Probability and Statistics

Chapter one : Descriptive Statistics

(١) **علم الاحصاء** : هو الطريقة العلمية التي تختص بجمع البيانات والحقائق عن ظاهرة أو فرضية (ظواهر أو فرضيات) معينة ، وتنظيم وترتيب وتبويب هذه البيانات والحقائق بالشكل الذي يسهل عملية تحليلها وتفسيرها ومن ثم استخلاص النتائج واتخاذ القرار على ضوء ذلك .

وبشكل عام ان هذا العلم يعتبر جمع لفرعين رئيسيين هما :

(أ) **الاحصاء الوصفي (Descriptive Statistics)**

وهو العلم الذي يتضمن الطرق والاساليب المستخدمة في جمع البيانات والمعلومات على ظاهرة معينة أو مجموعة من الظواهر ، وكيفية تنظيم وتصنيف وتبويب هذه البيانات مع امكانية عرضها في جداول ورسومات بيانية ، ومن ثم حساب بعض المؤشرات الاحصائية منها .

(ب) **الاحصاء الاستدلالي (الاستنتاجي) (Inferential Stat.)**

وهو العلم الذي يهتم بموضوعي التخمين (Estimation) واختبار الفرضيات (Testing of hypothesis) .

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(٢) **المجتمع (Population)** : هو جميع مفردات أو وحدات الظاهرة تحت البحث ، فقد يكون المجتمع مكون من مجموعة من الناس أو مجموعة من المنازل في منطقة معينة أو وحدات سلع معينة ينتجها معمل معين وهكذا .

والمجتمعات على انواع ، مجتمعات محددة (finite) تضم عدداً محدوداً من القيم مثل مجموعة النخيل في بستان معين أو مجتمعات غير محدودة تضم عدد غير منتهي مثل عدد النجوم في السماء .

وقد يكون المجتمع متماثل اي ان كل مفردة من مفرداته تحمل نفس الصفة مثل فصيلة الدم لدى الانسان أو غير متماثل مثل الاطوال والاوزان .

وقد يكون المجتمع صغيراً أو كبيراً ، معتمداً على حجم المجتمع ، يرمز لحجم المجتمع بالرمز (N) .

(٣) **العينة (Sample)** : هي جزء من المجتمع يجري اختيارها وفق قواعد خاصة لكي تمثل المجتمع تمثيلاً صحيحاً .

العينة العشوائية Random Sample

هي مجموعة وحدات احصائية تختار من المجتمع الاحصائي على ضوء اسس خاصة . فتؤخذ من كل طبقات المجتمع وبنسب معينة بحيث تمثل المجتمع تمثيلاً صحيحاً ، اي يجب ان تكون عينة متماثلة . ومن فوائد اختيار العينة انها تمثل المجتمع فتقلل بذلك الوقت والجهد والامكانيات المادية .
يرمز لحجم العينة ب (n) .

البيانات (Data) هي المعلومات والارقام والاعداد التي تجمع عن ظاهرة معينة قيد الدراسة مثل اطوال الطلبة للمرحلة الثالثة في قسم الرياضيات في جامعة بغداد .

تبويب البيانات هي عملية ترتيب وتكثيف البيانات الخام التي جمعت عن ظاهرة معينة قيد الدراسة بطرق معينة ، لكي تكون هذه البيانات واضحة وسهلة الفهم والوصف لاستخلاص النتائج منها .

البيانات على نوعين أ- بيانات مبوبة Group Data
ب- بيانات غير مبوبة Ungroup Data

البيانات غير المبوبة (Ungroup Data)

أ- مقاييس النزعة المركزية (Measure of Central Tendency)
ب- مقاييس التشتت والاختلاف (Measurement of Variations)

أ- مقاييس النزعة المركزية :

هناك بعض القيم تلتف حولها مفردات المجتمع تسمى هذه القيم (بنقاط الوسط) أو (نقاط التمركز) .. وتدعى هذه النقاط بمقاييس النزعة المركزية ومن اهم هذه المقاييس : الوسط الحسابي ، الوسط الهندسي ، الوسط الموزون ، الوسط التوافقي ، الوسط فوق الهندسي ، الوسيط ، المنوال .

① الوسط الحسابي (المعدل) (Average) : (Arithmetic Mean)

الوسط الحسابي للعينة : $\bar{X}$

لتكن n حجم العينة (عدد مفردات العينة)
X_1 و X_2 و ... و X_n مفردات العينة

$$\bar{X} = \frac{\text{مجموع المفردات}}{\text{عددها}}$$

$$\bar{X} = \frac{\sum_{i=1}^{n} x_i}{n}$$

(N.B) $\bar{X}$ متغير من عينة الى اخرى بتغير مفردات العينة .

الوسط الحسابي للمجتمع μ

لتكن $X_1, X_2, \dots, X_N$ مفردات المجتمع وان N حجم المجتمع (عدد مفردات المجتمع) ويعتبر μ ثابت (معلمة) لان عدد مفردات المجتمع ثابت لايتغير.

$$\mu = \frac{\text{مجموع المفردات}}{\text{عددها}}$$

$$\mu = \frac{\sum_{i=1}^{N} x_i}{N}$$

ان الوسط الحسابي $\bar{X}$ أو μ عندما تكون اوزان المجتمع متساوية. اما اذا كانت الاوزان غير متساوية فنستخدم (الوسط الحسابي الموزون)

(الوسط الموزون) : (Weighted Mean)

عندما تكون الاوزان لمفردات المجتمع غير متساوية ، نستخدم كل مفردة مع وزنها (w = (weight) ، ويدعى هذا المقياس (بالوسط الموزون أو المرجح).

$$\bar{X}_w = \frac{\sum_{i=1}^{n} w_i \, x_i}{\sum_{i=1}^{n} w_i}$$

x_i مفردات المجتمع

w_i اوزان المجتمع

$i = 1, 2, \dots, n$

مثال

البيانات الآتية تمثل الدرجات التي يحصل عليها أحد الطلبة وعدد الساعات الاسبوعية لكل مادة ، ويطلب ايجاد الوسط الحسابي أو معدل درجات هذا الطالب :

الدرجة	80	85	70	78	90	96	65
الساعات الاسبوعية	3	2	4	3	2	3	2

الساعات الاسبوعية تمثل الاوزان ، لذلك نستخدم الوسط الحسابي الموزون :

$$\bar{X}_w = \frac{\sum_{i=1}^{7} w_i x_i}{\sum_{i=1}^{7} w_i} = \frac{3(80) + 2(85) + \dots + 2(65)}{3 + 2 + \dots + 2} = 79.95$$

مميزات الوسط الحسابي (Mean)

١- سهولة حسابه.

٢- يدخل في حسابه جميع مفردات المجتمع التي تمثله.

٣- يتأثر بوجود القيم المتطرفة سواء كانت صغيرة أم كبيرة ، وهذا عيب فيه.

٤- مجموع انحرافات القيم عن الوسط الحسابي يساوي صفر.

$$\sum_{i=1}^{n} (x_i - \bar{X}) = \text{zero} \qquad \longleftarrow \text{(Prove that).}$$

② الوسيط (Median)

هو القيمة التي تحتل المرتبة الوسطى بعد ترتيب القيم قيد الدرس (البيانات) ترتيباً تصاعدياً أو تنازلياً.

أمثلة

① وسيط مجموعة الأرقام 10، 10، 8 و 8، 6 و 5 و 4 و 4 و 3 هو 6 -

② وسيط مجموعة الأرقام 18، 15، 12، 11 و 9 و 7 و 5 و 5 هو $\frac{9+11}{2} = 10$

وبصورة عامة: ان مرتبة الوسيط (R) هي:

$$R = \frac{n+1}{2}$$

وان الوسيط هو القيمة المتوسطة (اذا كان عدد المفردات n فردياً)

وان الوسيط هو متوسط القيمتين الوسيطتين (اذا كان عدد المفردات n زوجياً)

مثال ① اذا كانت البيانات الآتية تمثل الاطوال بالسنتمتر، جد الوسيط لها.

110 و 106 و 113 و 116 و 102 و 114 و 111

الحل: نرتب القيم تصاعدياً فنحصل على:

102 و 106 و 110 و 111 و 113 و 114 و 116 (عدد البيانات فردي) $n = 7$

$R =$ رتبة الوسيط $= \frac{n+1}{2} = \frac{7+1}{2} = 4$

∴ الوسيط $(M) = 111$ (المرتبة الرابعة)

مثال ② جد وسيط البيانات الآتية:

9 و 8 و 7 و 12 و 10 و 13 و 15 و 17

الحل: نرتب المفردات تصاعدياً:

7 و 8 و 9 و 10 و 12 و 13 و 15 و 17 (عدد البيانات زوجي) $n = 8$

$R =$ رتبة الوسيط $= \frac{n+1}{2} = \frac{8+1}{2} = 4.5$

∴ الوسيط $(M) =$ معدل القيمتين الوسيطتين (المرتبة الرابعة والخامسة)

∴ $M = \frac{10+12}{2} = 11$

③ المنوال (Mode)

هو القيمة التي تتكرر أكثر من غيرها أو هو القيمة الأكثر شيوعاً وتكراراً بين مجموعة قيم الظاهرة قيد الدرس.

وهناك منوال رئيسي ومنوال ثانوي.

وهناك مجموعة من القيم ذات منوال واحد وأخرى ذات منوالين أو قد تكون عديمة المنوال (ليس لها منوال).

امثلة

① مجموعة القيم: 18 و 12 و 11 و 10 و 10 و 9 و 9 و 9 و 7 و 2 و 2 و 5 .
لها منوال واحد هو (9)

$\therefore Mo = 9$

② مجموعة القيم: 16 و 15 و 12 و 10 و 8 و 5 و 3 .
ليس لها منوال .

③ مجموعة القيم: 9 و 7 و 7 و 7 و 5 و 5 و 5 و 5 و 4 و 4 و 3 و 2
لها منوالان هما 4 و 7 .

