then | i | : $i^2 = -1$, $i^4 = 1 \Longrightarrow$ | i | = 4

If *a* = -i , then |- i | : $-i^2 = -1$, $-i^3 = i$, $-i^4 = 1$

$\therefore$ | -i | = 4

((The group of integers modulo *n*))

Definition: Let *a* , *b* $\in$ Z , *n* $>$ 0 . Then *a* is congruent to *b* modulo *n* if *a* – *b* = *nk* , *k* $\in$ Z and denoted by *a* $\equiv$ *b* or *a* $\equiv$ *b* (mod *n*)

1- 17 $\equiv$ 5 (mod 6) , sine 17 – 5 = 12 = (6) (2)
2- 8 $\equiv$ 4 (mod 2) , since 8 – 4 = 4= (2) (2)
3- -12 $\equiv$3 (mod 3) , since -12 -3 = -15 = (3) (-5)
4- 5 $\not\equiv$ 2 (mod 2), since 5 – 2 = 3 $\neq$ (2)(*k*) , $\forall$ *k* $\in$ Z

Theorem: The congruence module *n* is an equivalence relation on the set of integers.

Definition:

Let a $\in$ Z , n $>$ o . The congruence class of a module *n*, denoted by [a] is the set of all integers that are congruent to a module *n* .

(i.e.)

$[a] = \{ z \in Z : z \equiv a \pmod n \}$

$= \{ z \in Z : z = a + kn , k \in Z \}$

Example(1):

If $n = 2$, find $[0]$, $[1]$

$[0] = \{ z \in Z : z \equiv 0 \pmod 2 \}$

$= \{ z \in Z : z = 0 + 2K , K \in Z \}$

$= \{ 0 , \mp 2 , \mp 4 , \ldots\ldots\ldots \}$

$[1] = \{ z \in Z : z \equiv 1 \pmod 2 \}$

$= \{ z \in Z : z = 1 + 2k , k \in \}$

$= \{ \mp 1 , \mp 3 , \mp 5 , \ldots . \}$.

Example(2):

If $n = 3$, find $[1]$, $[7]$

$[1] = \{ z \in Z : z \equiv 1 \pmod 3 \}$

$= \{ 1 , 1 \mp 3, 1 \mp 6 \ldots . \}$

$= \{ 1 , -2 , 4 , 7 , -5 , \ldots . \}$.

$[7]$ (H. W.)

Definition:

The set of all congruence classes module *n* is denoted by Z_n (which is read Z mod *n*). Thus

$Z_n = \{ [0] , [1] , [2] , \ldots\ldots , [n-1] \}$, or

$Z_n = \{\bar{0}, \bar{1}, \bar{2}, \ldots, \overline{n-1}\}$

Z_n has *n* elements.

Example:

$Z_1 = \{\bar{0}\}$

$Z_2 = \{ \bar{0}, \bar{1} \}$

$Z_3 = \{\bar{0}, \bar{1}, \bar{2}\}$

Now, we define addition on Z_n (write $+_n$) by the following : For any [a] , [b] $\in Z_n$ $[a] +_n [b] = [a +_n b]$

Similarly, we define multiplication on Z_n (write "$._n$ " by the following :

$[a] ._n [b] = [a ._n b] , \forall [a] , [b] \in Z_n$

It is easy to see that $(Z_n , +_n)$ is an abelian group with identity [0] and for every $[a] \in Z_n$, $[a]^{-1} = [n – a]$. This group is called the Additive Group of integers modulo n .

Also, $(Z_n , ._n)$ is abelian semi group with identity [1] . It is called the multiplicative semi group of integers modulo n.

Example (1): $(Z_4, +_4)$

$Z_4 = \{ \bar{0} , \bar{1} , \bar{2} , \bar{3} \}$

(1) Closure is true

(2) Asso. is true

(3) $\bar{0}$ is an identity element

(4) Inverse:

$\bar{1}^{-1} = \bar{4} - \bar{1} = \bar{3}$

$\bar{2}^{-1} = \bar{4} - \bar{2} = \bar{2}$

$\bar{3}^{-1} = \bar{4} - \bar{3} = \bar{1}$

(5) Comm : $\bar{1} + \bar{2} = \bar{3} = \bar{2} + \bar{1}$

$\bar{1} + \bar{3} = \bar{0} = \bar{3} + \bar{1}$

$+_4$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{0}$	$\bar{1}$
$\bar{3}$	$\bar{3}$	$\bar{0}$	$\bar{1}$	$\bar{2}$

:. $(Z_4, +_4)$ is a Comm. group.

Example (2): $(Z_4, \cdot_4)$

$\cdot_4$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
1	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{0}$	$\bar{2}$
$\bar{3}$	$\bar{0}$	$\bar{3}$	$\bar{2}$	$\bar{1}$

It is clear that we cannot have a group. Since the number $\bar{1}$ is identity but the numbers $\bar{0}$ and $\bar{2}$ have no inverse. It follows that

$(Z_4, \cdot_4)$ is not a group, but it is semi group.

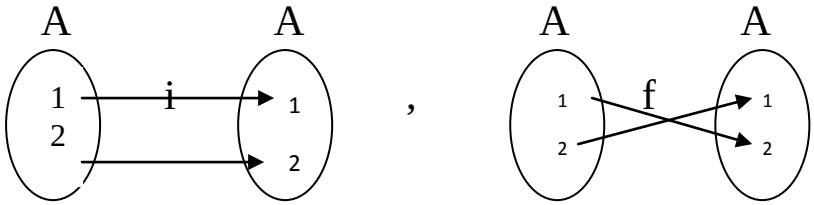
The Permutations : (التباديل)

Definition: A Permutation or symmetric of a set A is a function from A in to A that is both one to one and on to.

$$f: A \xrightarrow{1-1,onto} A$$

Symm $(A) = \{f \mid f: A \xrightarrow{1-1,onto} A\}$ the set of all permutation on A . If A is the finite set $\{1,2,\ldots, n\}$, then the set of all permutation of A is denoted by S_n or P_n and $o(S_n) = n!$, where $n! = n(n-1) \ldots (3)(2)(1)$

Example (1): Let A= {1,2} . Write all permutation on A.



Symm(A) = {i , f } = { $\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}$, $\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$ }.

Example (2): Let A = { 1, 2, 3} . Write all Perm. on A.

$f_1 = \begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}, f_2 = \begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}, f_3 = \begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}$

$f_4 = \begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix}, f_5 = \begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix}, f_6 = \begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix}.$

$P_3 = Symm(A) = \{f_1, f_2, f_3, f_4, f_5, f_6\}$

o(P_3)= 3 ! = (3)(2) = 6

<u>Theorem :</u> If $A \neq \varphi$, then the set of all permutation on A Forms agroup with composition of Mapps.

(i.e.) Let $\neq \varphi$, then (Symm(A) , o) is a group.

Proof :

Symm (A) = $\{f \mid f: A \xrightarrow{1-1,onto} A \ is\ a\ mapp.\}$,

T.P. (Symm(A) ,o) is a group.

since $\exists\ i_A: A \xrightarrow{1-1,onto} A$ a perm. on A

$\therefore i_A \in$ Symm(A) $\implies$ Symm(A) $\neq \varphi$.

(1) Closure : Let f , $g \in$ symm(A) , it follows that

$$f: A \xrightarrow{1-1,onto} A \ , g: A \xrightarrow{1-1,onto} A$$

$\Rightarrow fog: A \xrightarrow{1-1,onto} A \Rightarrow fog \in \text{Symm}(A)$

(2) Asso. : True since the composition of maps is an asso.

(3) The identity : since $i_A \in$ symm(A) and $i_A of = foi_A = f$ for all f in symm(A) $\Rightarrow i_A$ is an idenetity element

(4) The inverse : $\forall f: A \xrightarrow{1-1,onto} A$, $\exists f^{-1}: A \xrightarrow{1-1,onto} A$

$\therefore f^{-1} \in$ Symm(A) and $fof^{-1} = f^{-1}of = i_A$

$\therefore$ (Symm(A), o) is a group.

Is (Symm(A), o) comm. group ? (H.W.)

Example: Let A = {1,2,3}, then

S_3= {$f_1, f_2, f_3, f_4, f_5, f_6$} and ($S_3$, o) is a group.

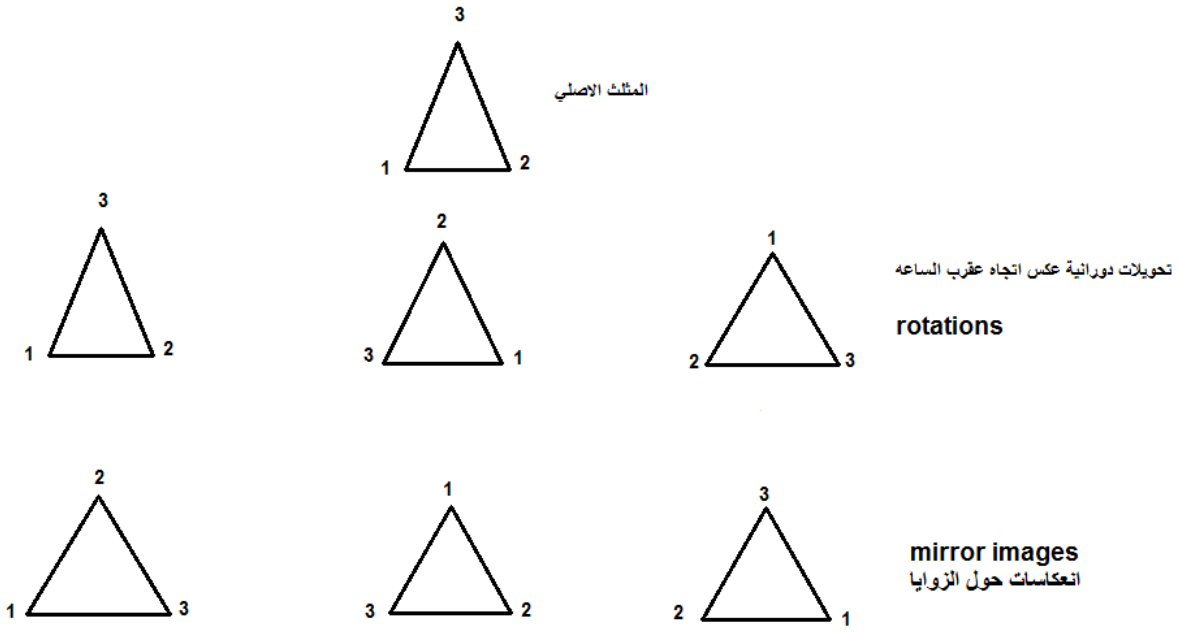
This group is called symmetric group.

O	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6
f_2	f_2	f_3	f_1	f_6	f_4	f_5
f_3	f_3	f_1	f_2	f_5	f_6	f_4
f_4	f_4	f_5	f_6	f_1	f_2	f_3
f_5	f_5	f_6	f_4	f_3	f_1	f_2
f_6	f_6	f_4	f_5	f_2	f_3	f_1

(S_3 , o) is not Comm. Group.

Also (S_3,o) is called the group of symmetries of on equilateral triangle .

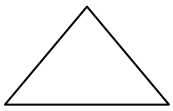
(زمرة تناظر المثلث متساوي الساقين)



Definition : (**The dihedral group D_n of order $2n$**)

The n^{th} dihedral group is the group of symmetries of the regular n-gon. $o(D_n) = 2n$

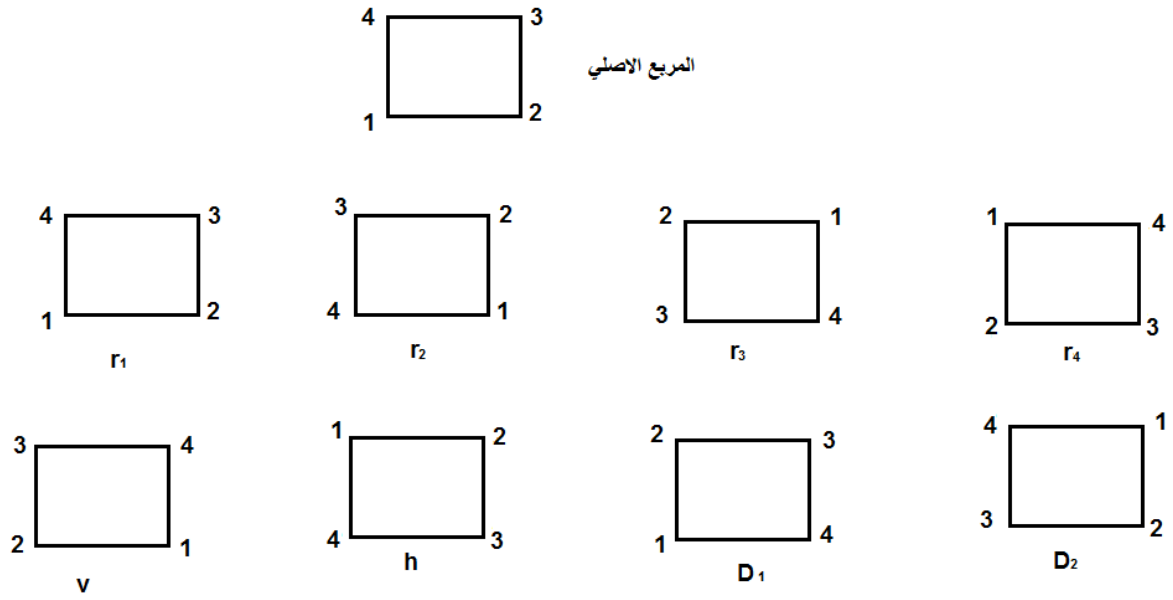
D_3 : is the third dihedral group.

 , O (D_3) = (2) (3) = 6 elements

Example . The group of symmetries of square D_4 or Gs, o (D_4)= 8

$G_s = D_4 = \{r_1, r_2, r_3, r_4, v, D_1, D_2\}$, where r_i are a clockwise rotation

V, h, D_1, D_2 are mirror images



(1) Write all elements of Gs as a permutation.

(2) Is (Gs , o) comm. group? Use table (H.W.)