ما العلاقة بين الوسط والوسيط والمنوال؟ Ⓜ

أ - في حالة التوزيعات المتماثلة (المنتظمة): كالتوزيع الطبيعي ، فان المقاييس الثلاثة تكون متساوية .

i.e. $Me = Mo = \mu$

ب - في حالة التوزيعات غير المتماثلة (غير المنتظمة): فان المنوال والوسيط هما الاكثر تمثيلاً للتمركز وان:

$$\text{الوسط الحسابي} = \frac{\text{٣ (الوسيط)} - \text{المنوال}}{\text{٢}}$$

i.e. $$\bar{X} = \frac{3Me - Mo}{2}$$

ب - مقاييس التشتت أو الاختلاف

وهي مقاييس تقيس مدى تشتت القيم المختلفة في ظاهرة طبيعية عن الوسط الحسابي أو مقاييس النزعة المركزية ، ومن هذه المقاييس :
المدى ، التباين ، الانحراف المعياري ، الخطأ المعياري ، معامل الاختلاف ، الانحراف الربيعي .

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① المدى : (Range) وهو احد مقاييس التشتت و يمثل الفرق بين القيمة العليا والقيمة الدنيا لمجموعة المفردات .

ومن عيوب هذا المقياس انه لا يهتم سوى بمفردتين فقط هي القيمة العليا والدنيا ونتيجة لذلك لا يهتم بالاختلافات الموجودة بين المفردات الباقية ، فهو لا يهتم بحجم العينة فلا يمكن استخدامه كمقياس للمقارنة بين العينات عند اختلاف حجم العينة . ومن ناحية اخرى قد تكون القيم المتطرفة هي القيم العليا أو الدنيا وهذا عيب كبير وبذلك لا يعتبر هذا المقياس كفوء كلما زاد عدد مفردات العينة .

المدى (Range) :

هو الفرق بين اكبر قيمة واصغر قيمة من قيم الظاهرة (المفردات) .

$$R = L - U$$

اكبر قيمة ↗ اقل قيمة ↗

امثلة

① جد مدى البيانات الآتية :

45 ، 50 ، 55 ، 60 و 65

$$R = L - U = 65 - 45 = 20$$

② مدى البيانات الآتية (الاوزان) :

45 و 100 ، 45 ، 55 و 95 و 50 كغم

$R = 45 - 100 = 55$ كغم

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③ التباين والانحراف المعياري (Variance & Standard Deviation)

ان مقاييس التشتت (المدى) تقيس مدى اقتراب أو ابتعاد القيم للعينة (المفردات) عن نقاط التمركز (الوسط الحسابي وغيره) .. ولحل مشكلة الاختلافات الموجودة في مفردات العينة أو المجتمع ، نستخدم مقاييس الوسط الحسابي لكن المشكلة في ان مجموع انحرافات القيم عن الوسط الحسابي = صفر) ، ولغرض التخلص من القيم السالبة في الانحرافات ، طرأت فكرة تربيع هذه الانحرافات ليدل على مدى الاختلافات ولاجل التخلص من مشكلة اخرى هي حجم المجتمع أو العينة ، تم القسمة على عدد مربعات انحرافات القيم . فنكوّن مقياس جديد يدعى التباين (Variance) الذي رمزه (S^2) ويقرأ (سكماسكوير) للعينة اما للمجتمع فان رمزه (σ^2) :

صيغة تباين المجتمع :

$$\sigma^2 = \sum_{i=1}^{N} \frac{(x_i - \mu)^2}{N - 1}$$

اما صيغة تباين العينة :

$$S^2 = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{n - 1}$$

وان استخدام ($n-1$) بدلاً من (n) الذي هو حجم العينة يعود الى اعتبارات رياضية تتعلق بالتقديرات غير المتحيزة وتسمى ($n-1$) بدرجات الحرية (degree of freedom) .

اما الانحراف المعياري (Standard deviation) : هو الجذر التربيعي للتباين

$\sigma = +\sqrt{\sigma^2}$ الانحراف للمجتمع

$S = +\sqrt{S^2}$ الانحراف للعينة

مثال) جد التباين للقيم الآتية التي تمثل وزن اللحم الصافي لستة رؤوس من الغنم .. تمثل مجتمعاً قائماً بذاته :

35 و 34 و 40 و 38 و 37 ، 32 (كغم)

sol.

الوسط الحسابي للمجتمع $\mu = \frac{35+34+40+38+37+32}{6} = \frac{216}{6} = 36$ كغم

(تباين المجتمع) ويعني تباين كل قيمة عن الوسط الحسابي μ للمجتمع .

$$\sigma^2 = \sum_{i=1}^{6}(x_i-36)^2/6-1 = \frac{42}{5} = 8.4 \text{(كغم}^2\text{)}$$

(نفس المثال اعلاه لكن للعينة)

الوسط الحسابي للعينة $\bar{X} = \frac{35+34+\cdots+32}{6} = 36$ كغم

$$S^2 = \sum_{i=1}^{6}(x_i-36)^2/6$$ (تباين العينة)

$$= \frac{42}{5} = 8.4$$

البيانات المبوبة (Group Data)

١- مقاييس النزعة المركزية ٢- مقاييس التشتت والاختلاف .

١- مقاييس النزعة المركزية :

① الوسط الحسابي :

$$\bar{X} = \frac{\sum_{i=1}^{k} x_i f_i}{\sum_{i=1}^{k} f_i}$$

x_i = يمثل مركز الفئة i

f_i = يمثل تكرار الفئة i

K = يمثل عدد الفئات .

② الوسيط : لايجاد الوسيط نتبع الخطوات الآتية :

① ترتيب الفئات تصاعدياً ، ثم ايجاد التكرار المتجمع الصاعد.

② ايجاد رتبة الوسيط بالصيغة :

$$R = \frac{\sum f_i + 1}{2} = \frac{n+1}{2}$$

③ تحديد الفئة الوسيطية من خلال مقارنة رتبة الوسيط وموقعها في أول تكرار متجمع صاعد.

④ نستخدم الصيغة التالية في ايجاد الوسيط :

$$Me = Le + \frac{R - G_1}{G_2} \times We$$

حيث ان :

L_e = الحد الادنى للفئة الوسيطية

W_e = طول الفئة الوسيطية

G_1 = التكرار المتجمع الصاعد للفئة قبل الوسيطية

G_2 = تكرار الفئة الوسيطية .

③ المنوال : لايجاد المنوال يجب تحديد الفئة المنوالية اولاً وهي الفئة التي تقابل اكبر تكرار ونستخدم الطريقة الآتية :

$$M_o = L_o + \frac{D_1}{D_1 + D_2} \times W_o$$

حيث ان :

L_o : الحد الادنى للفئة المنوالية

D_1 : الفرق بين تكرار الفئة المنوالية وتكرار الفئة قبل المنوالية .

D_2 : الفرق بين تكرار الفئة المنوالية وتكرار الفئة بعد المنوالية .

W_o : طول الفئة المنوالية .

ب - مقاييس التشتت والاختلاف :

① المدى :

$$R = X_L - X_s$$

حيث ان :

X_L = الحد الاعلى للفئة الاخيرة

X_s = الحد الادنى للفئة الاولى

② الانحراف المعياري والتباين :

$Var(X) = S^2 =$ التباين

$S = \sqrt{S^2} =$ الانحراف المعياري

ليكن :

f_i = التكرار المقابل لمركز الفئة x_i

x_i = مركز الفئة = $\left(\frac{\text{الحد الاعلى للفئة + الحد الادنى للفئة}}{2}\right)$

K = عدد الفئات

$$\text{التباين}\ S^2 = \frac{\sum_{i=1}^{k} f_i x_i^2}{\sum_{i=1}^{k} f_i - 1} - \frac{\left(\sum_{i=1}^{k} f_i x_i\right)^2}{\sum f_i\left(\sum f_i - 1\right)}$$

H.W. جد الوسط والوسيط والمنوال للبيانات الآتية :

Classes	frequency (fi)	xi (Mid point)	xifi	c.f. (التكرار المتجمع الصاعد)
25 - 35	2			
35 - 45	5			
45 - 55	12	?	?	?
55 - 65	13			
65 - 75	20			
75 - 85	15			
85 - 95	8			
95 - 1.5	5			

مثال جد الوسط (Mean) والوسيط (Median) والمنوال (Mode)
والتباين (Variance) والانحراف المعياري (S.d.) والمدى (Range)
لاطوال النباتات الآتية :

Classes	f_i	x_i	$x_i f_i$	c.f.
2 – 4	3	3	9	3
4 – 6	5	5	25	8
6 – 8	2	7	14	10

يساوي دائماً $n = \Sigma f_i = 10$

$$n = \Sigma f_i = 10 \ , \ \Sigma x_i = 15 , \ \Sigma x_i f_i = 48$$

$$\bar{X} = \frac{\Sigma f_i x_i}{\Sigma f_i} \simeq \bar{X}_w \qquad (\text{Mean})$$

$$Me = Le + \frac{R - G_1}{G_2} \times we \qquad (\text{Median})$$

$$= 4 + \frac{5.5 - 3}{5} \times 2 = 5 \in (4-6)$$ الفئة الوسيطية

$$Mo = Lo + \frac{D_1}{D_1 + D_2} \times W_o \qquad (\text{Mode})$$

$$= 4 + \frac{(5-3)}{(5-3)+(5-2)} \times (2)$$

$$= 4 + \frac{(2)(2)}{2+3} = 4 + \frac{4}{5} = \frac{24}{5} = 4.8$$

$$R = X_L - X_s = 8 - 2 = 6 \qquad (\text{Range})$$

$$S_x^2 = \frac{\Sigma x_i^2 f_i}{\Sigma f_i - 1} - \frac{(\Sigma x_i f_i)^2}{\Sigma f_i(\Sigma f_i - 1)} \qquad (\text{Variance})$$

نكمل عن طريق الجدول ،

x_i	x_i^2	$x_i f_i$	$x_i^2 f_i$	f_i
3	9	9	27	3
5	25	25	125	5
7	49	14	98	2

$\Sigma f_i = 10$
$\Sigma f_i - 1 = 9$
$\Sigma x_i f_i = 48$
$\Sigma x_i^2 f_i = 205$
$(\Sigma x_i f_i)^2 = (48)^2 = 2304$

الارتباط والانحدار

Correlation and Regression

الارتباط ((Correlation))

هو العلاقة التي تربط بين ظاهرتين أو أكثر ، وهو على انواع منها ؛ الارتباط البسيط ، الارتباط المتعدد ، الارتباط الجزئي .

الارتباط البسيط (Simple Correlation)

لمعرفة العلاقة بين ظاهرتين x و y ونوع هذه العلاقة ، نستخدم مؤشراً احصائياً يدعى (معامل الارتباط البسيط) (Simple Correlation Coefficient) حيث ان y يمثل المتغير المعتمد (التابع) و x هو المتغير المستقل . ويرمز لهذا المعامل بالرمز (r) حيث

$$-1 \leq r \leq +1$$

تعني الاشارة السالبة ان العلاقة بين الظاهرتين عكسية وضعيفة .
اما الاشارة الموجبة ان العلاقة بين الظاهرتين طردية وقوية .
والصيغة لهذا المعامل ؛

$$r_{x,y} = \frac{n \sum xy - \sum x \sum y}{\sqrt{[n \sum x^2 - (\sum x)^2][n \sum y^2 - (\sum y)^2}}$$

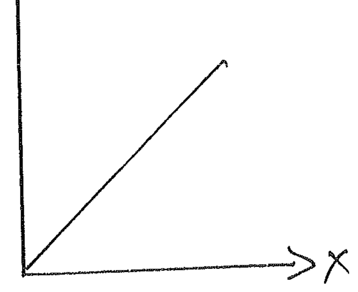
حيث ان n تمثل حجم العينة = عدد البيانات في كل عمود (عدد بيانات x = عدد بيانات y)

If $r_{x,y} \in (0,1] \longrightarrow$ معامل ارتباط قوي

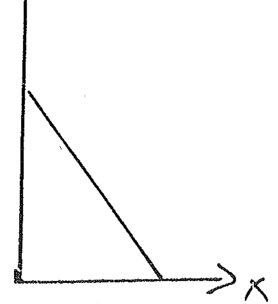
If $r_{x,y} \in [-1,0) \longrightarrow$ معامل ارتباط ضعيف

If $r_{x,y} = 1 \longrightarrow$ معامل ارتباط تام

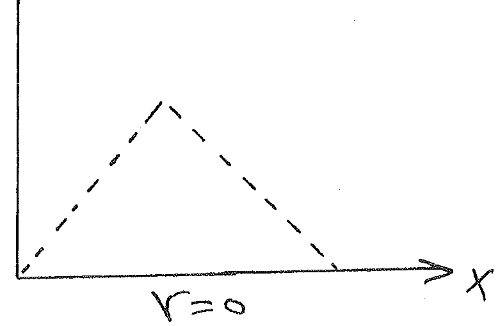
If $r_{x,y} = 0 \longrightarrow$ لايوجد ارتباط بين x و y



معامل ارتباط طردي



معامل ارتباط عكسي



$r = 0$

لاتوجد علاقة بين x و y

Note : $r_{x,y} = r_{y,x}$

اما معامل الارتباط البسيط للمجتمع :

$$\rho_{x,y} = \frac{N\sum XY - (\sum X)(\sum Y)}{\sqrt{N\sum X^2 - (\sum X)^2][N\sum Y^2 - (\sum Y)^2}}$$

مثال : صممت تجربة لدراسة تأثير زيارة جرعة منوم معين في وقت النوم ، وتم تكوين ثلاث قراءات لكل من المستويات الثلاثة للجرعة ، والمطلوب معرفة اذا كانت هناك علاقة بين الجرعة ووقت النوم :

Y^2	X^2	XY	وقت النوم (X)	الجرعة (Y)
9	16	12	4	3
9	36	18	6	3
9	25	15	5	3
100	81	90	9	10
100	64	80	8	10
100	49	70	7	10
225	169	195	13	15
225	121	165	11	15
225	81	135	9	15
1002	642	780	72	84

$$r_{x,y} = \frac{9(780) - (72)(84)}{\sqrt{[9(642)-(72)^2][9(1002)-(84)^2]}} = \frac{972}{1079.55}$$

$$= 0.9 \in (0,1]$$

∴ العلاقة قوية وطردية بين الجرعة ووقت النوم

ملاحظة: اذا كانت كلا الظاهرتين غير قابلة للقياس ، نستطيع ايجاد العلاقة بين الظاهرتين باستخدام (معامل الارتباط لسبيرمان) ، وذلك باعطاء رتب الى القيم أو البيانات الغير قابلة للقياس ثم نستخدم الصيغة الاتية .

$$r = 1 - \frac{6\sum D_i^2}{N(N^2-1)}$$ (معامل سبيرمان للرتب)

D_i : تمثل فرق الرتب لكل زوج

N : تمثل عدد القيم الغير قابلة للقياس (الغير كمية) .

مثال جد العلاقة بين درجة الطالبين A و B من البيانات الآتية:

A	B	$D_i (A-B)$	D_i^2
جيد (4)	مقبول (2)	2	4
مقبول (2)	جيد جداً (5)	−3	9
ضعيف (1)	مقبول (2)	−1	1
جيد (4)	متوسط (3)	1	1
جيد جداً (5)	جيد (4)	1	1
جيد (4)	متوسط (3)	1	1

* نرتب اولاً تصاعدياً ثم نضع الرتب

ضعيف (1)
مقبول (2)
متوسط (3)
جيد (4)
جيد جداً (5)

$N = 6$ (عدد البيانات)

$\sum D_i^2 = 17$

$$r_{A,B} = 1 - \frac{6.\sum D_i^2}{N(N^2-1)} = 1 - \frac{6.(17)}{6(6^2-1)}$$

$= 0.51 \in (0,1]$ (العلاقة قوية وطردية)

مثال آخر (H.W.)

A	B	D_i	D_i^2
متوسط (3)	جيد جداً (5)	−2	4
ضعيف (1)	متوسط (3)	−2	4
جيد (4)	مقبول (2)	2	4
متوسط (3)	جيد جداً (5)	−2	4
جيد جداً (5)	متوسط (3)	2	4

$N = 5$ و $\sum D_i^2 = 20$

$$r_{A,B} = 1 - \frac{6(20)}{5(5^2-1)} = ?$$

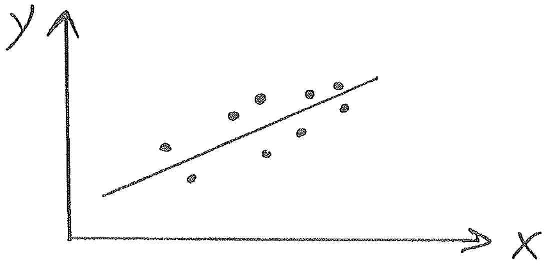
* ما نوع العلاقة بين A و B ؟

Regression الانحدار

شكل الانتشار :(Scatter diagram)

هو كل زوج من ازواج القيم التي تمثل متغيرات ما بين محورين متعامدين وعلى شكل نقط (points) :

وهذا الشكل في اغلب الاحيان يمثل علاقة دالية وذات اتجاه مستقيم (دالة خطية)



خط الانحدار (Linear Regression)

هو الخط الذي يمر بأكبر عدد ممكن من النقاط في شكل الانتشار .

اما معادلة خط الانحدار فهي :

$$\boxed{y = \alpha + \beta x + e}$$

حيث ان :

α : هي المسافة بين نقطة تقاطع خط الانحدار مع المحور y ونقطة الاصل .

β : هي ميل خط الانحدار أو ظل الزاوية التي يصنعها خط الانحدار مع المحور x .

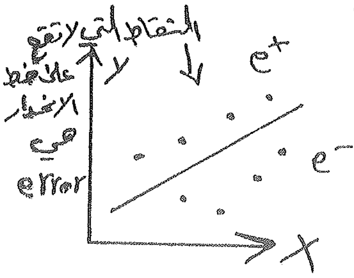
$$\underline{i.e.} \quad \beta = \tan\theta \Rightarrow \theta = \tan^{-1}\beta$$

e : هو الخطأ الذي يمثل الفرق بين القيمة الحقيقية والتقديرية لـ y

$$\underline{i.e.} \quad e_i = y_i - \hat{y}_i$$ (تقرأ ^ ب hat)

$$\text{if } e_i = 0 \Rightarrow y_i = \hat{y}_i$$

اما المعادلة التقديرية المستخدمة لحساب خط الانحدار هي :



$$\boxed{\hat{y} = \hat{\alpha} + \hat{\beta} x}$$

حيث ان :

$\hat{y}$: القيمة التقديرية لـ y .

$\hat{\beta}$: القيمة التقديرية لـ β .

$\hat{\alpha}$: القيمة التقديرية لـ α .

نحسب الدالة $\hat{y}$ بحيث يكون الخطأ e اقل ما يمكن ، ويمكن حساب قيمة $\hat{\alpha}$ و $\hat{\beta}$ باستخدام طريقة المربعات الصغرى في الحساب (Least Square) .

$\hat{\beta}$ تسمى (معامل الانحدار) حيث ان ($\hat{\beta} = 0$) يعني لا توجد علاقة بين x و y واذا كانت $\hat{\beta}$ قيمة سالبة يعني ان العلاقة عكسية بين x و y واذا كانت $\hat{\beta}$ قيمة موجبة يعني ان العلاقة بين x و y طردية .

ليكن $\bar{X}$ = mean of x & $\bar{Y}$ = mean of y فان :

لايجاد $\hat{\alpha}$ و $\hat{\beta}$ باستخدام طريقة المربعات الصغرى ، حيث ان $r_{x,y}$ يمثل (معامل الارتباط البسيط) وان δ هو الانحراف المعياري :

x | y	y | x
$\hat{\beta} = \frac{n\sum xy - \sum x \sum y}{}$ $= \frac{\delta_x}{\delta_y} \cdot r_{x,y}$ $\hat{\alpha} = \bar{x} - \hat{\beta}\,\bar{y}$	$\hat{\beta} = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}$ $= \frac{\delta_y}{\delta_x} \cdot r_{x,y}$ $\hat{\alpha} = \bar{y} - \hat{\beta}\,\bar{x}$

ملاحظة : ان مفهوم الانحدار الخطي على صلة وثيقة بمفهوم الارتباط الخطي ، حيث ان الانحدار يقدر العلاقة الخطية بين المتغيرات اما الارتباط الخطي فهو يصف العلاقة بين المتغيرات (يصف العلاقة نفسها) .

مثال (١) اجريت احدى التجارب لدراسة تأثير عقار معين في تخفيض دقات القلب لغير البالغين ، حيث ان المتغير المستقل (X) يمثل الجرعة بالملغم من العقار والمتغير المعتمد (y) يمثل الانخفاض في دقات القلب (دقة / الدقيقة) .