Definition: A permutation f of a set A is called a cycle of length n if there exist $a_1, a_2, \ldots\ldots, a_n \in A$ such that

$f(a_1) = a_2$, $f(a_2) = a_3, \ldots, f(a_{n-1}) = a_n$, $f(a_n) = a_1$ and $f(x) = x$,

for $x \in A$ but $x \notin \{ a_1, a_2, \ldots\ldots, a_n \}$. We write $f = (a_1, a_2, \ldots, a_n)$.

Example: If $A = \{1, 2, 3, 4, 5\}$, then

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 1 & 4 \end{pmatrix} = (1354)(2) = (1354)$$

Observe that

(1354) = (3541) = (5413) = (4135).

Example: (2) Let A = {1, 2, 3, 4, 5, 6} be a set of a group S_6 . Then

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 4 & 1 & 3 & 2 & 6 & 5\end{pmatrix} = (142)o(3)o(56) = (142)o(56)$$

And

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 6 & 4 & 3 & 5 & 2 & 1\end{pmatrix} = (16)o(245)o(3) = (16)o(245)$$

These permutations above are not cycles.

Theorem: Every permutation f of a finite set A is a product of disjoint cycles.

Definition: A cycle of length 2 is a transposition.

Example: The permutation

$f = \begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 4 & 3 & 2\end{pmatrix}$ = (24) is a transposition.

Property: any permutation can be expressed as the product of transpositions.

(i.e.) $(a_1a_2 \dots a_n) = (a_1a_2)\,(a_1a_3)$ $\dots..(a_1a_n)$

Therefore any cycle is a product of transpositions.

Example: We see that (16) (2 5 3) = (16) (2 5) (2 3).

Definition: A permutation is even or odd according as it can be written as the product of an even or odd number of transpositions .

Example (1) Let $f = \begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix} \in P_3$

Is f even or odd permutation .

Ans. $f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ = (1 3 2) = (13) (12)

f has 2 transpositions $\Rightarrow f$ is an even perm.

Example(2): Determine an even and odd permutations of P_4.

(H.W)

Definition: "Alternating group " زمرة التباديل

The Alternating group on *n* letters, denoted by A_n is the group consisting of all even permutations in the symmetric group S_n.

$$o(A_n) = \frac{n!}{2} \quad , \quad A_n \subset S_n$$

Example(1): Let $S_3 = \{ f_1, f_2, f_3, f_4, f_5, f_6 \}$, then

$A_3 = \{i , f_2, f_3\}$ is a sub group of S_3

$o(A_3) = \frac{6}{2} = 3$

Example(2): Find A_4 from S_4

(H. W.)

Chapter Two

Subgroups and Cyclic Groups الزمر الجزئية والزمر الدائرية

Definition (1):

Let (G,∗) be a group and H⊆G, H is anon-empty subset of G. Then (H,∗) is a subgroup of (G,∗) if (H,∗) is itself a group.

Definition (2)

Let (G,∗) be a group and H ⊆G, Then (H,∗) is subgroup of G if :

(1) ∀a,b∈ H⇒ a∗b∈H

(2) The identity element of G is an element of H. e∈G⇒ e∈H

(3)∀a∈H⇒a^{-1}∈H

Remark (1):

Each group (G,∗) has at least two subgroup ({e},∗) and (G,∗), these subgroups are known trivial subgroup and improper, any subgroup different from these subgroups known a proper subgroup.

Examples (1):

1. (Z,+) is a proper subgroup of (R,+)

2. H={1, – 1} ⊆ {1,–1,i , –i}, then (H,.) is a subgroup of ({1,–1, i ,–i}, .)

3. H={$\bar{0}, \bar{2}$}⊆Z_4

   (H,+$_4$) is a proper subgroup of (Z_4,+$_4$). But {$\bar{0}, \bar{3}$} is not subgroup of (Z_4,+$_4$). Since $\bar{3}+_4\bar{3} = \bar{6}$ (Mod4) $= \bar{2} \notin \{\bar{0}, \bar{3}\}$, it follows that closure is not true in {$\bar{0}, \bar{3}$}.

4. (Q\{0},×)is a subgroup of (R\{0},×).

Theorem (1): Let (G,∗) be a group and H≠ϕ, H⊆G. Then (H,∗) is a subgroup of (G,∗) iff $a*b^{-1}\in H$, ∀ a, b ∈H

Proof:

(⇒) let (H,∗) be a subgroup and a,b ∈ H, then

$a,b^{-1}\in H \Rightarrow a*b^{-1}\in H$ (since ∗closure)

(⇐) Let $a*b^{-1}\in H$ T.P. (H,∗) is subgroup

(1) Since H≠ϕ⟹∃ b∈H s.t. $b*b^{-1}\in H \Rightarrow e\in H$.

(2) Since b∈H and $e\in H \Rightarrow e*b^{-1}\in H \Rightarrow b^{-1}\in H$

(3) Let a∈ H and $b^{-1}\in H$ [by (2)] $\Rightarrow a*(b^{-1})^{-1}\in H \Rightarrow a*b\in H$

∴By definition (2) (H,∗) is a subgroup of (G,∗)

Example (2): Let (Z,+) be a group and H={5a: a∈Z}. Show that (H,+) is a subgroup of (Z,+)

Solution: By The above, let x , y ∈ H, T.P. $x+y^{-1}\in H$

$x\in H \Rightarrow x=5a, a\in Z$, $y\in H \Rightarrow y=5b, b\in Z$

$x+y^{-1}=5a+(5b)^{-1}=5a+5(-b)$

$=5\underbrace{(a-b)}_{\in Z}\in H$

⟹ (H,+) is a subgroup of (Z,+)

Theorem (2): If (H_i,∗) is the collection of subgroups of (G,∗), then ($\cap H_i$,∗)) is also subgroup of (G,∗)

Proof:

(1) Since ∃ e∈ H_i ,∀i ⟹e ∈ $\cap H_i \Rightarrow \cap H_i \neq \phi$

(2) Let x, y ∈$\cap H_i$ T.P. $x*y^{-1}\in \cap H_i$

Since x ,y∈ $\cap H_i \Rightarrow$ x,y ∈H_i ∀i

$\Rightarrow x*y^{-1}\in H_i$, ∀i (since H_i subgroups)

$\Rightarrow x*y^{-1}\in \cap H_i$

∴ ($\cap H_i$,∗) is subgroup of (G,∗)

Theorem (3): Let $(H_i,*)$ is the collection of subgroups of $(G,*)$ and let H_k and $H_j \in \{H_i\}$ such that $\exists H_\ell \in \{H_i\}$, $H_k \subseteq H_\ell$ and $H_j \subseteq H_\ell$ then $(\cup H_i,*)$ is also subgroup.

Proof. (1) Since $\exists\ e \in H_i$ for some i $\Longrightarrow e \in \cup H_i \Longrightarrow \cup H_i \neq \phi$

(2) Let $x,y \in \cup H_i$, then $x,y \in H_k$ or $x,y \in H_j$, so $x,y \in H_\ell$

$\Longrightarrow x * y^{-1} \in H_\ell$, (since $H_\ell$subgroup)

$\Longrightarrow x * y^{-1} \in \cup H_i$

$\therefore$ $(\cup H_i,*)$ is subgroup of $(G,*)$

Theorem (4): Let $(H_1,*)$ and $(H_2,*)$ are two subgroupsof $(G,*)$ then $(H_1 \cup H_2,*)$, is asubgroup of $(G,*)$ iff $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$.

Proof. $(\Rightarrow)$ Let$(H_1 \cup H_2,*)$ is a subgroup, T.P. $H_1 \subseteq H_2$or $H_2 \subseteq H_1$

Suppose that $H_1 \not\subset H_2$ and $H_2 \not\subset H_1$

$\therefore \exists\ a \in H_1$, $a \notin H_2$and $\exists\ b \in H_2$, $b \notin H_1$

$\therefore a * b \in H_1 \cup H_2 \Longrightarrow a * b^{-1} \in H_1 \cup H_2$

$\Longrightarrow a * b^{-1} \in H_1$or $a * b^{-1} \in H_2$

$\Longrightarrow$ a, b$\in H_1$or a, b$\in H_2$C! (تناقض)

$\therefore H_1 \subseteq H_2$ or $H_2 \subseteq H_1$

$(\Leftarrow)$ Let $H_1 \subseteq H_2$or $H_2 \subseteq H_1$T.P. $(H_1 \cup H_2,*)$ is a subgroup

If $H_1 \subseteq H_2 \Longrightarrow H_1 \cup H_2 = H_2$ is a subgroup.

If $H_2 \subseteq H_1 \Longrightarrow H_1 \cup H_2 = H_1$ is a subgroup

$\therefore H_1 \cup H_2$ is a subgroup in two cases.

Remark (2): $(H_1 \cup H_2,*)$ need not be a subgroup of $(G,*)$.

For example:$H_1 = \{r_1, r_3\}$ is a subgroup of G_s, and $H_2 = \{r_1, v\}$ is a subgroup of G_s.

But $H_1 \cup H_2 = \{r_1, r_3, v\}$ is not a subgroup of G_s, since $r_3 \circ v = h \notin H_1 \cup H_2$

Definition (3): Let $(G,*)$ be a group and $(H,*)$, $(K,*)$ be two subgroups of G, then the product of H and K is the set:

$$H*K=\{h*k : h \in H, k\in K\}$$

Notes(1):

(1) $H*H$ is write H^2

(2) If $H=\{a\}$, then $H*K=a*K$. If $K=\{b\}$, then $H*K=H*b$.

(3) $H\cup K \subseteq H*K$.

Theorem (5): Let $(G,*)$ be a group and $(H,*)$, $(K,*)$ are two subgroups of $(G,*)$, then

(1) $H*K \neq \phi \wedge H*K \subseteq G$

(2) $H\subseteq H*K$ and $K\subseteq H*K$

(3) $(H*K,*)$ is a subgroup of $(G,*)$ iff $H*K=K*H$

(4) If $(G,*)$ is commutative group, then $(H*K,*)$ is a subgroup of $(G,*)$. (منطوق فقط)

Proof:

(1)$\because e \in H \wedge e \in K \Rightarrow e * e= e\in H*K$

$$\therefore H*K \neq \phi$$

And let $x\in H*K \Longrightarrow x=a*b \ni a\in H \subseteq G$ and $b\in K\subseteq G$

$$\Longrightarrow a\in G \wedge b \in G$$

$$\Longrightarrow a*b =x\in G$$

$$\therefore H*K \subseteq G$$

(2) Let $x \in H \Longrightarrow x=x*e \in H*K$

$\Longrightarrow x\in H*K$

$$\therefore H\subseteq H*k$$

Similarly $K \subseteq H*K$

(3) (⇒) suppose (H∗K,∗) is asubgroup of (G,∗) T.P. H∗K=K∗H

(i.e.) H∗K ⊆ K∗H ∧ K∗H ⊆H∗K

Let x ∈ H∗K ⟹x =a∗b ∋ a∈H ∧ b ∈ K

Since H∗K is subgroupof G⟹x^{-1}∈ H∗K

Let x^{-1}= c ∗ d ∋ c ∈H∧ d∈K

x =$(x^{-1})^{-1}$ = $(c*d)^{-1}$ = $d^{-1}*c^{-1}$∋d^{-1} ∈K ∧c^{-1}∈ H

∴x = $d^{-1}*c^{-1}$∈ K∗H

∴H∗K⊆K∗H

K∗H ⊆ H∗K **(H.W.)**

(⇐) Let H∗K=K∗H T.P.(H∗K,∗) is subgroup of(G,∗)

H∗K≠ ϕand H∗K⊆G (by 1)

Let x,y ∈H∗K T.P.x∗y^{-1}∈H∗K

x∈H∗K ⟹x=a∗b ∋a∈ H ∧b ∈ K

y∈H∗K ⟹y=c∗d ∋ c ∈ H ∧d ∈ K

x∗y^{-1}= $(a*b)*(c*d)^{-1}$

$=(a*b)*(d^{-1}*c^{-1})$

$=a*(\underbrace{b*d^{-1}}_{\in K})*\underbrace{c^{-1}}_{\in H}$

∴$(b*d^{-1})*c^{-1}$∈ K∗H=H∗K

∴$(b*d^{-1})*c^{-1}$∈ H∗K

⇒∃ p∈ H , ℓ ∈K ∋ $(b*d^{-1})*c^{-1}$ = p∗ℓ

∴$a*(b*d^{-1})*c^{-1}$ =$\underbrace{a*p}_{\in H}*\underbrace{\ell}_{\in K}$ ∈H∗K

∴x∗y^{-1}∈H∗k

∴(H∗K,∗) is subgroup of (G,∗)

Example (3): In $(Z_8,+_8)$, Let $H=\{\bar{0},\bar{4}\}$ and $K=\{\bar{0},\bar{2},\bar{4},\bar{6}\}$. Find $H+_8K$

Solution: $H+_8K=\{\bar{0},\bar{2},\bar{4},\bar{6}\}$.

Notes (2): Let $(H,*)$ and $(K,*)$ are two subgroupof $(G,*)$, then :

(1) $H*K\neq K*H$

(2) $(H*K,*)$ need not be subgroup of $(G,*)$. Give example **(H.W.)**

Exercises: Is $(H,*)$ a subgroup of $(G,*)$ each of the following:

(1) $(Z_8, +_8)$, $H=\{\bar{0},\bar{6}\}$. Find H^2.

(2) $(Z_4, +_4)$, $H=\{\bar{0},\bar{1},\bar{2}\}$. Find H^2.

Definition (4): The center of a group $(G,*)$ denoted by cent(G) or C(G) is the set $C(G)=\{c\in G: c*x=x*c, \forall x\in G\}$ العناصر التي تتبادل مع كل عناصر الزمرة

Note (3): $C(G)\neq\phi$, since $\exists e\in G$ s.t.