① ارسم شكل الانتشار ② جد معادلة خط الانحدار التقديرية .

③ جد مقدار الانخفاضات في دقات القلب عندما تكون الجرعة (٣) ملغم .

X	y	x^2	X y	② لايجاد معادلة خط الانحدار التقديرية :
0.5	10	0.25	5	$\hat{\beta} = \frac{(8)(152.5) - 11(102)}{8(17.75) - (11)^2}$ $\hat{\beta} = 4.67$ $\hat{\alpha} = \frac{102}{8} - 4.67(\frac{11}{8})$ $= 6.33$ $\hat{y} = 6.33 + 4.67X$
0.75	8	0.5625	6	
1.00	12	1	12	
1.25	12	1.5625	15	
1.5	14	2.25	21	
1.75	12	3.0625	21	
2.00	14	4	32	
2.25	18	5.0625	40.5	
11	102	17.75	152.5	عندما (y | X)

② لايجاد $y(3)$ ،
i.e. $X = 3$

$$\hat{y}(3) = 6.33 + 4.7(3)$$
$$= 20.34$$

Chapter one
Descriptive Statistics

Ungrouped Data (البيانات غير المبوبة)

① جد الوسيط للارقام التالية :

3 ، 4 ، 4 ، 5 ، 6 ، 8 ، 8 ، 8 ، 10   وسيطها هو (6) .

10 ، 9 ، 8 ، 11 ، 12 ، 16 ، 9   نرتب الاعداد تصاعدياً (تنازلياً) :

16 ، 12 ، 11 ، 10 و 9 و 9 و 8

الوسيط هو (١٠) الذي يحتل المرتبة الرابعة .

10 ، 14 ، 7 ، 5 ، 20 ، 7 ، 22 ، 12   لايجاد الوسيط نرتب الاعداد تصاعدياً :

22 ، 20 ، 14 ، 12 ، 10 ، 7 ، 7 ، 5

هناك مرتبتان وسيطيتان هما المرتبة الرابعة والخامسة (عدد القيم زوجي) وهي بالطريقة التالية :

$$Me = (12+10)/2 = 11$$

الوسيط

---

② درجات طالب في ستة امتحانات كانت 78 ، 87 ، 68 ، 72 ، 91 ، 84 أوجد وسيط هذه الدرجات .

68 ، 72 ، 78 ، 84 ، 87 ، 91

عدد الدرجات زوجي فان هناك قيمتين في الوسط 78 ، 84 ووسطهما الحسابي هو $(84+78)/2 = 81$ وهو الوسيط .

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② درجات طالب في ستة امتحانات هي 78 ، 87 ، 68 ، 72 ، 91 ، 84 . أوجد الوسط الحسابي لهذه الدرجات .

$$\bar{X} = \frac{\sum_{i=1}^{6} x_i}{6} = \frac{84+91+72+68+87+78}{6} = \frac{480}{6} = 80$$

---

④ من مائة رقم (20) أربعة و (40) خمسة و (30) ستة والباقي كانوا سبعات . أوجد الوسط الحسابي لهذه الارقام .

$$\bar{X} = \frac{\sum w_i x_i}{\sum w_i}$$

$$= \frac{(20)(4)+(40)(5)+(30)(6)+(10)(7)}{100} = \frac{530}{100} = 5.30$$

(٥) رتبت الارقام 85 رقماً و 150 رقماً ، ما وسيطهما ؟

85 رقم فردي .. وهناك قيمة وسيطية واحدة هي (43) ، حيث قبلها (42) رقماً وبعدها (42) رقماً .. وترتيب الوسيط الثالث والاربعين.

150 رقم زوجي .. وهناك قيمتين في الوسط حيث يتواجد قبلهما (74) رقماً وبعدهما (74) رقماً وترتيبهما (75) و(76) ووسطهما الحسابي هو الوسيط :

$$(76+75)/2 = Me$$

---

(٦) اذا كانت درجات طالب في الرياضة والطبيعية واللغة الانكليزية والصحة العامة هي على الترتيب 82 ، 86 ، 90 و 70 وان عدد الساعات (ساعات المحاضرات) الاسبوعية لهذه المقررات هي 3 و 5 و 3 و 1 . أوجد متوسط الدرجات.

$$\bar{X} = \sum w_i x_i / \sum w_i$$

$$= \frac{(3)(82)+(5)(86)+(3)(90)+(1)(70)}{3+5+3+1} = 85$$

---

(٧) اوجد الوسط والمنوال والوسيط للارقام التالية:

3, 5, 2, 6, 5, 9, 5, 2, 8, 6

نرتب تصاعدياً :

$$X: 2, 2, 3, 5, \underline{5, 5}, 6, 6, 8, 9 \qquad (n=10)$$

$$\mu = Mean = \sum x_i/10 = 5.1$$

$$Me = Median = (5+5)/2 = 5$$

$$Mo = Mode = 5$$

وللارقام التالية :

50.3 , 49.5 , 48.9 , 51.6 , 48.7

نرتب تصاعدياً (أو تنازلياً) :

$$487, 48.9, \underline{49.5}, 50.3, 51.6$$

$$\mu = \sum x_i/5 = 49.8$$

$$Me = 49.5$$

$Mo =$ لايوجد

---

(٨) حصل طالب على درجات 96 و 82 و 93 و 76 و 85 في (5) مواد . جد الوسط والوسيط والمنوال.

$$\bar{X} = \sum x_i/5 = 86 \;, \quad R = \frac{n+1}{2} = \frac{5+1}{2} = 3$$ (الوسيط في المرتبة الثالثة)

نرتبهم ← $Me = 85$ ، $Mo =$ لايوجد منوال

(البيانات المبوبة) Grouped Data

① في تجربة لتحديد تأثير عقار معين لمستوى الكوليسترول في الدم مقاساً بـ (Mg/100Ml) لأشخاص اعمارهم (30) سنة . سُجلت البيانات التالية لمجموعة عولجت بهذا الدواء والمطلوب معرفة المنوال لمستوى الكولسترول في الدم .

مستوى الكوسترول (الفئات)		f_i عدد المرضى (التكرار)	
120 − 160		2	
$\boxed{160 - 200}$	← الفئة المنوالية	$\boxed{5}$	← اعلى تكرار
200 − 240		3	
		$n = \sum_{i=1}^{3} f_i = 10$	

$D_1 = 5 - 2 = 3$

$D_2 = 5 - 3 = 2$

$w_0 = 200 - 160 = 40$

$L_0 = 160$

$$M_0 = L_0 + \frac{D_1}{D_1 + D_2} * w_0$$

$$= 160 + \frac{3}{3+2}(40) = 160 + \frac{120}{5} = 160 + 24 = 184$$

---

② في دراسة لتحديد تأثير التدخين على الارجين تم قياس هذه المادة في البلازما (Mg/Ml) وبعد ساعتين من تناول الدواء من قبل (16) من المدخنين تم الحصول على البيانات التالية والمطلوب ايجاد معدل مستوى هذه المادة (الوسيط) .

Classes / مستوى الارجين (الفئات)	عدد المدخنين (التكرارات) f_i	C.f.d. التكرار المتجمع الصاعد
0.505 − 1.005	4	4
1.005 − 1.505	3	7
الفئة الوسيطة $\boxed{1.505 - 2.000}$	6	13
2.000 − 2.505	3	16

$n = \sum f_i = 16$

$R = \frac{n+1}{2} = \frac{16+1}{2} = \frac{17}{2} = 8.5$ رتبة الوسيط

$We = 2.000 - 1.505 = 0.495$, $Le = 1.505$,

$$Me = Le + \frac{R - G_1}{G_2}(We) = 1.505 + \frac{8.5 - 7}{3}(0.495) = ?$$

⑨ جد الوسط والوسيط والمنوال لمجموعة الارقام والاعداد التالية :

5 ، 4 ، 8 ، 3 ، 7 ، 2 ، 9

$\mu_x = \Sigma x_i / 7 = 5.4$

$2 , 3 , 4 , 5 , 7 , 8 , 9$ نرتب الارقام

رتبة الوسيط $R = \frac{n+1}{2} = \frac{7+1}{2} = 4$ الرتبة الرابعة

$Me = 5$ ، $Mo =$ لايوجد منوال

---

⑩ جد المنوال للارقام التالية :

$4 , 7 , 7 , 7 , 9 , 9 , 10 , 12 , 15$

$Mo = 7$

$1 , 2 , 3 , 4 , 4 , 4 , 5 , 5 , 5 , 6 , 6 , 6 , 7 , 8 , 8 , 8 , 10 , 10 , 10 , 11 , 12$

هناك (5) مناويل هي

$Mo = 4 , 5 , 6 , 8 , 10$

منوال الرئيسي $Mo = 10$ و $M_1 = 4 , M_2 = 5 , M_3 = 6 , M_4 = 8$ (مناويل ثانوية)

---

⑪ في شركة بها (80) عاملاً .. (60) يحصلون على 3.0 $ في الساعة وان (20) يحصلون على 2.0 $ في الساعة . أوجد متوسط دخولهم في الساعة .

$$\bar{X} = \frac{\Sigma w_i x_i}{\Sigma w_i} = \frac{(60)(3.0) + (20)(2.0)}{30 + 20} = \frac{220.0}{50} = \text{[illegible]}$$

---

⑫ جد منوال الارقام التالية :

$2 , 2 , 5 , 7 , 9 , 9 , 9 , 10 , 10 , 11 , 12 , 18$

$Mo = 9$

$3 , 5 , 8 , 10 , 12 , 15 , 16 , 20$

$Mo =$ لايوجد منوال (لايوجد تكرار)

$2 , 3 , 4 , 4 , 4 , 5 , 5 , 7 , 7 , 7 , 9 , 10$

$Mo = 7$ منوال رئيسي

$M_1 = 4$ منوال ثانوي

H.W بيّن من العينتين X و Y أفضل للدراسة وفق البيانات التالية :

$\bar{X} = 145$ ، $\bar{Y} = 80$, $S_x = 10$, $S_y = 10$ (استخدم معامل الاختلاف C.V.) .

② الجدول التالي يمثل أوزان خمسين طفل مصابين بمرض فقر الدم والمطلوب ايجاد الانحراف المعياري والتباين والمدى.