$e*x=x*e \ \forall x\in G \Rightarrow e\in C(G)$

Examples (4):

(1) The group $(R\backslash\{0\}, .)$

$C(R)=R$ since R with multiplication is commutative

(2) The group $(S_3,\circ)$, $C(S_3)=\{f_1\}$

Since $C(S_3)=\{f\in S_3: f\circ g=g\circ f \ \forall g\in S_3\}=\{f_1\}$

Theorem (6): Let $(G,*)$ be a group. Then $(cent(G),*)$ is a subgroup of $(G,*)$.

Proof:

$cent(G)\neq\phi$ (by note (3))

$C(G)=\{a\in G: x*a=a*x, \forall x\in G\}\subseteq G$

Let $a,b\in cent(G)$ T.P. $a*b^{-1}\in cent(G)$

$a\in cent(G)\Rightarrow a*x=x*a, \forall x\in G$

$b\in cent(G)\Rightarrow b*x=x*b, \forall x\in G$

T.P. $(a*b^{-1})*x=x*(a*b^{-1}) \ \forall x\in G$

$(a*b^{-1})*x= a*(b^{-1}*x)$

$= a*(x^{-1}*b)^{-1}$

$=a*(b*x^{-1})^{-1}$ (since b ∈ cent (G))

$= a*(x*b^{-1})$

$=(a*x)*b^{-1}$

$=(x*a)*b^{-1}$ (since b ∈ cent (G))

$= x*(a*b^{-1})$

∴ $(a*b^{-1}) \in$ cent (G)

∴ (cent (G),∗) is a subgroup of (G,∗).

Theorem(7): Let (G,∗) be a group. Then

cent(G)=G⇔G is a commutative group.

Proof.

(⇒) ∀a∈G ⇒a∈ cent(G)

∴ a∗x =x∗a, ∀x∈G

∴ a∗x =x∗a, ∀x, a∈G

∴ G is commutative group

(⇐) suppose that G is commutative group T.P. cent(G) =G

(i.e) T.P. cent (G) ⊆G∧G⊆ cent(G)

By definition of cent(G) we have cent(G)⊆G.

T.P. G⊆ cent(G)

Let x∈G, G is commutative group ⇒x∗a = a ∗x, ∀a∈G

∴x∈ cent(G)⇒G⊆ cent(G)

∴ cent G=G

Cyclic Groups الزمر الدوارة أو (الزمر الدائرية)

Definition (5): Let (G,∗) be a group and $a \in G$, the cyclic subgroup of G generated by the a is denoted by $\langle a \rangle$ and defined as

$$\langle a \rangle = \{a^k : k \in Z\} = \{\ldots, a^{-1}, a^0, a^1, \ldots\}$$

G=$\langle a \rangle$ is called cyclic group.

- تسمى الزمرة دائرية او دوارة اذا امكن توليدها من عنصر واحد او اذا وجد عنصر يولدها

Definition (6):A group (G,∗) is called cyclic group generated by a iff $\exists a \in G$ such that

$$G=<a>=\{a^k : k \in Z\}$$

Examples (5): In $(Z_9,+_9)$find the cyclic subgroup generated by$\bar{2},\bar{3},\bar{1}$

$<\bar{2}>=\{a^k : k\in Z\}=\{\ldots,(\bar{2})^{-3},(\bar{2})^{-2},(\bar{2})^{-1},(\bar{2})^{0},(\bar{2})^{1},(\bar{2})^{2},(\bar{2})^{3},\ldots\}$

$=\{\ldots,\bar{3},\bar{5},\bar{7},\bar{0},\bar{2},\bar{4},\bar{6},\ldots\} = \{\bar{0},\bar{1},\bar{2},,\ldots,\bar{8}\} = Z_9$

$\therefore Z_9$ is cyclic group generated by $\bar{2}$

$<\bar{3}>=\{\ldots,(\bar{3})^{-3},(\bar{3})^{-2},(\bar{3})^{-1},(\bar{3})^{0},(\bar{3})^{1},(\bar{3})^{2},(\bar{3})^{3},\ldots\}$

$=\{\ldots,\bar{3},\bar{6},\bar{0},\bar{3},\bar{6},\bar{0},\ldots\} = \{\bar{0},\bar{3},\bar{6}\}$is a cyclic subgroup of Z_9

$<\bar{1}>=\{\ldots,(\bar{1})^{-3},(\bar{1})^{-2},(\bar{1})^{-1},(\bar{1})^{0},(\bar{1})^{1},(\bar{1})^{2},(\bar{1})^{3},\ldots\}$

$=\{\ldots,\bar{6},\bar{7},\bar{8},\bar{0},\bar{1},\bar{2},\bar{3},\ldots\} = \{\bar{0},\bar{1},\bar{2},,\ldots,\bar{8}\} = Z_9$

$\therefore Z_9$ is a cyclic group generated by $\bar{1}$

Examples (6): In (Z,+) fined a cyclic group generated by 1, 2, – 1

$<1>=\{1^k : k\in Z\}=\{\ldots,1^{-3},1^{-2},1^{-1},1^{0},1^{1},1^{2},1^{3},\ldots\}$

$=\{\ldots,-3,-2,-1,0,1,2,3,\ldots\} = Z$

$<2>=\{2^k : k \in Z\}=\{\ldots,2^{-3},2^{-2},2^{-1},2^{0},2^{1},2^{2},2^{3},\ldots\}$

$=\{\ldots,-6,-4,-2,0,2,4,6,\ldots \neq Z$

$<-1>=\{(-1)^k : k \in Z\}$

$=\{\ldots,(-1)^{-3},(-1)^{-2},(-1)^{-1},(-1)^{0},(-1)^{1},(-1)^{2},(-1)^{3},\ldots\}$

$=\{\ldots,2,1,0,-1,-2,\ldots\}=Z$

$\therefore$(Z,+) is cyclic group generated by 1 and – 1

Examples (7): Is $(S_3,\circ)$ cyclic group ?

$<f_1>=\{f_1\}\neq S_3$

$<f_2>=\{f_2^k : k\in Z\}=\{\ldots,f_2^{-2},f_2^{-1},f_2^{0},f_2^{1},f_2^{2},\ldots..\}$

$=\{\ldots,f_2,f_3,f_1,f_2,f_3,\ldots\}=\{f_1,f_2,f_3\}\neq S_3$

$<f_3>=\{f_1,f_2,f_3\}\neq S_3$

$<f_4>=\{f_1,f_4\}\neq S_3$

$<f_5>=\{f_1, f_5\}\neq S_3$

$<f_6>=\{f_1, f_6\}\neq S_3$

$\therefore (S_3,\circ)$ is not cyclic group.

Examples (8): In$(Z_6,+_6)$ find cyclic group generated by$\bar{1},\bar{2},\bar{5}$**(H.W.)**

Theorem (8): Every cyclic group is commutative.

Proof: Let (G,∗) be acyclic group

$\therefore \exists a\in$G s.t. G=$<a>=\{a^k: k\in Z\}$T.P. G is commutative group

Let $x,y\in$G T.P. $x*y=y*x$, $\forall x,y\in$G

$\because x\in$G $=<a>\Rightarrow x=a^m \ni m\in Z$ and$y\in$G $=<a>\Rightarrow$y$=a^n \ni n\in Z$

$x*y=a^m * a^n = a^{m+n} = a^{n+m} = a^n * a^m = y * x$

$\therefore$G is commutative group

The converse of this theorem is not true, for example:

(G=$\{e,a,b,c\}$,∗) s.t. $a^2 = b^2 = c^2=e$

$a^2=e\Rightarrow a*a=e\Rightarrow a^{-1}=a$

$b^2=e\Rightarrow b*b=e\Rightarrow b^{-1}=b$

$c^2=e\Rightarrow c*$c$=e\Rightarrow c^{-1}=$c

$e^{-1}=e\Rightarrow x^{-1}=x$ $\forall x\in$G

$\therefore$ (G,∗) is commutative group

But (G,∗) is not cyclic group since:

$< e >=\{e\}\neq$G

$< a >=\{a^k: k \in Z\}=\{e,a\}\neq$G

$< b >=\{b^k: k\in Z\}=\{e,b\}\neq$G

$< c >=\{c^k: k\in Z\}=\{e,c\}\neq$G

$\therefore$(G,∗) is not cyclic

Theorem (9):$<a>=<a^{-1}>\forall a\in$G

Proof:

$< a >=\{a^k: k\in Z\}=\{ (a^{-1})^{-k} \because -$ k$\in Z\}$

$=\{ (a^{-1})^m \because$m $= -$ k$\in Z\}$

$=<a^{-1}>$

Theorem (10): If (G,∗) is a finite group of order *n* generated by *a*, then G= $<a>=\{a^k : k\in Z\} = \{a^1, a^2, \ldots\ldots, a^n = e\}$such that *n* is least positive integer $\ni a^n = e\,, (i.e.)$

o(*a*)=*n*= o(G) (رتبة العنصر الذي يولد الزمرة = رتبة الزمرة)

Examples (9): Show that (Z_n,+$_n$) is cyclic group.

$$Z_n = \{\bar{0}, \bar{1}, \bar{2}, \ldots, \overline{n-1}\}$$

بما ان الزمرة مرتبهية فتكتب بالشكل :

o(Z_n) = *n* ,T.P. $Z_n . =<\bar{1}>$

$<\bar{1}>=\{(\bar{1})^k : k\in Z\}=\{(\bar{1})^1, (\bar{1})^2, (\bar{1})^3, (\bar{1})^n = \bar{0}\}$

$=\{\bar{1}, \bar{2}, \bar{3}, \ldots, \bar{n} = \bar{0}\}=Z_n$

$Z_n . =<\bar{1}>$ and o(Z_n)= o($\bar{1}$)=*n*.

Definition (7):(Division Algorithm for Z) **خوارزمية القسمة**

Let a and b be two integer numbers with b>0, then there is a unique pair of integers q and r such that:

a =bq + r where $0 \le r < b$

The number q is called the quotient and r is called the remainder when a is divided by b.

Examples (10): Find the quotient q and remainder r when 38 is divided by 7 according to the division algorithm.

Answer: 38=7(5)+3 $0\le 3\le 7$

∴q =5 and r=3.

Examples (11):a=23 ,b=7

23=7(3)+2 $0\le 2\le 7$

q =3 , r=2.

Examples (12): a=15 , b=2

15=(2)(7)+1 $0\leq 1\leq 2$

q =7 , r=1

Theorem (11): A subgroup of acyclic group is cyclic.(للاطلاع)

Corollary (1): If (G,∗) is a finite cyclic group of order *n* generated by *a* , then every subgroup of G is cyclic generated by$a^m \ni m|n$

Proof: Suppose (G,∗) is a finite, then o(G)=*n*

G= < a >=$\{a^1, a^2, \ldots\ldots, a^n=e\}$

Let (H,∗)be a subgroup of (G,∗). Then (H,∗) is cyclic (by Theorem 11) such that H=<a^m>

T.P. *m*|*n*(*n*=*m*g , g ∈ Z)

e ∈H ⇒a^n∈H,a^m∈H, by division algorithm of *n* and *m*

⇒*n*=*m*g+r $0\leq r<m$

r =n – mg⇒$a^r = a^n * (a^m)^{-g}$

⇒$a^r = (a^m)^{-g}$∈H

But $0\leq r<m$

⇒ If r=0 ⇒*n*=*m*g

∴*m*|*n*

Examples (13): Find all subgroup of (Z_{15},$+_{15}$)

Answer: o(Z_{15})=15 , H=<$(\bar{1})^m$>∋*m*|*n*

H=<$(\bar{1})^m$>∋*m*|15

m=1,3,5,15

If *m*=1 ⇒H_1=<$\bar{1}$>= Z_{15}

If *m*=3⇒ H_2=<$(\bar{1})^3$>=$\{\bar{3}, \bar{6}, \bar{9}, \overline{12}, \bar{0}\}$

If *m*= 5⇒ H_3=<$(\bar{1})^5$>=$\{\bar{5}, \overline{10}, \bar{0}\}$

If *m*= 15⇒ H_4=<$(\bar{1})^{15}$>=$\{\bar{0}\}$ = <$\bar{0}$>

(H.W.) Find all subgroup of (Z_8,$+_8$).

Corollary (2): If (G,∗) is finite cyclic group of prime order, then G has no proper subgroup.

Proof: Let (G,∗) be a finite group such that

o(G)=p (p prime number)

G= <a>=$\{a^1, a^2, \ldots\ldots, a^p$=e$\}$

Let (H,∗) be cyclic subgroup

∴H=< a^m>∋m|p⇒m = 1 or m = p

If m=1 ⇒ H=<a>= G (not proper subgroup)

If m=p ⇒H=<a^p=e>= {e} (not proper subgroup)

∴G has no proper subgroup.

Examples (14): Find all subgroup of (Z_7,$+_7$)

Answer: o(Z_7)=7, let H=<$(\bar{1})^m$>∋m|7

∴ m=1 , m =7

If m=1 ⇒H_1=<$\bar{1}$>= Z_7

If m=7⇒ H_2=<$(\bar{1})^7$>=$\{\bar{0}\}$

Definition (8): [g.c.d(x,y)] القاسم المشترك الاكبر

A positive integer c is said to be a greatest common divisor of two non-zero number x and y iff

(1) c|x ∧c|y

(2) if a|x ∧a|y ⇒a|c

(g.c.d(x,y) =c)

Examples (15): Find (g.c.d.(12,18))

Answer: g.c.d(12,18)=6 since

(1) 6|12 ∧ 6|18

(2) 3|12 ∧ 3|18 ⇒ 3|6

or 2|12 ∧2|18 ⇒2|16

Remark (3): If $(G,*)$ is finite cyclic group of order n generated by a ,then the generators of G is a^k such that g.c.d $(k,n)=1$.

Examples (16): Find all generators of $(Z_6,+_6)$

Answer: o $(Z_6)=6$, $Z_6=<\bar{1}>$

$Z_6=<<(\bar{1})^k>$ s.t. g.c.d(k,6)=1,k=1,2,3,4,5

k=1 $\Rightarrow$ g.c.d(1,6)= 1 $\Rightarrow Z_6=<\bar{1}>$

k=2 $\Rightarrow$ g.c.d(2,6)$\neq$ 1$\Rightarrow Z_6\neq<(\bar{1})^2>=<\bar{2}>$

k=3$\Rightarrow$ g.c.d(3,6)$\neq$ 1 $\Rightarrow Z_6\neq<(\bar{1})^3>=<\bar{3}>$

k=4$\Rightarrow$ g.c.d(4,6)$\neq$1 $\Rightarrow Z_6\neq<<(\bar{1})^4>=<\bar{4}>$

k=5 $\Rightarrow$ g.c.d(5,6)=1 $\Rightarrow Z_6=<(\bar{1})^5>=<\bar{5}>$

The generators of Z_6 are $\{\bar{1},\bar{5}\}$

Theorem (12): If $(G,*)$ is an infinite cyclic group generated by a, then:

(1) a and a^{-1} are only generators of G

(2) Every subgroup of G except {e} is an infinite subgroup.

Definition (9): **المجموعات المشاركة للزمرة الجزيئة H**

Let $(H,*)$ be a subgroup of a group $(G,*)$.The set

$a*H=\{a*h : h\in H\}$of G is the left coset of H containing a , while the subset $H*a=\{h*a : h\in H\}$ is the right coset of H containing a.

Examples (17): If $(Z_6,+_6)$,a=$\bar{1}$, H=$\{\bar{0},\bar{2},\bar{4}\}$, then

$\bar{1}+_6H=\{\bar{1},\bar{3},\bar{5}\}$, $H+_6\bar{1}=\{\bar{1},\bar{3},\bar{5}\}$

$\bar{3}+_6H=\{\bar{3},\bar{5},\bar{1}\}$, $H+_6\bar{3}=\{\bar{3},\bar{5},\bar{1}\}$.

Notes(4):

(1) $a*H$ is not subgroup in general. Give an example **(H.W.)**

(2) $a*H \neq H*a$ in general, for example

$(S_3,\circ), H=\{f_1,f_4\}$, $a=f_2$

$f_2\circ H=\{f_2,f_5\}$, $H\circ f_2=\{f_2,f_6\}$

$f_2\circ H\neq H\circ f_2$

Theorem (13): Let $(H,*)$ be a subgroup of $(G,*)$ and $a\in G$, then

(1) H is itself left coset of H in G.

Proof: (1) $e\in G$, $e*H=\{e*h:h\in H\}=H$

(2) If $(G,*)$ is abelian group, then $a*H=H*a$

Proof: $a*H=\{a*h:h\in H\}=\{h*a:h\in H\}=H*a$

The converse is not true, for example: $(S_3,\circ)$, $H=\{f_1,f_2,f_3\}$ $a=f_4$

$f_4\circ H=\{f_4,f_5,f_6\}$ and $H\circ f_4=\{f_4,f_6,f_5\}$

$\therefore f_4\circ H=H\circ f_4$ but $(S_3,\circ)$ is not abelian group.

(3) $a\in a*H$

Proof: $a=a*e\in a*H$

(4) $a*H=H \Leftrightarrow a\in H$

Proof: $(\Rightarrow)$ Suppose $a*H=H$, then by (3) we get $a\in H$

$(\Leftarrow)$ Suppose $a\in H$ T.P. $a*H=H$

We must prove that $a*H\subseteq H \wedge H\subseteq a*H$

T.P. $a*H\subseteq H$

Let $x\in a*H\Rightarrow x=a*h\in H$ (since $a\in H\wedge h\in H$)

$\therefore a*H\subseteq H$

T.P.$H\subseteq a*H$

Let $b\in H\Rightarrow b=e*b$

$=(a*a^{-1})*b$

$=a*\underbrace{(a^{-1}*b)}_{\in H}\Rightarrow b\in a*H$

$\therefore H \subseteq a*H$

Thus $a*H=H$

(5) $a*H = b*H \Leftrightarrow a^{-1}*b \in H$

Proof: $(\Rightarrow) a*H = b*H$

$a^{-1}*(a*H)=a^{-1}*(b*H)$

$(a^{-1}*a)*H=(a^{-1}*b)*H$

$H=(a^{-1}*b)*H$

By (4) $\Rightarrow a^{-1}*b \in H$

$(\Leftarrow)$ Suppose that $a^{-1}*b \in H$

By (4) $\Rightarrow (a^{-1}*b)*H = H$

$\Rightarrow b*H=a*H$

Remark (4): Every coset (left or right) of a subgroup H of a group $(G,*)$ has the same number of elements as H.

(6) $a*H = b*H \vee (a*H) \cap (b*H)=\phi$

(7) The set of all distinct left coset of H in G form a partition on G.

Proof: T.P. $G=\cup_{a\in G} a* H$ and $a_i*H \cap a_j*H = \phi$

$\because a_i*H , a_j*H$ are distinct

$\therefore a_i*H \cap a_j*H = \phi$ T.P. $G=\cup_{a\in G} a* H$

$a* H \subseteq G \quad \forall a \in G$ (by definition of coset)

$\Rightarrow \cup_{a\in G} a* H \subseteq G \quad \ldots(1)$

$\forall a \in G \Rightarrow a \in a*H \Rightarrow a \in \cup_{a\in G} a* H$

$\therefore G \subseteq \cup_{a\in G} a* H \quad \ldots(2)$

From (1) and (2) $\Rightarrow G=\cup_{a\in G} a* H$

Example (17): The group $(Z_6,+_6)$ is abelian. Find the partition of Z_6 into coset of the subgroup $H=\{\bar{0},\bar{3}\}$

Answer: $Z_6=\{\bar{0},\bar{1},\bar{2}, \bar{3},\bar{4},\bar{5}\}$

$\bar{0} +_6 H=\{\bar{0},\bar{3}\} =H$

$\bar{1} +_6 H=\{\bar{1},\bar{4}\}$

$\bar{2} +_6 H=\{\bar{2},\bar{5}\}$

$\bar{3} +_6 H=\{\bar{3},\bar{0}\}$

$\bar{4} +_6 H=\{\bar{4},\bar{1}\}$

$\bar{5} +_6 H=\{\bar{5},\bar{2}\}$

∴All the cosets of H are :$\{\bar{0},\bar{3}\}$, $\{\bar{1},\bar{4}\}$, $\{\bar{2},\bar{5}\}$ and since $(Z_6,+_6)$ is abelian group, then the left coset is equal the right coset.

Example (18):(H.W.)

In $(S_3,\circ)$,let H= $\{f_1,f_4\}$.Find the partitions of S_3 into left cosets of H and the partitions into right cosets of H.

Definition (10): Let $(H,*)$ be a subgroup of a group $(G,*)$.The number of left cosets or right cosets of H in G is called the index of H in G and denoted by [G:H].

Remark (5): If $(G,*)$ is a finite group. Then $[G:H]=\frac{o(G)}{o(H)}$.

Example (19):$(S_3,\circ)$,H= $\{f_1,f_2, f_3\}$

∴ $[S_3:H] =\frac{o(S_3)}{o(H)} = \frac{6}{3} =2$.

Example (20):$(Z_6,+_6)$, H=$\{\bar{0},\bar{3}\}$

∴$[Z_6:H]=\frac{6}{2} =3$

Theorem (14): (Lagrange Theorem)

Let H be a subgroup of a finite group $(G,*)$.Then the order of H is a divisor of the order of G .

Proof:

Let G be a finite group ∋o(G)=n and H be a subgroup of G ∋ o(H)=m.

T.P. o(H) | o(G) (T.P. m|n , n=mk)

Since G is finite⇒[G:H] = k

Let $a_1 * H, a_2 * H, \ldots, a_k * H$ are left cosets of H

$a_1 * H \cup a_2 * H \cup \ldots \cup a_k * H = G$ and

$a_i * H \cap a_j * H = \phi$

$o(a_1 * H) + o(a_2 * H) + \ldots + o(a_k * H) = o(G)$

$$\underbrace{m + m + \cdots + m}_{k-\text{times}} = n$$

$mk=n \Rightarrow m|n \Rightarrow o(H)| o(G)$

Corollary (1): If $(G,*)$ is finite group, then the order of any element of G divides the order of G.

Proof. Suppose that $(G,*)$ is finite $\ni o(G) = n$.

Let $a \in G \Rightarrow$ a is finite order such that $o(a) = m$ T.P.$o(a)| o(G)$.

Since $a \in G \Rightarrow H = \langle a \rangle$ cyclic group.

$H = \{a, a^2, \ldots, a^m = e\}$

$o(H) = o(a) = m \Rightarrow o(H)| o(G)$ (by Lagrange theorem)

$\therefore o(a)| o(G)$

Corollary (2): If $(G,*)$ is a finite group, then $a^{0(G)} = e \quad \forall a \in G$.

Proof.

Suppose that $o(G) = n$, let $a \in G \ni o(a) = m$

By Corollary (1) of Lagrange theorem⇒o(a) | o(G)

⇒m|n

⇒ n=mk

$a^{o(G)} = a^n = (a^m)^k = e^k = e$

$\therefore a^{0(G)} = e \forall a \in G$.

Corollary (3): Every group of prime order is cyclic.

Corollary (4): Every group of order less than 6 is commutative.

Exercises:

(1) Find all subgroupsof $(Z_5,+_5)$.

(2) Let $(Z_8,+_8)$ be agroup and $H=<\bar{2}>$. Is H a subgroup of Z_8?

(3) If $H=\{\bar{0},\bar{6},\overline{12},\overline{18}\}$, show that $(H,+_{24})$ is a cyclic subgroupof $(Z_{24},+_{24})$. Also list the elements of each coset of H in Z_{24}.

Chapter Three

Normal Subgroups and Quotient Groups

(الزمر الجزئية الطبيعية وزمر القسمة)

Def (1). Let (G,*) be a group and $a, b \in G$, then "a is conjugate to b " and denoted by $a \sim b$ iff $\exists x \in G \ni b = x*a*x^{-1}$ and $b \sim a$ iff $\exists x \in G \ni a = x*b*x^{-1}$.

$a \not\sim b \Leftrightarrow b \neq x*a*x^{-1} \ \forall x \in G$.

ex.(1) In (S_3,o) . Is $f_3 \sim f_2$?

Ans. $f_3 \sim f_2 \Leftrightarrow \exists x \in S_3 \ni f_2 = x \ o \ f_3 \ o x^{-1}$

$x=f_1 \Rightarrow f_1 \ o \ f_3 o \ f_1^{-1}=f_3 \neq f_2$

$x=f_2 \Rightarrow f_2 \ o \ f_3 \ o f_2^{-1}=f_1 \ o f_2^{-1}=f_3 \neq f_2$

$x=f_3 \Rightarrow f_3 \ o \ f_3 \ o \ f_3^{-1}=f_2 \ o \ f_2=f_3 \neq f_2$

$x=f_4 \Rightarrow f_4 \ o \ f_3 o f_4^{-1}=f_5 \ o \ f_4=f_2$

$x=f_5 \Rightarrow f_5 \ o \ f_3 \ o f_5^{-1}=f_6 \ o \ f_5=f_2$

$x=f_6 \Rightarrow f_6 \ o \ f_3 \ o \ f_6^{-1}=f_4 \ o \ f_6=f_2$

$\therefore \exists x \in S_3 \ni x o \ f_3 \ o \ x^{-1}=f_2$

$\therefore f_3 \sim f_2$

IS $f_1 \sim f_2$ and $f_1 \sim f_1$? (H.W)

ex.(2) In $(z_4,+_4)$. Is $\bar{1} \sim \bar{2}$? $\bar{2} = \bar{x} +_4 \bar{1} +_4 x^{-1}$

$x = \bar{1} \Rightarrow \bar{1} +_4 \bar{1} +_4 (\bar{1})^{-1} = \bar{2} +_4 \bar{3} = \bar{5} = \bar{1} \neq \bar{2}$

$x=\bar{2} \Rightarrow \bar{2}+_4\bar{1}+_4(\bar{2})^{-1}=\bar{3}+_4\bar{2}=\bar{5}=\bar{1}\neq\bar{2}$

$x=\bar{3} \Rightarrow \bar{3}+_4\bar{1}+_4(\bar{3})^{-1}=\bar{0}+_4\bar{1}=\bar{1}\neq\bar{2}$

$x=\bar{0} \Rightarrow \bar{0}+_4\bar{1}+_4(\bar{0})^{-1}=\bar{1}\neq\bar{2}$

$\therefore \bar{1}\neq\bar{2}$

Remark(1): if (G,*) is a belian group and a,b ∈ G , then a~ b ⇔ a=b

Proof: Suppose a~b⇔ $\exists x \in G \ni b=x*a*x^{-1}$

⇔b= $x * x^{-1} * a = e * a$

⇔ b=a

Theorem (1): The relation (Conjugate) is an equivalent relation.

Proof: (1) Reflexive (الانعكاسية)

Let $a \in$G , T.P $a \sim a$

∃ e ∈ G ∋ a=e*a*e^{-1}

$\therefore a \sim a$

(2) Symmetric (التناظر)

Let a,b ∈ G and a~b ,T.P b ~a

a~b⇒ ∃ x ∈ G ∋ b=$x*a*x^{-1}$

⇒ x^{-1}*b=$a*x^{-1}$

⇒ x^{-1}*b*x=a

⇒b~a

(3) Transitive (متعدية)

Let a,b,c ∈ G s.t.a~b∧ b~c ,T.P a~c

a~b ⇒ ∃x ∈ G s.t b=x*a*x^{-1}……(1)

b~c⇒ ∃y ∈ G s.t c=y*b*y^{-1}……(2)

put (1) in (2)

c=y*($x * a * x^{-1}$)*y^{-1}

c=($y * x$) $* a *$ ($y * x$)$^{-1}$

$c = z * a * z^{-1}$ (where z=$y * x$ ∈ G)

∴ a~c

Def(2). Let (G,*) be a group and a ∈ G , then the conjugate of a is denoted by c(a) and defined as

c(a)={b∈ G: a~b} (a مجموعة العناصر التي ترافق)

or c(a)={b∈ G:b=$x * a * x^{-1}$ }

or c(a)={$x * a * x^{-1}$,∀x ∈ G} .

The set of all elements conjugate to a is called the conjugate class of a.