وزن الطفل classes	التكرار (f_i) عدد الاطفال	وسط الفئة (x_i)	x_i^2	x_if_i	$x_i^2f_i$
12-14	20	13	169	260	3380
14-16	18	15	225	270	4050
16-18	12	17	289	204	3468

$n = \sum_{i=1}^{3} f_i = 50$, $\sum x_if_i = 734$, $\sum x_i^2 f_i = 10898$

$$Range = R = 18 - 12 = 6$$

$$\delta = \sqrt{\frac{10898}{50-1} - \frac{(743)^2}{50(50-1)}} = 1.58$$

$$\delta^2 = Var(x) = (1.58)^2 \simeq 2.5$$

(H.W) اذا كانت اوزان رؤوس الغنم في قطيع معين موزعة على النحو التالي لعينة مأخوذ من هذا القطيع. احسب قيمة كل من الوسط والوسيط والمنوال والتباين والانحراف المعياري لوزن الرأس الواحد.

الوزن (كغم)	(التكرارات) عدد رؤوس الغنم
20-25	20
25-30	60
30-35	10
35-40	6

Ungrouped Data

① جد تباين القيم التالية التي تمثل وزن اللحم الصافي لستة رؤوس من الغنم وتمثل مجتمعاً قائماً بذاته :

(كغم) 32 , 37 , 38 , 40 , 34 , 35

$$\delta_x^2 = \sum_{i=1}^{6} (x_i - \mu_x)^2 / N$$

$$\mu_x = \frac{35+34+40+38+37+32}{6} = 36$$

$$\delta^2 = \sum_{i=1}^{6} (x_i - 36)^2 / 6-1 = \frac{42}{6-1} = \underline{7} (\text{كغم}^2)$$

اي ان تباين كل قيمة عن الوسط الحسابي للمجتمع μ_x هو (7) كغم^2 .

نفس السؤال اعلاه لكن للعينة :

$$\bar{X} = \frac{35+34+\cdots+32}{6} = 36 \text{ كغم}$$

$$S^2 = \sum_{i=1}^{n} (x_i - \bar{X})^2 / n-1 = \sum_{i=1}^{6} (x_i - 36)^2 / 5$$

$$= \frac{42}{5} = \underline{8.4} (\text{كغم}^2)$$

∴ تباين العينة هو تقدير جيد لتباين المجتمع .

---

② جد الوسط والوسيط والمنوال للارقام المكونة من ست ستات و سبع سبعات و ثماني ثمانينات و تسع تسعات وعشر عشرات .

$$\text{Mean} = \mu = \frac{6(6)+7(7)+8(8)+9(9)+10(10)}{6+7+8+9+10} = ?$$

$$n = \sum w_i = 10+9+8+7+6 = \underset{\text{عدد زوجي}}{\underline{40}}$$

$$R = \frac{n+1}{2} \quad , \quad n = \sum w_i$$

$$R = \frac{40+1}{2} = 20.5$$

اي المرتبة 20 و 21

والوسيط هو

$$Me = (8+8)/2 = 8$$

ويمكن ايجاد الوسيط في هذا السؤال بطريقة اخرى كالاتي:

$$666666 \quad 7777777 \quad 888888\underset{\text{الوسيط}}{\underline{88}} \quad 99999999$$

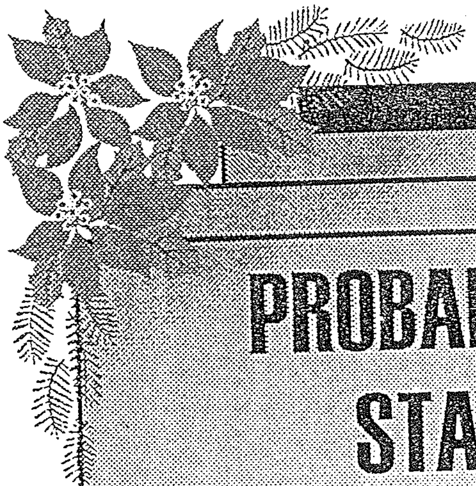
$$10\,10\,10\,10\,10\,10\,10\,10\,10\,10 .$$

المنوال (Mo) : القيمة التي تكرر اكثر من غيرها ولان اكثر قيمة تكررت هي (10)

$$\therefore Mo = 10$$

معامل الارتباط البسيط Simple Correlation Coefficient

$$r_{x,y} = \frac{n\sum xy - \sum x \sum y}{\sqrt{[n\sum x^2 - (\sum x)^2][n\sum y^2 - (\sum y)^2]}}$$



PROBABILITY AND STATISTICS

الاحتمالية والاحصاء

الفصل الثاني

الأستاذ : عباس نجم الجبوري

Probability and Statistics

الاحتماليـــــة والاحصــــاء

Chapter Two : Introduction to Probability

<u>Def :-</u> A random (Statistical) experiment is an experiment with :-

1. All out comes (results) of the experiment are known in advance
2. Any performance of the experiment results in an out comes is not known in advance .
3. The experimant can be repeated under the identical conditions .

ويمكن تعريف التجربة العشوائية بأنها تلك التجربة التي ينتج عنها مجموعة من الاحـــداث كل حدث منها مستقل عن الاخر وان وقوع ذلك الحدث يرجع الى عامل الصدفة وحـــده (لا يمكن التنبؤ بحدوثه)

ملاحظة: ستطرق الى جملة من التجارب كأمثلة تساعدنا في تفسير بعض المفاهيم مثل :

1. Tossing a coin	تجربة رمي العملة
2. Rolling adice	تجربة رمي الزار
3.Playing cards	تجربة ورق اللعب

وهكذا

Def: (Sample Space)

A sample space of an experiment is a set of all possible outcomes denoted by (S)

فضاء العينة : هو كل النتائج المحتملة من تجربة عشوائية معينة.

Def: (Events)

Any events is a (proper) subset of a sample space

الحوادث: هي مجموعه جزئيه من(S) ويكون الحادث بسيطاً اذا تكون من عنصر واحد فقط او مركب اذا تكون من اكثر من عنصر ومستحيلاً اذا لم يحوي على اي عنصر واكيداً اذا احتوى على عناصر(S) جميعا

ie: If A is an Event, then $A \subseteq S$

ex: Toss a coin once

sol. S={ H,T }

S has (2) elts since acoin has two faces

Let A: to get H

A=$\{H\} \subseteq S$

Let B: to get T

B= $\{T\} \subseteq S$

$\therefore$ A and B are Events

ex: Roll adice once

a dice has (6) faces

each face has adots

$S = \{1,2,3,4,5,6\}$

$= \{d : 1 \le d \le 6\}$

A: To get one odd no.

$A = \{1,3,5\} \subseteq S$

A is an Event

B: To get one even no.

$B = \{2,4,6\} \subseteq S$

$\therefore$ B is an event

C: $d \le 3$

$C = \{1,2,3\} \subseteq S$

$\therefore$ C is an Event

ملاحظة : سيكون فضاء العينة في هذا الفصل من النوع المنتهي والقابل للعد .

Def:- Empty set (Φ) الحادثة التي لا تحدث

Φ is an Impossible event

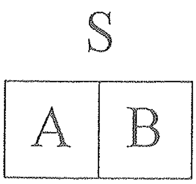
ex Toss a dice once

Let A: to get 7

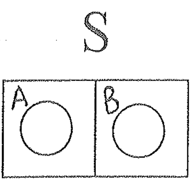
$\therefore A = \Phi$

Def :- (Disjoint events) الحوادث المنفصلة

If A and B are events, then A and B are disjoint iff $A \cap B = \Phi$



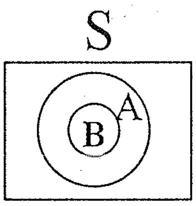
$AB = \Phi$



$AB = \Phi$

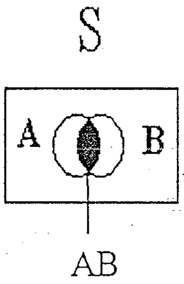
Def. (Joint events) الحوادث المتصلة

If A and B are event, then A and B are Joint iff $A \cap B \neq \Phi$



$B \subset A$

$AB = B$

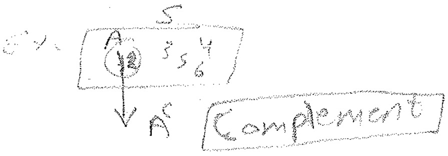


Note :- We shall use the symbol (AB) to denote of $A \cap B$

Def :- If A and B are events , then $(A \cup B)$ & $(A \cap B)$, $(A - B)$, (B/A) $A^C \cdots etc$ are also events .

$A^c = SlA = S - A$

$= \{ x ; x \notin A \}$



ex. Roll a dice once .

Let

$$A = \{d : d \geq 2\} = \{2,3,4,5,6\}$$
$$B = \{d : d \leq 3\} = \{1,2,3\}$$
$$C = \{d : d \leq 1\} = \{1\}$$

Find $A \cup B$, $A \cap B$, $A \cup C, \cdots$

A^c, B^c, AB^c, BA^c, ...

Ex.