Ex(3). Find the conjugate class of each element in the following groups:

1) (G={1,-1,i,-i},·) ∋ i^2= -1

Ans. c(i)={x.i.x^{-1}, ∀x ∈ G}

={1.i.1^{-1},(-1).i.$(-1)^{-1}$, i.i.i^{-1} , (-i).i.$(-i)^{-1}$}

$= \{1.\,i.\,1, (-1).\,i.\,(-1), i.\,i.\,(-i), (-i).\,i.\,i\}$

$=\{i, i, i, i\}=\{i\}$

$\therefore$ c(1)={1} , c(-1)={-1} , c(-i) ={-i} .

2) $(S_3,\circ)$ **(H.W)**

3) $(G_S,\circ)$ **(H.W)**

Ex(4). Find $c(\bar{3})$ in $(z_4,+_4)$

Ans. $c(\bar{3}) = \{\bar{0}, +_4\bar{3}+_4\bar{0}^{-1}\ , \bar{1}+_4\bar{3}+_4\bar{1}^{-1}, \bar{2}+_4\bar{3}+_4\bar{2}^{-1}, \bar{3}+_4\bar{3}+_4\bar{3}^{-1}\}$

$= \{\bar{0}+_4\bar{3}+_4\bar{0}, \bar{1}+_4\bar{3}+_4\bar{3}, \bar{2}+_4\bar{3}+_4\bar{2}, \bar{3}+_4\bar{3}+_4\bar{1}\}$

$= \{\bar{3}, \bar{3}, \bar{3}, \bar{3}\}$

$\therefore$ $c(\bar{3})=\{\bar{3}\}$ (by remark if G is comm. group and a$\sim b$ then a=b)

Note. Let $(G,*)$be a group and a$\in G$, then c(a) need not be a subgroup of $(G,*)$.

For example: In (S_3,o)

$c(f_3)=\{f_2,f_3\}$ is not subgroup of S_3

Theorem(2): Let $(G,*)$be a group and a,b$\in G$,then

1) $a \in c(a)\ \forall a \in G$
2) $c(a)=c(b) \Leftrightarrow a\sim b\ \forall a,b \in G$
3) $c(a)\cap c(b)=\emptyset$ iff $a\sim b$ **(H.W)**
4) $c(a)\cap c(b)=\emptyset$ or $c(a)=c(b)$ **(H.W)**
5) $b\in c(a)\Leftrightarrow c(a)=c(b)$
6) $c(a)=\{a\}\ \forall a \in G \Leftrightarrow$ G is a comm. group.

7) c(a)={a}⇔ a ∈cent (G) **(H.W)**

8) c(e) ={e} **(H.W)**

Proof: (1) a ∈ c(a) ∀a ∈G

Since $a\sim a$ ∀a ∈G (~ is ref.)

⟹ a ∈ c(a)⇒ c(a)≠Ø

(2) c(a)=c(b) ⇔ $a\sim b$ ∀a,b ∈ G

(⇒) Suppose c(a)=c(b) T.P $a\sim b$

By (1) ,a ∈ c(a)=c(b)⇒ $a \in c(b) \Rightarrow a\sim b$

(⇐) suppose $a\sim b$ T.P c(a)=c(b)

(i.e.) c(a) ⊆c(b) ∧ c(b) ⊆ c(a) ?

Let x ∈ c(a)⇒ x~a ∧ $a\sim b$ ⇒ x~b

⇒ x∈ c(b) ⇒ c(a)⊆c(b) ……(1)

Let x ∈ c(b) ⇒ x~b ∧ $a\sim b$

⇒ x~a ⇒ x ∈c(a) ⇒ c(b)⊆ c(a) .....(2)

By (1) and (2)⇒ c(a)=c(b)

(5) b∈ c(a)⇔ c(a)=c(b)

(⇒) Let b ∈c(a)⇒ b~a ⇒c(a)=c(b)(by Th.)

(⇐) c(a)=c(b)⇒ $a\sim b$ ⇒ b~a ⇒ b∈ c(a).

(6) c(a) ={a} ∀a ∈ G⇔ G is a comm. group.

c(a)=$\{a\}$ $\forall a \in$ G

$\Leftrightarrow x * a * x^{-1}=a \quad \forall x \in G$

$\Leftrightarrow x * a = a * x$

$\Leftrightarrow$ G is a comm. group.

Def(3). Let (G,$*$) be a group and $a \in G$, then the normalizer of a is denoted by N(a) and defined as: N(a) $= \{x \in G: x * a = a * x\}$

(مجموعة العناصر التي تتبادل مع a)

Ex 5. In (z_8,$+_8$) . find N($\bar{3}$)

N($\bar{3}$)=$\{\bar{x} \in z_8: \bar{x} +_8 \bar{3} = \bar{3} +_8 \bar{x}\}$ =$\{\bar{0},\bar{1},\bar{2},\bar{3},\bar{4},\bar{5},\bar{6},\bar{7}\}$ =Z_8.

Theorem(3): Let (G,$*$) be a groub and $a \in G$, then

1) (N(a) , $*$) is a subgroup of (G,$*$) .
2) cent (G)= $\cap N(a)$ $\forall a \in$ G **(H.W)**
3) N(a) $= G$, $\forall a \in G \Leftrightarrow (G,*)$ *is a comm.*
4) N(a) $= G \Leftrightarrow a \in G$ **(H.W)**
5) The Cardinal number of c(a)= the index of N(a) in G (منطوق فقط)
6) If (G,$*$) is a finite group, then o (c(a)) / o(G)). (منطوق فقط)

Proof: (1)N(a) $= \{x \in G : x * a = a * x\} \subseteq$ G

Since $e * a = a * e \Rightarrow$ e$\in$ N(a)

$\therefore$ N(a)$\neq \emptyset$

(i) Closure: Let x, y $\in$ N(a) T. P x$*$y $\in$ N(a)

$x \in N(a) \Rightarrow x * a = a * x$

$y \in N(a) \Rightarrow y * a = a * y$

$(x, y) * a = x * (y * a) = x * (a * y)$

$= (x * a) * y = (a * x) * y = a * (x * y)$

$\therefore x * y \in N(a)$

(ii) $Let\ x \in N(a)\ T.P.\ x^{-1} \in N(a)$

$Since\ x \in N(a) \Rightarrow x * a = a * x$

$\Rightarrow x * a * x^{-1} = a$

$\Rightarrow a * x^{-1} = x^{-1} * a$

$\Rightarrow x^{-1} \in N(a)$

$\therefore (N(a), *)\ is\ a\ subgroup$.

(3)($\Rightarrow$) Suppose $N(a) = G \forall a \in G$ T.P G is a comm.

$\forall x \in G = N(a) \Rightarrow x \in N(a)\ \ \forall a \in G$

$\Rightarrow x \in N(a) \forall x, a \in G$

$\Rightarrow x * a = a * x \forall x, a \in G$

$\therefore (G, *)\ is\ a\ comm.\ group$.

($\Leftarrow$)$Suppose (G, *)\ is\ a\ comm.group \quad T.P. \quad N(a) = G$

$T.P. \quad N(a) \subseteq G \ \wedge \ G \subseteq N(a)$

$N(a) \subseteq G \quad (bydef.)$

$T.P.\ \ G \subseteq N(a)$

Let $x \in G \wedge$ G is a comm. $\Rightarrow x * a = a * x \quad \forall x, a \in G$

$\Rightarrow$ x$\in N(a) \ \ \forall a \in G$

$\Rightarrow$ G$\subseteq N(a) \ \Rightarrow \ N(a) = G \ \ \forall a \in G.$

Def(4). Let (H,$*$), (K,$*$) be two subgroups of (G,$*$), then H is a conjugate subgroup K iff $\exists x \in G \ni K = x * H * x^{-1}$ and denoted by $H \sim K$.

$H \not\sim K \Leftrightarrow K \neq x * H * x^{-1} \ \ \forall x \in G$.

Ex.(6) In (S_3,o), H={f_1,f_6}, K ={f_1,f_5} . Is $H \sim K$?

Ans.(i.e.) Is $\exists x \in S_3 \ni x$o Ho x^{-1} =K ?

$x=f_1 \Rightarrow f_1$ o{f_1,f_6} o f_1^{-1}={f_1o f_1o f_1^{-1},f_1o f_6o f_1^{-1}}={f_1,f_6}$\neq K$

$x=f_2 \Rightarrow f_2$o{f_1,f_6} o f_2^{-1}={f_2o f_1o f_2^{-1},f_2o f_6o f_2^{-1}}={f_1,f_4}$\neq K$

$x = f_3 \Rightarrow$ f_3o{f_1,f_6} o f_3^{-1}={ f_3o f_1o f_3^{-1}, f_3o f_6o f_3^{-1}}={f_1,f_5}$=K$

$\therefore \exists x = f_3 \ni H \sim K$.

ex(7): In (Z_{12},$+_{12}$), H={$\bar{0},\bar{4},\bar{8}$}, K ={$\bar{0},\bar{3},\bar{6},\bar{9}$} Is H$\sim K$?

Ans. (i.e.) $\exists \bar{x} \in Z_{12} \ni \bar{x} +_{12}$ H$+_{12} \bar{x}^{-1}$=K

$\bar{x}=\bar{1} \Rightarrow \bar{1} +_{12} \{\bar{0},\bar{4},\bar{8}\} +_{12} \bar{1}^{-1} = \bar{1} +_{12} \{\bar{0},\bar{4},\bar{8}\} +_{12} \overline{11}$=H ={$\bar{0},\bar{4},\bar{8}$} $\neq$K

(.i.e.) $\bar{1} +_{12} \{\bar{0},\bar{4},\bar{8}\} +_{12} \bar{1}^{-1} = \bar{1} +_{12} \bar{1}^{-1} +_{12} \{\bar{0},\bar{4},\bar{8}\} = \{\bar{0},\bar{4},\bar{8}\}$=H

$\therefore$H$\not\sim$ K $\ \forall \bar{x} \in Z_{12}$

Since $\bar{x} +_{12}$ H$+_{12} \bar{x}^{-1} = \bar{x} +_{12} \bar{x}^{-1} +_{12}$ H= H $\neq$ K

ex.(8): In (G_s,o) , let $H=\{r_1,r_4\}$, $K=\{r_1,r_2\}$. Is $H\sim K$? **(H.W)**

Theorem (4) : Let $(H,*)$, $(K,*)$be two finite subgroups of$(G,*)$ and $H\sim K$, then $o(H)=o(K)$. (منطوق فقط)

Theorem(5) : Let $(H,*)$be a subgroup of $(G,*)$ and $x\in G$ then $(x*H*x^{-1},*)$

is a subgroup of $(G,*)$.

Proof: Since $e*H*e^{-1}=H\neq\emptyset \Rightarrow x*H*x^{-1}\neq\emptyset$

$x*H*x^{-1}=\{x*h*x^{-1} : \forall h\in H\}\subseteq G$

Let $a,b\in x*H*x^{-1}$ T.P $a*b^{-1}\in x*H*x^{-1}$

$a\in x*H*x^{-1}\Rightarrow a=x*h_1*x^{-1}\ni h_1\in H$

$b\in x*H*x^{-1}\Rightarrow b=x*h_2*x^{-1}\ni h_2\in H$

$a*b^{-1}=(x*h_1*x^{-1})*(x*h_2*x^{-1})^{-1}$

$=(x*h_1*x^{-1})*(x*h_2^{-1}*x^{-1})$

$=(x*h_1)*(x^{-1}*x)*(h_2^{-1}*x^{-1})$

$=x*(h_1*h_2^{-1})*x^{-1}\in x*H*x^{-1}$

$\therefore (x*H*x^{-1})$ is a subgroup of $(G,*)$.

Remark(2): The relation "conjugate" is an equivalent relation on the set of all subgroups of G. **(H.W)**

Def(5). Let $(H,*)$ be a subgroup of $(G,*)$, then the conjugate class of H is denoted by C(H) and defined as

$C(H)=\{x*H*x^{-1} : \forall x\in G\}$

(مجموعة الزمر الجزئية التي ترافقH)

Ex(9). (S_3,o), $H=\{f_1,f_4\}$, find C(H)

Ans. $C(H)=\{xoHox^{-1} : \forall x \in S_3\}$

$=\{f_1o\{f_1,f_4\}of_1^{-1},f_2o\{f_1,f_4\}o\ f_2^{-1},\ldots.,f_6o\{f_1,f_4\}of_6^{-1}\}$

$=\{\ \{f_1,f_4\},\{f_1,f_6\},\ldots..,\{f_1,f_5\}\ \}$

Ex.(10) : $(G=\{e,a,b,c\},*)$ is a four Klien group.

G is comm. group , $a^2=b^2=c^2=e$.

$H=\{e,a\}\subseteq G$, find C(H)

Ans. $C(H)=\{x*H*x^{-1} : \forall x \in G\}$

$=\{x*x^{-1}*H : \forall x \in G\ \}=\{H\}$

Def(6). Let $(H,*)$ be a subgroup of $(G,*)$, then the normalizer of H is denoted by N(H) and defined as: $N(H)=\{x \in G : x*H=H*x\}$

Ex(11). (Gs,o) , $H=\{r_2,r_3\}$, find N(H)

$N(H)=\{x \in G_5 \ni xoH=Hox\}$

$x=r_1 \Rightarrow r_1oH=H\ o\ r_1$

$x=r_2 \Rightarrow r_2oH=H\ o\ r_2$ …. → N(H) = { r_1, r_2, r_3, r_4, h, v, D_1, D_2 } = Gs

Exercises: Find C(H), N(H) to each of the following:

1)(S_3,o) ,$H_1=\{f_1,f_5\}$, $H_2=\{f_1,f_4\}$

2)(G_5,o) , $H_1=\{r_3,r_1,v,h\}$, $H_2=\{r_1,D_1\}$

3)$(Z_{12}, +_{12})$, H=$\{\bar{0},\bar{4},\bar{8}\}$

Theorem(6): Let (H,∗) be a subgroup of a group (G,∗), then

1) (N(H),∗) is a subgroup of(G,∗) contianing H.
2) If (G,∗) is a commutative group, then N(H)=G
3) The cardinal number of C(H) =the index of N(H) in G . **(H.W)**
4) If (G,∗) is finite group , then o(C(H))/(o(G)). **(H.W)**

Proof: (1) Since e∗ $H = H * e \Rightarrow e \in N(H) \neq \emptyset$

$$N(H) = \{x \in G \ni x * H = H * x\} \subseteq G$$

Let a,b ∈ $N(H), to\ prove\ a * b^{-1} \in N(H)$

(i.e) $(a * b^{-1}) * H = H * (a * b^{-1})$

Since a∈ $N(H) \Rightarrow a * H = H * a$

b∈ $N(H) \Rightarrow b * H = H * b$

$b * H * b^{-1}$=H

$H * b^{-1}$=$b^{-1} * H$

∴$b^{-1} \in N(H)$

$(a * b^{-1}) * H = a * (b^{-1} * H)$

$= a * (H * b^{-1})$ ($b^{-1} \in N(H)$)

$= (a * H) * b^{-1}$

$= (H * a) * b^{-1}$

$= H * (a * b^{-1})$

$\Rightarrow a * b^{-1} \in N(H) \Rightarrow (N(H),*)$ is a subgroup of (G,$*$) T.P $H \subseteq N(H)$

Let a$\in H \Rightarrow a * H = H \ \wedge H * a = H$

$\therefore a * H = H * a \Rightarrow$ a$\in N(H)$

$\therefore$H$\subseteq N(H)$

(2) Suppose G is comm. To prove N(H)=G i.e. N(H) $\subseteq G \wedge G \subseteq N(H)$.

By definition N(H) $\subseteq G$.

Let x$\in G \Rightarrow x * H = H * x \Rightarrow x \in N(H)$

$\therefore G \subseteq N(H)$, $\therefore$G=N(H)

Note : If N(H)=G, then (G,$*$) is comm. group ? **(H.W)**

Def(7). A subgroup (H,$*$) is called self-conjugate iff C(H)=H.

(i.e.) $x * H * x^{-1}$=H $\forall x \in G$.

Ex(12). In (S_3,o) ,$H_1=\{f_1,f_2,f_3\}$, $H_2=\{f_1,f_5,f_6\}$

$C(H_1)=H_1 \Rightarrow H_1$ is self-conjugate

$C(H_2)\neq H_2 \Rightarrow H_2$ is not self-conjugate.

Def(8). : A subgroup (H,$*$) is called normal subgroup of(G,$*$) and denoted by H$\Delta G \Leftrightarrow$H is self-conjugate

Or H$\Delta G \Leftrightarrow x * H * x^{-1}=H \ \forall x \in G$.

H$\not\Delta$G $\Leftrightarrow \exists x \in G \ \ni$ x*H*x$^{-1}\neq$H

Ex(13). (1) (G_s,o) , $H = \{r_1,r_4,v,h\}$

$C(H)=H \Rightarrow H \Delta G_s$

(2) $(S_3,o), \Rightarrow H_1=\{f_1,f_5\}$, $H_2=\{f_1,f_2,f_3\}$

$C(H_1)\neq H_1 \Leftrightarrow H \not\Delta S_3$, $C(H_2)=H_2 \Rightarrow H_2 \Delta S_3$

1) $(Z_4,+_4)$, $H=\{\bar{0},\bar{2}\}$

$C(H)=H \Rightarrow H \Delta Z_4$

Theorem(7): Let $(H,*)$ be a subgroup of $(G,*)$, then

1) $H \Delta G \Leftrightarrow x * H = H * x \quad \forall x \in G$
2) $H \Delta G \Leftrightarrow N(H)=G$
3) $H \Delta G \Leftrightarrow c(a) \subseteq H \quad \forall a \in H$
4) $H \Delta G \Leftrightarrow (x * H) * (y * H) = (x * y) * H \quad \forall x,y \in$ (منطوق فقط)

Proof: (1) $H \Delta G \Leftrightarrow x * H * x^{-1}=H \quad \forall x \in G$

$\Leftrightarrow x * H = H * x \quad \forall x \in G$

(2) ($\Rightarrow$) Suppose that $H \Delta G$ T.P. $N(H)=G$

T.P. $N(H) \subseteq G \quad \wedge \quad G \subseteq N(H)$

$N(H) \subseteq G$ by definition

T.P. $G \subseteq N(H)$

Let $x \in G \Rightarrow x * H = H * x \Rightarrow x \in N(H)$

$\therefore G \subseteq N(H)$

$\Rightarrow G=N(H)$

($\Leftarrow$) Suppose $N(H)=G$, T.P. $H \Delta G$

$\forall x \in G \Rightarrow x \in N(H) \Rightarrow \quad x * H = H * x \ \forall x \in G \Rightarrow H \Delta G$ (by(1))

(3) ($\Rightarrow$) Suppose $H \Delta G$. T.P. c(a) $\subseteq H \quad \forall a \in H$

Since $H \Delta G$ so by definition

$x * H * x^{-1} = H$

$x * H * x^{-1} \subseteq H$

$\therefore c(a) = \{x * a * x^{-1} \forall\ a \in H\ \} \subseteq H$

($\Leftarrow$) Suppose c(a) $\subseteq H \quad \forall a \in H$ T.P. $H \Delta G$ (i.e.) $x*H*x^{-1} = H$

T.P. $x*H*x^{-1} \subseteq H \wedge H \subseteq x * H * x^{-1}$

c(a) $\subseteq H \Rightarrow x * H * x^{-1} \subseteq H$ …..(1)

T.P. $H \subseteq x*H*x^{-1}$

Let $b \in H \Rightarrow b = e * b * e^{-1}$

$b = (x * x^{-1}) * b * (x * x^{-1})$

$= x * (x^{-1} * b * x) * x^{-1}$

$b = x * h * x^{-1} \in x * H * x^{-1} \qquad (h = x^{-1} * b * x)$

$\therefore H \subseteq x * H * x^{-1}$…….(2)

From (1) and (2) $\Rightarrow H = x * H * x^{-1} \quad \forall x \in G \quad \Rightarrow H \Delta G$

Theorem(8): Let $(G,*)$ be agroup , then

1) $\{e\} \Delta G$

2) $G \Delta G$

3) cent (G) ΔG .

Theorem(9): Every subgroup of a comm. group is a normal subgroup.

Proof: Let (G,$*$) be a comm. group and (H,$*$) be a subgroup of (G,$*$).

To prove $x * H * x^{-1}$=H $\quad \forall x \in G$

$x * H * x^{-1}=(x * x^{-1}) * H = e * H = H \quad \forall x \in$ G $\quad \therefore$H$\Delta\, G$

Note: The converse of this theorem is not true

For example:

(G=$[\pm 1, \pm i, \pm j, \pm k]$,.) $\ni$ $i^2=j^2=k^2=-1$

ij=K , ji= -K $\Rightarrow$ ij$\neq$ji

G is not a comm.

The subgroup of G are :{1} , G,$\{\pm 1\}$, {1,-1,i,-i}, {1,-1,j,-j},{1,-1,k,-k}and

All subgroup of G are normal of G, since $C(H_i) = H_i \ \ \forall i$

Theorem(10): Let (H,$*$) be a subgroup of (G,$*$) such that[G:H]=2 , then HΔG.

(منطوق فقط)

Note: The converse of this theorem is not true

For example : (G_s,o) , H=$\{r_1, r_4\}$

H Δ G_s but $[G_s:H] = 4 \neq 2$

Remark(3): If H Δ G, then H$\cap$ K$\not\Delta$ G $\wedge$ (H$*$K)$\not\Delta$ G

Where H and K are two subgroups of the group (G,$*$)

Consider (S_3,o)

$H=\{f_1\} \Delta S_3$, $K=\{f_1,f_4\} \not\Delta S_3$, then

$H*K=\{f_1,f_4\} \not\Delta S_3$

In (G_s,o) , $H=\{r_1,r_3,h,v\}$, $K=\{r_1,v\}$ s.t. $H\Delta G_s$, $K \not\Delta G_s$, then

$H\cap K=\{r_1,v\} \not\Delta G_s$.

Def(9). A group $(G,*)$ is called simple group iff G has no proper normal subgroup.

Ex(14).

1) (S_3,o) is not simple $H=\{f_1,f_2,f_3\} \Delta s_3$

2) (G_s,o) not simple since $H=\{r_2,r_4,v,h\}$ proper subgroup and $H\Delta G_s$.

3) $(Z_6,+_6)$ is not simple , $H=\{\bar{0},\bar{3}\} \Delta Z_6$

4) $(Z_3,+_3)$ is simple group since Z_3 has no proper normal subgroup. $\{0\} \Delta Z_3$ and $Z_3 \Delta Z_3$.

Def(10). Let $H\Delta G$ and $G\backslash H=\{x*H: x\in G\}$ define $\otimes$ on G/H as follows:

$$(x*H)\otimes(y*H)=(x*y)*H \quad \forall x,y\in G$$

$(G/H,\otimes)$ is called quotient group of G by H .

Theorem(10): Let $H\Delta G$, then $(G|H,\otimes)$ is a group .

Proof: $G|H=\{a*H: a\in G\}$

Since $e*H=H\in G|H\neq\varphi$

Closure: Let $a * H , b * H \in G|H$

$$(a * H)\otimes(b * H) = (a * b) * H \quad \forall x, y \in G|H$$

Asso. Let $a * H , b * H , c * H \in G|H$

$$[(a * H) \otimes (b * H)]\otimes (c * H) = [(a * b) * H]\otimes (c * H)$$

$$= ((a * b) * c) * H$$

$$= (a * (b * c)) * H$$

$$= (a * H)\otimes[(b * c) * H]$$

$$= (a * H)\otimes[(b * H) \otimes (c * H)]$$

Identity: $e * H = H \in G|H$

$$(a * H)\otimes(e * H) = (a * e) * H = a * H \quad \forall a * H \in G|H$$

$$(e * H)\otimes(a * H) = (e * a) * H = a*H$$

$\therefore$ e *H is an identity element of G|H.

Inverse: Let a* $H \in G|H$ $T.P.(a * H^{-1})=a^{-1}* H$

$(a * H) \otimes (a^{-1}* H) = (a * a^{-1}) * H = e * H = H$

$(a^{-1}* H) \otimes (a * H)=(a^{-1}* a) * H = e * H = H$

$\therefore \forall a * H \in G|H \quad \exists a^{-1}* H \in G|H.$

$\therefore$(G|H,$\otimes$) is a group.

Ex(15). (Z_6,$+_6$) , H={$\bar{0},\bar{3}$} , find Z_6|H **(if exist)**

$\because H\Delta\ Z_6 \Rightarrow Z_6|H$ exist

$\bar{0}+_6H=H$

$\bar{1}+_6H=\{\bar{1}, \bar{4}\}$

$\bar{2}+_6H=\{\bar{2}, \bar{5}\}$

$\bar{3}+_6H= \{ \bar{3},\bar{0}\} =H$

$\bar{4}+_6H=\{\bar{4},\bar{1}\}=\bar{1}+_6H$

$\bar{5}+_6H=\{\bar{5},\bar{2}\}=\bar{2}+_6H$

So, $Z_6|H=\{H,\bar{1}+_6H,\bar{2}+_6H \}$

$O(Z_6|H)=3$

$(Z_6|H,\otimes)$ is a quotient group.

$\otimes$	H	$\bar{1}+_6H$	$\bar{2}+_6H$
H	H	$\bar{1}+_6H$	$\bar{2}+_6H$
$\bar{1}+_6H$	$\bar{1}+_6H$	$\bar{2}+_6H$	H
$\bar{2}+_6H$	$\bar{2}+_6H$	H	$\bar{1}+_6H$

H is an identity .

$(\bar{1}+_6H)^{-1}=(\bar{1})^{-1}+_6H=\bar{5}+_6H= \bar{2}+_6H$

$(\bar{2}+_6H)^{-1}=(\bar{2})^{-1}+_6H=\bar{4}+_6H=\bar{1}+_6H$

Ex(16). 1) $(Z_{20},+_{20})$, $H=<\bar{5}>$

Find $Z_{20}|H$ (if exist) **(H.W)**

2) (S_3,o) ,$H=\{f_1,f_2,f_3\}$

Since $H\Delta S_3 \Rightarrow S_3|H$ (exist)

$f_1 o\ H=H$

$f_2 o\ H=\{f_2,f_3,f_1\} =H$

$f_3 o\ H= \{f_3,f_1,f_2\}=H$

f_4 o H={f_4,f_6,f_5}

f_5 o H={f_5,f_4,f_6}=f_4oH

f_6 oH= {f_6,f_5,f_4}=f_4oH

∴S_3|H={H,f_4o H}

But if H={f_1,f_4} , H⊿S_3

∴S_3|H is not exist

Theorem(11): The quotient group of comm. group is comm.

Proof: Suppose (G,∗) is a comm. group and (H,∗) is a subgroup of (G,∗) such that HΔG.

∴G|H is a group.

Let $a * H , b * H \in G|H$

$(a * H)\otimes(b * H) = (a * b) * H$

$= (b * a) * H$ (since G is comm.)

$= (b * H)\otimes(a * H)$

∴ (G|H,∗) is a comm. group.

Theorem(12): If (G,∗) is a cyclic group , then (G|H,∗) is a cyclic group. منطوق فقط

The converse of this theorem is not true.

For example: (S_3,0) , H={f_1,f_2,f_3}ΔS_3

$\therefore S_3|H$ is a group.

$S_3|H=\{H, f_4 oH\}$

$o(S_3|H)=2$ (prime order)

$S_3|H$ is a cyclic group but $(S_3,0)$ is not cyclic.

$S_3|H=<f_4 oH>=\{f_4 \text{ o } H, (f_4 oH)^2\}$

$=\{f_4 oH, f_1 oH\}$

Theorem(13): Let $(G,*)$ be a group and $(G\backslash cent(G),\otimes)$ is a cyclic group. Then $(G,*)$ is comm. (بدون برهان)

The converse of this theorem is not true.

For example:

$G=\{e,a,b,c\} \ni a^2=b^2=c^2=e$

G is comm. (not cyclic)

$Cent(G)=G \Rightarrow G\backslash cent(G)= (G\backslash G) =\{G\}=\{e,a,b,c\}$

$\therefore G\backslash cent(G)$ is not cyclic.