$AB = \{ x ; x \in A \wedge x \in B \} \subseteq S$

$A \cup B = \{ x ; x \in A \vee x \in B \} \subseteq S$

$A - B = \{ x ; x \in A \wedge x \notin B \} \subseteq S$

$B - A = \{ x ; x \in B \wedge x \notin A \} \subseteq S$

$A \cup B - AB$

$A \cup B = \{1,2,3,4,5,6\}$

$A \cap B = \{2,3\}$

$A^c \cup C = \{1,2,3,4,5,6\}$

$A^c = \{1\}$

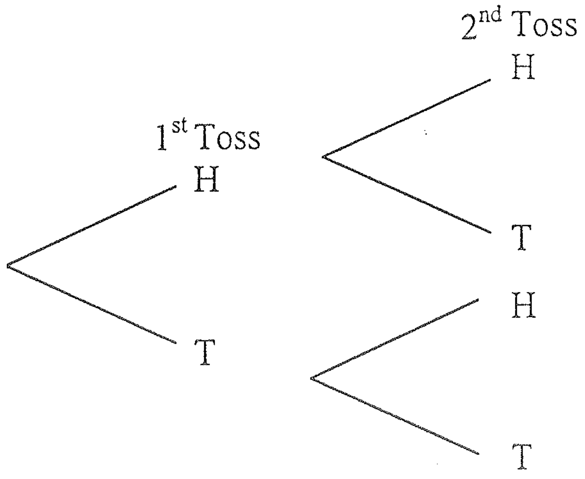
$B^c = \{4,5,6\}$

$AB^c = \{4,5,6\}$

$BA^c = \{1\}$

Toss a coin twice (2- time)

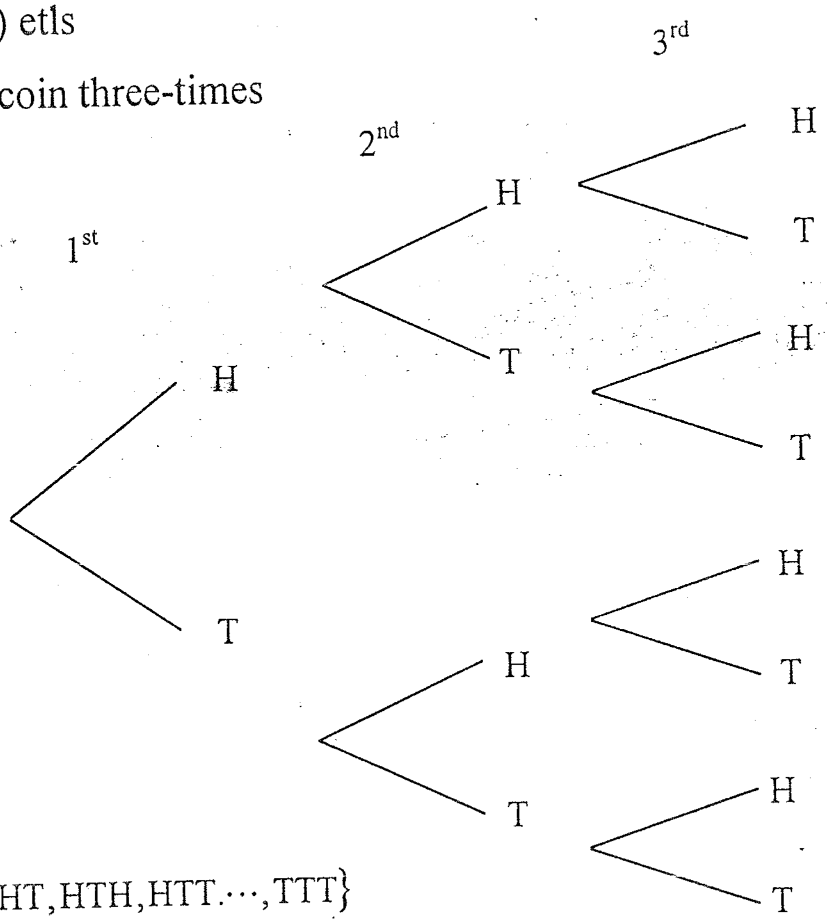
When toss a coin twice (2-time)



$S = \{HH, HT, TH, TT\}$

S has ($2^2 = 4$) etls

When toss a coin three-times



$S = \{HHH, HHT, HTH, HTT.\cdots, TTT\}$

$\therefore$ S has (2^3=8) elts

when toss a coin n-times , then the sample space (S) has 2^n elts

ex. Roll a dice twice (2-times)

$S = \{(d_1, d_2); 1 \le d_1 \le 6; 1 \le d_2 \le 6\}$

S= has (6^2=36) elts

When roll a dice n-times , then the sample space (S) has (6^n) elts

Simple Probability الاحتمالية البسيطة

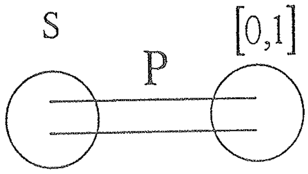
Def:- If A be an events , P(A)= probability of event A = Pr.(A) that is mean (pr) that event A happence . If S has (n) elts & A has (m) elts , then

$$p(A) = \frac{\text{no. of elts of event A}}{\text{no. of elts of S}} = \frac{|A|}{|S|} = \frac{m}{n}$$

ملاحظة : عند احتساب الاحتمال الرياضي لاي حادثة يجب ان تتوفر الحالة التي تكون فيها الاحداث مستبعدة لبعضها الاخر وان كل حدث ياخذ الفرصة التي تاخذها الاحداث الاخرى في الوقوع .

ويعرف احتمال الحصول على صفة معينة (حادثة معينة) مثل (A) من تجربة عشوائية معينة بأنه عدد مرات حدوث الصفة (A) مقسوما على الحالات المتوقعة .

ملاحظة : نجد ان قيمة الاحتمال هي عبارة عن دالة (تطبيق) P مجالها (منطلقها) S ومداها $R_{[0,1]}$



$P: S \rightarrow R_{[0,1]}$

تسمى دالة الاحتمال P دالة احتمال منتظم اذا اعطى نفس القيمة الاحتمالية لكل عنصر من عناصر فضاء العينة .

ملاحظة: نجد ان قيمه الاحتمال تعتمد بالدرجة الاساس على معرفة كل الحالات الممكنة لـ(S) وان هذه الحالات يمكن حصرها بسهوله في الحالات البسيطة ولكن عند زيادة

عدد الاحداث يؤدي الى وجود صعوبة في تحديد عدد الحوادث الممكنة ولذلك لابد من اللجوء الى بعض الطرق الرياضية التي تساعد في تحديد مثل هذه الحالات ومهما زاد عددها واهم هذه الطرق

1. permutation التباديل

وهي عملية ترتيب n من الاشياء في مجاميع كل منها يتالف من r من الاشياء وحسب القاعدة التالية

$$P_r^n = P_{n,r} = \frac{n!}{(n-r)!}$$

$$P_r^n = P_{n,r} = \frac{n!}{(n-r)!}, n, r \in I^+$$

مثال/ جد عدد الطرق الممكنة لترتيب اربع كرات مرقمة من 1-4

$$P_4^4 = \frac{4!}{(4-4)!} = \frac{4!}{0!} = 4 \times 3 \times 2 \times 1 = 24 \text{ or } 4! = 4 \times 3 \times 2 \times 1 = 24$$

مثال / جد عدد الطرق الممكنة التي يمكن وضع خمس كرات في صندوقين بحيث ان كل صندوق يحتوي على كرة واحدة فقط

$$P_2^5 = \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{5 \times 4 \times 3!}{3!} = 20 \text{ samples}$$

في حالة وجود مكررات في مفردات المجموعة فان عدد الطرق الممكنة للترتيب يمكن حسابها بالطريقة التالية :

$$P_{n1,n2,\cdots nk}^n = \frac{n!}{n_1!n_2!,\cdots n_k!}$$

where $n = n_1 + n_2 + \cdots + n_k$

مثال / جد عدد الطرق الممكنة لترتيب سبع كرات اربعة منها بيضاء واثنان حمراء والباقي الوان اخرى .

$$P^{7}_{4,2,1} = \frac{7!}{4!2!1!} = 105 \text{ Methods}$$

2. Combination التوافيق

هي عملية اختيار او انتخاب (Selection) عدد من المفردات بحجم r من مجموعة كبيرة بحجم n وبدون ترتيب وتستخدم الصيغة التالية :

$$\begin{pmatrix} n \\ r \end{pmatrix} = \left(\begin{matrix} n \\ r \end{matrix}\right. = \frac{n!}{r!(n-r)!}, n, r \in I^{+}$$

مثال / ما هو عدد العينات التي يمكن تكوينها من مجتمع مؤلف من ست مفردات بحيث يكون حجم العينة مفردتين اثنين فقط .

$$\begin{pmatrix} 6 \\ 2 \end{pmatrix} = \frac{6!}{2!(6-2)!} = \frac{6!}{2!4!} = \frac{6 \times 5 \times 4!}{2!4!} = 15 \text{ samples.}$$

مثال / ما هو عدد اللجان التي يمكن تأليفها من اربعة افراد بحيث ان كل لجنة تحتوي على أ. فردين اثنين ب. ثلاثة افراد

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \frac{4!}{2!(4-2)!} = \left(\frac{4!}{2!2!}\right) = \frac{24}{2} = 6$$

$$\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \frac{4!}{3!(4-3)!} = \frac{4!}{3!} = 4$$

<u>ملاحظة :</u> هناك علاقة بين التباديل والتوافيق وهي :

$$\binom{n}{r} = \overset{\mathrm{P}^{\,n}}{\frac{r}{r!}},$$

Binomial theorm نظرية ذات الحدين

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^{r}$$

binomial coefficients معادلات ذات الحدين

$$= a^{n} + n\, a^{n-1} b + \frac{n(n-1)}{1-2} a^{n-2} b^{2} + \cdots + b^{n}$$

ex/ Prove that

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

Sol. R.S

Facts about the sets حقائق حول المجموعات

1. $A \cap \Phi = \Phi, A \cup \Phi = A, \left(\overset{c}{A}\right)^c = A$

2. $$\left.\begin{array}{l} A^c \cup B^c = (A \cap B)^c \\ A^c \cap B^c = (A \cup B)^c \end{array}\right\} Demo.Law$$

3. $\Phi^c = S, S^c = \Phi$

4. $A \cup A^c = S, A \cap A^c = \Phi$

5. $A_1 \cap A_2^c = A_1 - A_2$ A_1 happence but A_2 not happence

6. $A_1 \cup A_2$ A_1 or A_2 happence

7. $(A_1 \cup A_2) - A_1 \cap A_2$ A_1or A_2 happ , but not both

8. $A_1 \cap A_2$ Both A , and A_2 happence

Axioms of Probability :بديهيات الاحتمالية

1. If $A \subseteq S$, then $0 \leq P(A) \leq 1$

2. $P(S) = 1$

3. If $A_1 A_2, \cdots A_n, \cdots$ are sequence of disjoint events , then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n \cdots) = P(A_1) + P(A_2) + \cdots + P(A_n) \cdots$$

ie. $P\left(\bigcup_{i=1}^{\infty} A\right) = \sum_{i=1}^{\infty} P(Ai)$

Note:- Special case of Ax.3

If A and B are disjoint events , then $P(A \cup B) = P(A) + P(B)$

ex. Toss a coin 3-times

a. Find the Pr. to get 2-H

b. Find the Pr. to get no-H

S has 8 elts

Sol. $S = \{HHH, HHT, \cdots, TTT\}$

a. let A to get 2-H

$A = \{HHT, HTH, THH\}$ A has 3 elts

$$P(A) = \frac{3}{8} \in [0,1]$$

b. let B to get no-H

$B = \{TTT\}$ $P(B) = \frac{1}{8} \in [0,1]$

A^c : to get less (2-H) or (3-H) = { S − A ; A = (2-H) }

$$= P(A^c) = 1 - P(A) \Rightarrow P(A^c) = 1 - \frac{3}{8} = \frac{5}{8} \in [0,1]$$

H.W ex/ toss a ~~die~~ twice a: Find the pr. That sum. of dots is equal to (8), D = $d_1 + d_2 = 8$
D = {(4,4), (2,6), (6,2), (5,3), (3,5)}
$P(D) = \frac{5}{36} \in [0,1]$

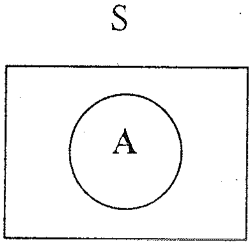
$d_2 \geq 3$

b. To get ~~one~~ [illegible]', P(c) = ? , P (c^c) = ?

c. Find the pr. That $d_2 < 3$ = {(1,2), (2,1), (3,1), (4,1), (5,1), (6,1), (3,2), (4,2), (5,2), (6,2)} ⇒ $P(C) = \frac{10}{36} \in [0,1]$

Theorem 1 :- $P(\Phi) = 0$

Proof :- let A be any event



$A\Phi = \Phi$

A & Φ are disj.