Chapter 4

Isomorphic groups

(الزمر المتشاكلة او تشاكل الزمر)

Definition(1). Let $(G,*)$ and $(G',\circ)$ be two groups and $f:(G,*)\to(G',\circ)$ be a mapping, then f is called a homomorphism iff

$$f(\chi*\gamma)=f(\chi)\circ f(\gamma),\ \forall\chi,\gamma\in G.$$

Example(1). Let $f:(R,+)\to(R^+,.)$, s.t. $f(a)=2^a,\forall a\in R$. Is f a homomorphism map.?

Ans. Let $a,b\in R\Rightarrow f(a+b)=2^{a+b}=2^a.2^b=f(a).f(b)$

$\therefore f$ is a homomorphism map.

Example(2). Let $f:(Z,+)\to(Z,+)$, s.t. $f(x)=3x+2,\forall x\in Z$. Is f a homomorphism map.?

Ans. Let $x,y\in Z\Rightarrow f(x+y)=3(x+y)+2=3x+3y+2$ ... ... (1)

$$f(x)+f(y)=(3x+2)+(3y+2)=3x+3y+4\ldots..(2)$$

$$\Rightarrow(1)\neq(2)$$

$$\Rightarrow f(x+y)\neq f(x)+f(y)$$

$\therefore f$ is not a homomorphism map.

Example(3). Let $g:(S_3,\circ)\to(S_3,\circ)$, s.t. $g(x)=x,\forall x\in S_3$. Is g a homomorphism map.? **(H.W.)**

Example(4). Let $f:(Z_6,+_6)\to(Z_6,+_6)$, s.t. $f(\bar{x})=\bar{x},\forall x\in Z_6$. Is f a homomorphism map.? **(H.W.)**

Example(5). Let $f:(\mathrm{Z},+)\to(\{1,-1\},.)$, s.t.

$f(a)=\begin{cases}1 & if\ a\ is\ even\\ -1 & if\ \ a\ is\ odd\end{cases}\},\forall a\in Z$. Is f a homomorphism map.?

Ans. Let $a,b\in Z\Rightarrow a,b\in E\ \ or\ \ a,b\in O\ \ or\ \ a\in E\wedge b\in O$

(1) If $a,b\in E$:

$f(a+b)=1\quad$ (since $a+b\in E$)

$f(a).f(b)=1.1=1$

(2) If $a,b\in O\ \Rightarrow a+b\in E$

$f(a+b)=1$

$f(a).f(b)=-1.-1=1$

(3) If $a\in E\wedge b\in O\Rightarrow a+b\in O$

$f(a+b)=-1$

$f(a).f(b)=1.-1=-1$

In all cases $f(a+b)=f(a).f(b)$

$\therefore f$ is a homomorphism map.

Example(6). Let $f:(\mathrm{G},*)\to(\mathrm{G},*)$, s.t. $f(a)=x*a*x^{-1},\forall a\in G$. Is f a homomorphism map.?

Ans. Let $a,b\in G\Rightarrow f(a*b)=x*(a*b)*x^{-1}$ ... ... (1)

$f(a)*f(b)=(x*a*x^{-1})*(x*b*x^{-1})$

$=x*a*(x^{-1}*x)*b*x^{-1}$

$=x*(a*b)*x^{-1}$ ... ... (2)

$\Rightarrow(1)=(2)$

$\therefore f$ is a homomorphism map.

Example(7). Let $f:(\mathrm{G},*)\to(\mathrm{G}',\circ)$, s.t. $f(a)=e',\forall a\in G$. Is f a homomorphism map.?

Ans. Let $a, b \in G \Rightarrow f(a * b) = e' \ldots\ldots(1)$

And $f(a) \circ f(b) = e' \circ e' = e' \ldots\ldots(2)$

Then f is a trivial homomorphism map.

Example(8). Let HΔG and $f : (G,*) \to (G/H, \otimes)$, s.t.

$f(a) = a * H, \forall a \in G$. Is f a homomorphism map.?

Ans. Let $a, b \in G \Rightarrow f(a * b) = (a * b) * H \ldots\ldots(1)$

And $f(a) \otimes f(b) = (a * H) \otimes (b * H) = (a * b) * H \ldots\ldots(2)$

$\Rightarrow (1) = (2)$

$\therefore f$ is a homomorphism map.

Definition(2). Let $(G,*)$ and $(G',\circ)$ be two groups and $f : (G,*) \to (G',\circ)$ be a mapping, then

(1) f is called a monomorphism (mono.) iff f is a homomorphism and (1-1) map.

(2) f is called an epimorphism (epi.) iff f is a homomorphism and (onto) map.

(3) f is called an isomorphism (iso.) iff f is a homomorphism, (1-1) and (onto) map.

Definition(3). Any two groups $(G,*)$ and $(G',\circ)$ are called isomorphic iff there exist an isomorphism map between them and denoted by $G \cong G'$.

(i.e.) $G \cong G' \Leftrightarrow \exists\, f : (G,*) \to (G',\circ)$ and f is an isomorphism.

Example(9). Let $(G = \{2^x : x \in Z\}, .)$, show that $(Z, +) \cong (G, .)$.

Ans. Define $f : (Z, +) \to (G, .) \quad s.t. \quad f(x) = 2^x, \forall x \in Z$

homo.? Let $x, y \in Z$, then

$$f(x + y) = 2^{x+y} = 2^x . 2^y = f(x) . f(y).$$

$\therefore f$ is homo.

(1-1) ? Let $f(x) = f(y)$ T.P. $x = y$

$f(x) = f(y)$

$2^x = 2^y \Longrightarrow x = y \Longrightarrow f$ is (1-1)

(onto)?

$R_f = \{f(x): x \in Z\} = \{2^x : x \in Z\} = G$

$\therefore f$ is (onto)

$\therefore f$ is an isomo.

$\therefore (Z, +) \cong (G, .)$.

Theorem(1). Let $f : (G, *) \to (G', .)$ be an isomorphism map. Then

(1) $f(e) = e'$, s.t. e is an identity of G and
e' is an identity of G′.

(2) $f(a^{-1}) = (f(a))^{-1}, \forall a \in G$.

(3) If $(H, *)$ is a subgroup of a group $(G, *)$, then $(f(H), .)$ is a subgroup of a group $(G', .)$.

(4) If $(K, .)$ is a subgroup of a group $(G', .)$, then $(f^{-1}(K), *)$ is a subgroup of a group $(G, *)$.

(5) If H Δ G and f is onto , then $f(H)$ Δ G′.

(6) If K Δ G′, then $f^{-1}(K)$ Δ G.

Proof.

(1) $f(e) = e'$

Let $a \in G \Rightarrow a * e = a$

$\Rightarrow f(a*e)=f(a)$ (f is map)

$\Rightarrow f(a).f(e)=f(a)$ (f is homo.).....(1)

Let $f(a)\in \mathrm{G}'$

$\Rightarrow f(a).e'=f(a)$ (def. of identity).....(2)

$(1)=(2)$, then

$f(a).f(e)=f(a).e'$

$\therefore f(e)=e'$ (by cancellation law).

... ...

(2) $f(a^{-1})=(f(a))^{-1}, \forall a\in G.$

Let $a\in G\Rightarrow a*a^{-1}=e$ (def. of inverse)

$\Rightarrow f(a*a^{-1})=f(e)=e'$

$\Rightarrow f(a).f(a^{-1})=f(e)=e'$ ... ... (1)

Let $f(a)\in \mathrm{G}'$

$\Rightarrow f(a).\big(f(a)\big)^{-1}=e'$ ... ... (2)

$(1)=(2)$, then

$f(a).f(a^{-1})=f(a).\big(f(a)\big)^{-1}$

$\therefore f(a^{-1})=(f(a))^{-1}$ (by cancellation law).

... ...

(3)If $(H,*)$ is a subgroup of a group $(\mathrm{G},*)$, then $(f(\mathrm{H}),.)$ is a subgroup of a group $(\mathrm{G}',.)$.

Proof. $f(H)=\{f(x): x\in H\}\subseteq \mathrm{G}'$

$f(e)=e'\in f(H)\Rightarrow f(H)\neq \varphi$

Let $f(x),f(y)\in f(H)$ T.P. $f(x).f(y)^{-1}\in f(H)$

$f(x).f(y)^{-1}=f(x).f(y^{-1})$ (by (2))

$= f(x * y^{-1})$ (f is homo.)

$since(H,*)$ is subgroup, then $x * y^{-1} \in H \Rightarrow f(x * y^{-1}) \in f(H)$

So, $f(x).f(y)^{-1} \in f(H)$

$\therefore$ $(f(\mathrm{H}),.)$ is a subgroup of a group $(\mathrm{G}',.)$.

… … … … … … … … … ..

(4)If $(K,.)$ is a subgroup of a group $(\mathrm{G}',.)$, then $(f^{-1}(\mathrm{K}),*)$ is a subgroup of a group $(\mathrm{G},*)$.

Proof. $f^{-1}(K) = \{x \in G: f(x) \in K\} \subseteq G$

Since $(K,.)$is a subgroup of a group $\mathrm{G}' \Rightarrow e' = f(e) \in K \Rightarrow e \in f^{-1}(K)$

So, $f^{-1}(K) \neq \varphi$

Let $x, y \in f^{-1}(\mathrm{K})$ T.P. $x * y^{-1} \in f^{-1}(K)$

$x \in f^{-1}(K) \Rightarrow f(x) \in K$

$y \in f^{-1}(K) \Rightarrow f(y) \in K$

Since $(K,.)$is a subgroup of a group G'

$\Rightarrow f(x).f(y)^{-1} \in K$

$\Rightarrow f(x).f(y^{-1}) \in K$

$\Rightarrow f(x * y^{-1}) \in K \Rightarrow x * y^{-1} \in f^{-1}(K)$

$\therefore$ $(f^{-1}(\mathrm{K}),*)$ is a subgroup of a group $(\mathrm{G},*)$.

… … … … … … … … … ..

(5)If $H \Delta \mathrm{G}$ and f is onto, then $f(\mathrm{H}) \Delta \mathrm{G}'$.

Proof. Suppose that $H \Delta \mathrm{G}$ and f is onto T.P. $f(H) \Delta \mathrm{G}'$

By (3), $(f(\mathrm{H}),.)$ is a subgroup of a group $(\mathrm{G}',.)$.

Let $y \in \mathrm{G}' \wedge a \in f(H)$ T.P. $y.a.y^{-1} \in f(H)$

$y \in \mathrm{G}'$ and f is onto, then $\exists x \in G \;\; s.t. \;\; f(x) = y$

$a \in f(\mathrm{H})$, then $a = f(h) \;\; s.t. \;\; h \in H.$

$y.a.y^{-1} = f(x).f(h).f(x)^{-1} = f(x).f(h).f(x^{-1}) = f(x * h * x^{-1})$

Since $H \;\Delta\; \mathrm{G}$, then $x * h * x^{-1} \in H$. It follows that $f(x * h * x^{-1}) \in f(H)$

$\therefore y.a.y^{-1} \in f(H) \Rightarrow f(\mathrm{H}) \;\Delta\; \mathrm{G}'.$

… … … … … … … … … ..

Theorem(2). The relation "isomorphic" is an equivalent relation.

Theorem(3).

(1) Every two finite cyclic groups of the same order are isomorphic.

(2) Every finite cyclic group is isomorphic to $(\mathrm{Z_n}, +_{\mathrm{n}})$.

(3) Every two infinite cyclic groups are isomorphic.

(4) Every infinite cyclic group is isomorphic to $(\mathrm{Z}, +)$.

Definition(4). Let $(\mathrm{G},*)$ be a group, then

(1) $Hom(G) = \{ f \mid f: (\mathrm{G},*) \to (\mathrm{G},*) \;\ni f \; is \; homo. \}$

(2) $Aut(G) = \{ f \mid f: (\mathrm{G},*) \to (\mathrm{G},*) \;\ni f \; is \; isomo. \}$

Theorem (4). Let $(\mathrm{G},*)$ be a group, then

(1) $(Hom(G),\circ)$ is a semigroup with identity.

(2) $(Aut(G),\circ)$ is a group **(H.W.)**

(3) $(Aut(G),\circ)$ is a subgroup of $(symm(G),\circ)$.

Proof. (1) $Hom(G) = \{ f \mid f: (\mathrm{G},*) \to (\mathrm{G},*) \;\ni f \; is \; homo. \}$

$\exists \;\; i: (\mathrm{G},*) \to (\mathrm{G},*) \;\; \ni \;\; i(x) = x \;\; \forall x \in G$, and i is homo.

$\therefore \; Hom(G) \neq \varphi.$

Closure: let $f, g \in Hom(G)$ T.P. $f \circ g \in Hom(G)$

Since $f:(\mathrm{G},*)\to(\mathrm{G},*)\ \ni f\ is\ homo.$ and

$g:(\mathrm{G},*)\to(\mathrm{G},*)\ \ni g\ is\ homo.$

$\therefore\ f\circ g:(\mathrm{G},*)\to(\mathrm{G},*)\ \ is\ homo.$

$\therefore f\circ g\in Hom(G)$

Asso. Is true since $(f\circ g)\circ h=f\circ(g\circ h)$

Identity: $\exists\ i\in Hom(G)$ and $f\circ i=i\circ f=f\ \ \forall f\in Hom(G)$

It follows that $(Hom(G),\circ)$ is semigroup with identity.

… … … … … … … … ..

(3) $T.P.(Aut(G),\circ)$ is a subgroup of $(symm(G),\circ)$.

$Aut(G)=\{f|\ f:(\mathrm{G},*)\to(\mathrm{G},*)\ \ni f\ is\ isomo.\}$

$Symm(G)=\{f|f:(\mathrm{G},*)\to(\mathrm{G},*)\ \ni f\ is\ bij.\}$

Since $\exists\ i:(\mathrm{G},*)\to(\mathrm{G},*)\ \ni\ i(x)=x\ \ \forall x\in G$, and i is homo.