$A \cup \Phi = A$

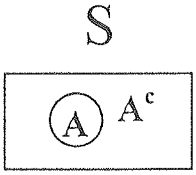
$P(A \cup \Phi) = P(A)$

$P(A) + P(\Phi) = P(A)$ by AX.3

$P(\Phi) = 0$

Theorem 2 :- $P(A^c) = 1 - P(A)$

Proof :-



$\because AA^c = \Phi$

$\therefore$ A & A^c are disj

$A \cup A^c = S \Rightarrow P(A \cup A^c) = P(S)$

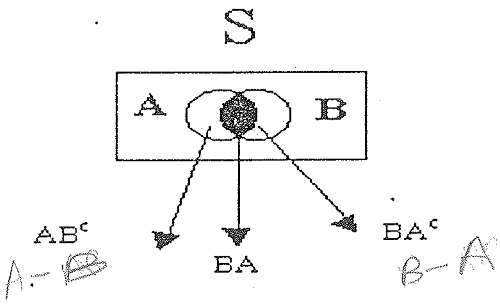
$P(A) + P(A^c) = 1$ $\quad$ *by*AX,2, AX,3

$\therefore P(A^c) = 1 - P(A)$

Theorem 3 :- If A and B are joint events , then

$$P(A \cup B) = P(A) + P(B) - P(AB)$$

Proof :-



AB^c & AB $\quad$ *are* disjoint $\Rightarrow A = AB^c \cup AB$

$P(A) = P(AB^c) + P(AB)$ $\quad$ *by* $\quad$ AX.3 $\quad \cdots(1)$

BA^c & AB $\quad$ *are* $\quad$ disjoint

$B = BA^c \cup AB$

$P(B) = P(BA^c) + P(AB)$ $\quad$ *by* $\quad$ AX.3

$P(BA^c) = P(B) - P(AB) \cdots\cdots\cdots\cdots\cdots\cdots\cdots(2)$

$\because AB^c, AB$ & BA^c $\quad$ *are* $\quad$ *disjo* int

$A \cup B = AB^c \cup AB \cup BA^c$

by AX-3

$P(A \cup B) = P(AB^c) + P(AB) + P(BA^c)$

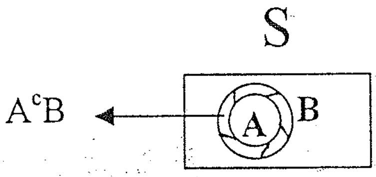
$\therefore P(A \cup B) = [P(A) - \cancel{P(AB)}] + \cancel{P(AB)} + [P(B) - P(AB)]$

$P(A \cup B) = P(A) + P(B) - P(AB)$

Theorem 4 :- If A & B are events such that $A \subseteq B$, then $P(A) \leq P(B)$

Proof :- A and A^cB are disj.

$$B = A \cup A^cB$$
$$P(B) = P(A \cup A^cB)$$
$$P(B) = P(A) + P(A^cB) \quad \textit{by} \quad AX_3$$
$$P(B) - P(A) = P(A^cB) \geq 0$$
$$P(B) - P(A) \geq 0 \Rightarrow P(A) \leq P(B)$$



H.W For any events A and B , show that

1. $P(AB) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$

2. If A and B are joint events , when $P(A) = 0.8; P(B) = 0.5$

Find the conditions and the value of Max $P(AB)$ and Min $P(AB)$

3. If $P(A) = \frac{1}{3}$ & $P(B) = \frac{1}{2}$ Find the value of $P(BA^c)$ when

a. A & B are disj. events b. $A \subseteq B$ c. $P(AB) = \frac{1}{8}$

④ If A,B and c are dis J. events find

1. $P[(A \cup B) \cap C]$ 2. $P[A^c \cup B^c]$

Theorem 5 :- (H.W.)

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$$

Theorem 6 :- Convergent of pr. → Th. 6 → Th. 7

If $A_1, A_2 \cdots,$ $A_n, \ldots$ be asequence of infinite events such that

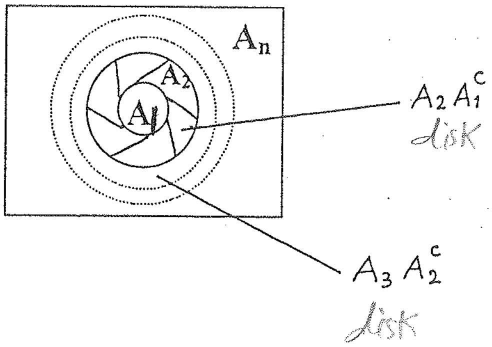
$A_1 \subset A_2 \subset A_3 \subset \ldots\ldots \subset A_n \subset \ldots\ldots$

Then $P\left[\bigcup_{i=1}^{\infty} A_i\right] = \lim_{n \to \infty} P(A_n)$ (Conv. from above)

Proof:- $A_1, A_2A_1^c, A_3A_2^c, \dots A_nA_{n-1}^c$ are disj

$$A_2 = A_1 \cup A_2 = A_1 \cup A_2A_1^c$$

$$A_3 = A_1 \cup A_2 \cup A_3 = A_2 \cup A_3A_2^c$$



$$A_n = \bigcup_{i=1}^{n} Ai\,A_{i-1}^c = \bigcup_{i=1}^{n} Ai$$

$$\therefore P(A_n) = P\left(\bigcup_{i=1}^{n} Ai \quad A_{i-1}^c\right)$$

$$= \sum_{i=1}^{n} P\left(AiA_{i-1}^c\right) \quad by AX.3$$

$$\underset{n\to\infty}{Lim} P(A_n) = \underset{n\to\infty}{Lim} \sum_{i=1}^{n} P\left(Ai \quad A_{i-1}^c\right)$$

$$= \sum_{i=1}^{\infty} P\left(Ai \quad A_{i-1}^c\right) \dots (1)$$

$$\bigcup_{i=1}^{\infty} Ai = \bigcup_{i=1}^{\infty} Ai \quad A_{i-1}^c$$

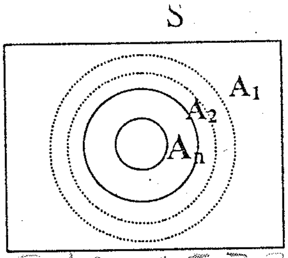
$$P\left(\bigcup_{i=1}^{\infty} Ai\right) = P\left(\bigcup_{i=1}^{\infty} Ai \quad A_{i-1}^c\right)$$

$$= \sum_{i=1}^{\infty} P\left(Ai \quad A_{i-1}^c\right) \dots (2)$$

$$\because (1) = (2) \quad \therefore P\left(\bigcup_{i=1}^{\infty} Ai\right) = \underset{n\to\infty}{Lim} \quad P(A_n)$$

(Conv- from below)

Theorem 7 : Let $A_1, A_2, A_3, \ldots, A_n, \ldots$ *be an* infinite sequence of events such that



$$A_1 \supset A_2 \supset A_3 \supset \ldots\ldots \supset A_n \supset \ldots\ldots$$

Then $P\left(\bigcap_{i=1}^{\infty} Ai\right) = \lim\limits_{n\to\infty} \quad P(A_n)$

تعتمد كليا على
$A \subset B \Leftrightarrow$
$A^c \supset B^c$

Proof :- $A_1^c \subset A_2^c \subset A_3^c \ldots\ldots \subset A_n^c \subset \ldots\ldots$ $\therefore$ By theor. 6

$$P\left(\bigcup_{i=1}^{\infty} A^c i\right) = \lim\limits_{n\to\infty} \quad P(A_n^c)$$

$$P\left[\left(\bigcap_{i=1}^{\infty} Ai\right)^c\right] = \lim\limits_{n\to\infty} \quad [1 - P(A_n)] \quad \textit{by the th2. \& D.Law.}$$

[illegible]

$$1 - P\left[\bigcap_{i=1}^{\infty} Ai\right] 1 = \lim\limits_{n\to\infty} \quad P(A_n)$$

$$P\left[\bigcap_{i=1}^{\infty} Ai\right] = \lim\limits_{n\to\infty} \quad P(A_n)$$

Random Sampling العينات العشوائية

Suppose a population of n-elts .$(a_1, a_2, \ldots\ldots a_n)$ We want to choose a subset of this population has (K) elts$(a_1, a_2, \ldots\ldots a_k)$at random $(k \le n)$ These subset is called random samples . there aretow kinds of random sample :-

1. Un Ordered Sample العينات غير المرتبة

Select (k) elts from (n) elts at once (at the same time)

2. Ordered Sample العينات المرتبة

a. one by one without replacement selecte (k)elts. From (n) elts .

one by one without repl .

b. one by one with replacement selecte (k) elts , From (n) elts.

one by one with replac .

Case 1 الحالة الاولى

Choose (k) elts . at the same time from (n) elts .

We use , (combination n , K) to find the number of all samples

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Ex/ Given a set of (4) elts. $\{a,b,c,d\}$

Choose a sample of (2) elts.

a. Find the sample space of all samples .

b. Find the pro. That a sample has elts . (b)

Sol/ a.