$\therefore Aut(G)\neq\varphi\ ,\ Aut(G)\subseteq Symm(G)$ and by (2) $(Aut(G),\circ)$ is a group.

$\therefore (Aut(G),\circ)$ is a subgroup of $(symm(G),\circ)$.

Definition(5). Let $(\mathrm{G},*)$ be a group and $x\in\mathrm{G}$. Define

$f_x:(\mathrm{G},*)\to(\mathrm{G},*)\ \ni f_x(a)=x*a*x^{-1}\ \ \forall a\in G$. Then f_x is called an inner automorphism of G and the set

$Inn(\mathrm{G})=\{f_x:\forall x\in\mathrm{G}\}$ or $I(\mathrm{G})=\{f_x:\forall x\in\mathrm{G}\}$ (تشاكل تقابلي داخلي)

Theorem (5). Let $(\mathrm{G},*)$ be a group and $x\in\mathrm{G}$, then

(1) f_x is an isomorphism map.

(2) $(I(\mathrm{G}),\circ)$ is a subgroup of $(Aut(G),\circ)$

(3) $I(\mathrm{G})\ \Delta\ Aut(G)$.

Proof. (1)T.P. f_x is (1-1) , onto and homo.

Let $f_x(a) = f_x(b),\ \forall a, b \in G$, then

$x * a * x^{-1} = x * b * x^{-1}$

$\Rightarrow a = b$ $\quad\therefore$ f_x is (1-1)

$R_{f_x} = \{ f_x(a): \forall\ a \in G\} = \{x * a * x^{-1}: \forall\ a \in G\} = G$

$\therefore$ f_x is onto

$f_x(a) * f_x(b) = (x * a * x^{-1}) * (x * b * x^{-1})$

$= x * a * (x^{-1} * x) * b * x^{-1}$

$= x * a * b * x^{-1} = f_x(a * b)$

$\therefore$ f_x is homo. $\Rightarrow$ f_x is an isomo. Map.

… … … … … … … … ..

(2) $(I(\mathrm{G}),\circ)$ is a subgroup of $(Aut(G),\circ)$

Proof. T.P. $(I(\mathrm{G}),\circ)$ is a subgroup of $(Aut(G),\circ)$

$I(\mathrm{G}) = \{ f_x | f_x: (\mathrm{G},*) \to (\mathrm{G},*) \quad is\ an\ isomo. \}$

$Aut(G) = \{ f | f: (\mathrm{G},*) \to (\mathrm{G},*)\ \ni f\ is\ an\ isomo. \}$

Since $e \in \mathrm{G} \Rightarrow f_e \in I(G) \neq \varphi$

$f_e(a) = e * a * e^{-1} = a$

$\therefore I(\mathrm{G}) \subseteq Aut(\mathrm{G})$

Closure: Let f_x , $f_y \in I(\mathrm{G})$, T.P. $f_x \circ f_y \in I(\mathrm{G})$

$(f_x \circ f_y)(a) = f_x\left(f_y(a)\right) = f_x(y * a * y^{-1})$

$= x * (y * a * y^{-1}) * x^{-1}$

$$= (x * y) * a * (x * y)^{-1}$$

$$= f_{x*y}(a) \in I(\mathrm{G})$$

Inverse: Let $f_x \in I(\mathrm{G})$

Since $x^{-1} \in G \Rightarrow f_{x^{-1}} \in I(\mathrm{G})$

$f_x \circ f_{x^{-1}} = f_{x*x^{-1}} = f_e \Rightarrow f_{x^{-1}} \circ f_x = f_{x^{-1}*x} = f_e$

$\therefore (f_x)^{-1} = f_{x^{-1}}$ $\therefore (I(\mathrm{G}),\circ)$ is a subgroup of $(Aut(G),\circ)$.

… … … … … … … … ..

(3) $I(\mathrm{G}) \Delta Aut(G)$.

Proof. We have $(I(\mathrm{G}),\circ)$ is a subgroup of $(Aut(G),\circ)$ and

$Aut(G) = \{ g | g: (\mathrm{G},*) \to (\mathrm{G},*) \ \ni g \ is \ an \ isomo. \}$,

$I(\mathrm{G}) = \{ f_x | f_x: (\mathrm{G},*) \to (\mathrm{G},*) \quad is \ an \ isomo. \}$.

Let $g \in Aut(G)$, $f_x \in I(\mathrm{G})$ T.P. $g \circ f_x \circ g^{-1} \in I(\mathrm{G})$

$(g \circ f_x \circ g^{-1})(a) = g \circ f_x\left(g^{-1}(a)\right)$

$= g[\, f_x\left(g^{-1}(a)\right)]$

$= g(x * g^{-1}(a) * x^{-1})$

$= g(x) * a * g(x^{-1})$

$= f_{g(x)}(a) \in I(\mathrm{G})$

$\therefore I(\mathrm{G}) \Delta Aut(G)$

Definition(6). Let $f: (\mathrm{G},*) \to (\mathrm{G}',.)$ be a homomorphism, then the kernel of f is denoted by $kerf$ and defined as follows $kerf = \{x \in G | \ f(x) = e'\}$.

Example(9). Find $kerf$ for the following mappings:

(1) $f:(\mathrm{R},+)\to(\mathrm{R}^{+},.)\quad \ni f(x)=3^{x}$

Ans. f is homo. (Check)

$\Rightarrow kerf$ exist

$kerf=\{x\in R: f(x)=1\}=\{x\in R: 3^{x}=1\}=\{0\}.$

(2) $f:(\mathrm{G},*)\to(\mathrm{G}',.)\ \ni f$ is a trivial homo.

$f(x)=e'\quad \forall x\in G$

Since f is homo., then $kerf$ is exist

$kerf=\{x\in G\mid f(x)=e'\}=G$

(3) $f:(\mathrm{Z},+)\to(\mathrm{Z}_3,+_3)\quad \ni f(x)=[x]\ \forall x\in Z$

Ans. f is homo. (Check)

$kerf=\{x\in Z: f(x)=[0]\}=\{x\in Z: [x]=[0]\}$

$=\{x\in Z: x\equiv_3 0\}$

$=\{x\in Z: x=0+3k\ \forall k\in Z\}$

$=\{0,\bar{+}3,\bar{+}6,\ldots\}\subseteq Z$

Theorem (6). Let $f:(\mathrm{G},*)\to(\mathrm{G}',.)$ be a homomorphism, then

(1) $(kerf,*)$ is a subgroup of $(\mathrm{G},*)$

(2) $kerf\ \Delta\ \mathrm{G}$

(3) $kerf=\{e\}$ iff f is (1-1).

Proof. **(1)** $kerf=\{x\in G\mid f(x)=e'\}\subseteq G$

Since $f(e)=e'\Rightarrow e\in kerf\neq\varphi$

Let $a,b\in kerf$ T.P. $a*b^{-1}\in kerf$ T.P. $f(a*b^{-1})=e'$

$f(a * b^{-1}) = f(a).f(b^{-1})$

$= f(a).\big(f(b)\big)^{-1}$

$= e'.(e')^{-1} = e'$

$\therefore\ f(a * b^{-1}) = e' \Rightarrow a * b^{-1} \in kerf$

$\therefore$ $(kerf,*)$ is a subgroup of $(G,*)$.

(2) T. P. $kerf\ \Delta\ G$

By (1), $(kerf,*)$ is a subgroup of $(G,*)$

Let $x \in G$ and $a \in kerf$ T.P. $x * a * x^{-1} \in kerf$

(i.e.) T.P. $f(x * a * x^{-1}) = e'$

$f(x * a * x^{-1}) = f(x).f(a).f(x^{-1})$

$= f(x).e'.\big(f(x)\big)^{-1}$

$= e'$

$\therefore\ x * a * x^{-1} \in kerf \Rightarrow kerf\ \Delta\ G$

(3) $kerf = \{e\}$ iff f is (1-1).

($\Rightarrow$) Suppose $kerf = \{e\}$ T.P. f is (1-1)

Let $f(a) = f(b)$

$\Rightarrow f(a).(f(b))^{-1} = f(b).(f(b))^{-1}$

$\Rightarrow f(a).f(b^{-1}) = e'$

$\Rightarrow f(a * b^{-1}) = e'$ (since f is a homo.)

$\Rightarrow a * b^{-1} \in kerf$

Since $kerf = \{e\} \Rightarrow a * b^{-1} = e \Rightarrow a = b$

$\therefore$ f is (1-1)

($\Leftarrow$) Suppose f is (1-1) T.P. $kerf = \{e\}$

Let $a \in kerf$ T.P. $a = e$

Since f is (1-1) , then $f(a) = f(e) \Rightarrow a = e$

$\therefore$ $kerf = \{e\}$.

The first fundamental theorem of isomorphism:

(النظرية الأساسية الأولى للتشاكل)

Let f: (G,$*$) $\rightarrow$ (G$'$,.) be an onto and homomorphism mapping, then

(G/$kerf$,$\otimes$) $\cong$ (G$'$,.) .

Proof. Since f is onto $\Rightarrow R_f = \{ f(a): a \in G\} =$ G$'$

Since $kerf$ Δ G $\Rightarrow$ (G/$kerf$,$\otimes$) is a group.

Define g: (G/$kerf$,$\otimes$) $\rightarrow$ (G$'$,.), such that $g(a * kerf) = f(a)$, $\forall a \in G$

T.P. g is map., (1-1), onto and homo.

g is map.

Let $a * kerf = b * kerf \Rightarrow a^{-1} * b \in kerf$

$\Rightarrow f(a^{-1} * b) = e'$

$\Rightarrow f(a^{-1}).f(b) = e'$

$\Rightarrow (f(a))^{-1}.f(b) = e'$

$\Rightarrow f(b) = f(a)$

$\Rightarrow g(b * kerf) = g(a * kerf)$

$\therefore$ g is map.

g is 1-1 ?

Let $g(a * kerf) = g(b * kerf)$

$\Rightarrow f(a) = f(b)$

$\Rightarrow e' = (f(a))^{-1}.f(b)$

$\Rightarrow e' = f(a^{-1}).f(b)$

$\Rightarrow e' = f(a^{-1} * b)$

$\Rightarrow a^{-1} * b \in kerf$

$\Rightarrow a * kerf = b * kerf$

$\therefore$ g is 1-1

g is onto ?

$R_g = \{\ g(a * kerf) : a \in G\} = \{f(a) : a \in G\} = G'$

$\therefore$ g is onto

g is homo. ?

$$g[(a * kerf) \otimes (b * kerf)] = g[(a * b) * kerf]$$

$$= f(a * b)$$

$$= f(a).f(b)$$

$$= g(a * kerf).g(b * kerf)$$

$\therefore$ g is homo.

$\therefore$ g is isomo.

$\Rightarrow (\mathrm{G}/ker f, \otimes) \cong (\mathrm{G}', .)$.

... ...

Corollary: Let $(\mathrm{G},*)$ be a group, then $(\mathrm{G}/cent(G), \otimes) \cong (\mathrm{I}(\mathrm{G}),\circ)$.

The second theorem of isomorphism:

Let $(H,*)$ and $(K,*)$ be two subgroups of $(G,*)$ such that $K \Delta G$, $(H * K,*)$ is subgroup of $(G,*)$, $K \Delta (H * K)$ and $(H \cap K) \Delta H$. Then

$(H * K/K, \otimes) \cong (\mathrm{H}/H \cap K, \otimes)$.

Proof. Since $K \Delta (H * K)$, then $(H * K/K, \otimes)$ is a group.

And since $(H \cap K) \Delta H$, then $(\mathrm{H}/H \cap K, \otimes)$ is a group.

Define $f: (H * K,*) \to (\mathrm{H}/H \cap K, \otimes)$ such that

$f(a * b) = a * (H \cap K) \quad \forall a \in H, b \in K$

f is map.?

Let $a * b = c * d$ such that $a, c \in H, b, d \in K$

$\Rightarrow c^{-1} * a = d * b^{-1}$

$\Rightarrow c^{-1} * a \in H \ \wedge \ c^{-1} * a \in K$

$\Rightarrow c^{-1} * a \in H \cap K$

$\Rightarrow c * (H \cap K) = a * (H \cap K)$

$\Rightarrow f(c * d) = f(a * b)$

$\therefore$ f is map.

f is onto ?

$R_f = \{\ f(a*b): \forall a \in H\} = \{a*(H\cap K): \forall a \in H\} = \mathrm{H}/H\cap K$

$\therefore\ f$ is onto

f is homo. ?

$f[(a*b)*(c*d)] = f[(a*(c*c^{-1})*b)*(c*d)]$

$= f[(a*c)*(c^{-1}*b*c)*d]$

Since $c \in G\ \wedge\ b \in K\ \wedge\ K\Delta G$, then $(c^{-1}*b*c) \in K$

Let $(c^{-1}*b*c) = r \in K$

$\therefore f[(a*b)*(c*d)] = f[(a*c)*(r*d)]$

$= (a*c)*(H\cap K)$

$= [a*(H\cap K)]\otimes[c*(H\cap K)]$

$= f(a*b)\otimes f(c*d)$

$\therefore\ f$ is homo.

By the first theorem of isomo.

$\Rightarrow (H*K/kerf\ ,\otimes) \cong (\mathrm{H}/H\cap K\ ,\otimes)$.

$kerf = \{a*b \in H*K: f(a*b) = H\cap K\}$

$= \{a*b \in H*K: a*H\cap K = H\cap K\}$

$= \{a*b \in H*K: a \in H\cap K\}$

$= \{a*b \in H*K: a \in H\ \wedge a \in K\}$

$= \{a*b \in H*K: a \in k\ \wedge b \in K\} = K$

$\therefore \ (H * K/K, \otimes) \cong (\mathrm{H}/H \cap K, \otimes)$.

The third theorem of isomorphism:

Let $(H,*)$ and $(K,*)$ be two normal subgroups of $(G,*)$ such that $H \subseteq K$, then

(1) $H \Delta K$,

(2) $(\mathrm{K}/H, \otimes)\Delta(\mathrm{G}/H, \otimes)$

(3) $\left(\mathrm{G}/H \Big/ \mathrm{K}/H, \otimes\right) \cong (\mathrm{G}/K, \otimes)$.