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4\times3\times2!}{2!\times2!} = \frac{12}{2} = 6 \text{ samples}$$

$S = \{(a,b),(a,c),(a,d),(b,c),(b,d),(c,d)\}$

b. Let A be a sample has elts (b)

$A = \{(a,b),(b,c),(d,d)\}$

$\therefore P(A) = \frac{3}{6} = \frac{1}{2}$

i.e/

طريقة عامة . في السؤال السابق يكون هناك اربعة عناصر اما لو اخذنــا 50 عنصــر

يكون الاستخدام بالشكل التالي :

$$(\square\ ,\ \square)$$

$$\overset{3}{(a,c,d)}\quad b$$

$$P_{3,1}\times b_{1,1}$$

$$\frac{3!}{2!}\times\frac{1!}{0!}$$

$$3\times1=3$$

$$(\square\ ,\ \square)$$

$$(a_2,a_3,\ldots a_{50})\ 49\qquad a_1$$

$$P_{49,1}\times\ P_{1,1}$$

$$\frac{49!}{48!}\times\frac{1!}{0!}$$

Ex/ Given a set of (3) boys and (4) girls students. Choose a sample of (3) students

a. Find S

b. Find the pr. That a sample has 2 boys

c. Find the pr. That a sample has at least (2) girls .

sol/ a.

$$\binom{7}{3}=\frac{7!}{3!(4!)}=35$$

$$\therefore S\quad has\quad 35\, samples$$

b. $\left(\frac{3b}{3g}\right)\rightarrow\left(\frac{2b}{1g}\right)$

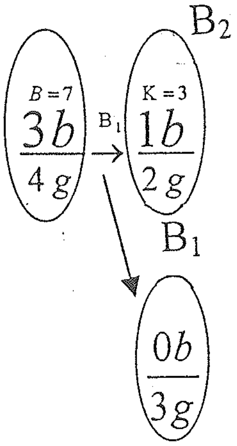
A: be a sample that has (2) boys

$$\binom{3}{2}\binom{4}{1}=\left[\frac{3!}{2!(3-2)!}\right]\times\left[\frac{4!}{1!(4-1)!}\right]=12$$

$3\times 4=12$ [12 *samples has 2boys and girl*]

$$\therefore P(A)=\frac{\binom{3}{2}\binom{4}{1}}{\binom{7}{3}}=\frac{12}{35}$$

c.



$\binom{3}{1}\binom{4}{2}^{B}$ *be asample has at least (2)girls*

$\binom{3}{0}\left(\frac{4}{3}\right)^{B=B_1\cup B_2}$

$B=B_1\cup B_2$ [B_1 *and* B_2 *are disj.*]

$$P(B)=P(B_1)+P(B_2)$$

$$P(B)=\frac{\binom{3}{1}\binom{4}{2}}{\binom{7}{3}}+\frac{\binom{3}{0}\binom{4}{3}}{\binom{7}{3}}=0.5$$

Case 2 :

a. Choose (k) elts. From (n) . elts. one by one with out replacement .

In this case we use (permutation n, k) to find the number of samples in S.

Where $P_k^n = \dfrac{n!}{(n-K)!}$

Ex/ Given a set of (4) elts. $\{a,b,c,d\}$

Choose a sample of (2) elts. one by one without replacement

a. Finds

b. Find the pr. That a sample has elts. (b) .

Sol/

$$P_2^4 = \frac{4!}{(4-2)!} = 12$$

S *has* (12) *elts.*

$$S = \begin{Bmatrix} (a,b), & (b,a), & (c,d), & (d,a) \\ (a,c), & (b,c), & (c,b), & (d,b) \\ (a,d), & (b,d), & (c,d), & (d,c) \end{Bmatrix}$$

$$A = \{(a,b),(b,a),(b,c),(b,d),(c,b),(d,b)\}$$

$$P(A) = \frac{6}{12} = \frac{1}{2}$$

ex/ Given $\{2,3,5,6,8\}$ a set of (5) integers choose a sample of (3) integers one by one without replacement .

a. Find the pr. That the sample con be divided by 5

b. divided by (2) .

Sol/ $P_3^5 = \frac{5!}{2!} = 60$

S has (60) samples

a. let A be a sample which divided by (5)

$\boxed{2}$	☐	☐
2	3	5
3	6	
6	8	
8		

$4 \times 3 \times 1 = 12$

$\therefore$ A has (12) samples

$$P(A) = \frac{12}{60}$$

b. let B be a sample which divided by (2)

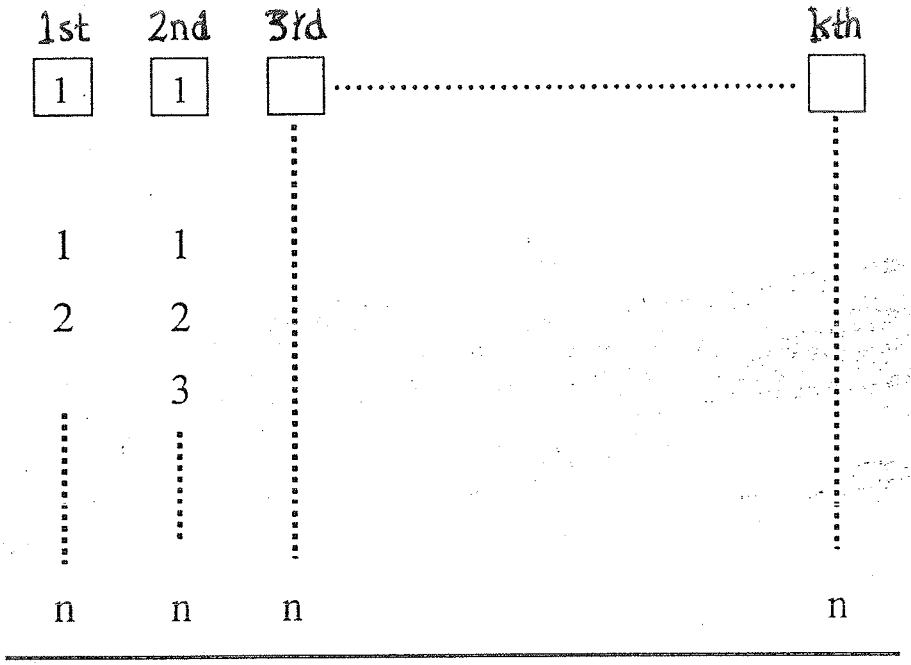
B has (36) samples

$$P(B) = \frac{36}{60} = \in [0,1]$$

☐	☐	☐
3	5	2
5	6	6
6	8	8
8		

$4 \times 3 \times 3 = 36$

b. Choose (k) elts. From (n) elts , one by one with replacement.



$$n \times n \times n \ldots \ldots \ldots \times n = n^k$$

$\therefore S \quad has \left(n^k\right) \quad samples$

ex/ Given 4 elts $\{a,b,c,d\}$ choose a sample of 2 elts. One by one with replacement

a. Find S

b. Find the pr. That a sample has elt (b)

Sol/

a. $4^2=16$ S has (16) samples

$$S=\begin{Bmatrix} (a,a), & (b,a), & (c,a), & (d,a) \\ (a,b), & (b,b), & (c,b), & (d,b) \\ (a,c), & (b,c), & (c,c), & (d,c) \\ (a,d), & (b,d), & (c,d), & (d,d) \end{Bmatrix}$$

b. Let A be a sample has (b).

$$A=\{(a,b),(b,a),(b,b),(b,c),(b,d),(c,b),(d,b)\}$$

$$P(A)=\frac{7}{16}$$

Exercises :-

1. A box has (24) bulbs of which (4) are defective .Choose 4 bulbs , find the pr. That they are defective .

2. A set of (11) integers ; (5) of them are negative and the others are positive . Choose a sample of (4) integers and multiply them , then find the pr. That the product is .

a. negative b. positive

3. Given a set of (12) transistors of which (3) are defective،choose a sample of (4) transistors then find the pr. that .

a. Two transistor are defective .

b. at lest one transistors is defective .

4. Find the pr. That two people of (K) people will have the same birthday .

Probability Space

Def :- (δ-Field)

A non-empty collection $\wp$ subsets of a set (S) is called σ -field of subsets of (S) provided the following two properties holds .

1. If $A \in \wp$, *then* $A^c \in \wp$

2. If $A_n \in \wp$, $n = 1,2,\ldots$

$$\text{then } \bigcap_{n-1}^{\infty} A_n \wp \ \& \bigcup_{n-1}^{\infty} A_n \in \wp$$

Def:- A probability measure (p) on a (δ)-field of subsets $(\wp)$is a real valued function having a domain$(\wp)$and satisfying the following properties

1. $P(S) = 1$

2. $P(A) \geq 0, \forall A \in \wp$

3. If $A_1, A_2, \ldots\ldots A_n$, are disjoint in $\wp$ then

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

Def:- (Probability Space)

The triple $(S, \wp, P)$ is called a probability space .

Remarks :- the elements of S are called sample points .

Any $A \in \wp$ is know as event clearly A is a collection of sample points.

ex/ Toss a coin once

$$S = \{H, T\}$$

$$\wp = \{\{H\}, \{T\}, S, \Phi\} 2^n$$

ex/ Toss a coin twice

$$S = \left\{\begin{matrix} HH, & HT, & TH, & TT \\ a & b & c & d \end{matrix}\right\}$$

$$\wp = \left\{\begin{matrix} \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\} \\ (b,d), \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\} \\ \{b,c,d\}, S, \Phi \end{matrix}\right\}$$

$$\genfrac{}{}{0pt}{}{4}{2} = 16$$

Independent Events : الحوادث المستقلة

Def :- If A and B are events . We say A and B are independent events iff

$$P(A) \times P(B) = P(AB)$$

At the same time A and B are dependent events iff

$$P(A) \times P(B) \neq P(AB)$$

Ex/ Choose (2) integers From $\{1,2,3,4\}$ one by one without (with) replacement.

If A: 1st chosen int. is (2)

B: 2nd chosen int. is (1)

Are A and B ind. Events ? why

Joint ⇒ Dep.

Ex.

[illegible] العكس

Joint ⇏ indep.

while the indep. ⇒ Joint

Sol/

1. Without repl.

$$P^{4}_{2} = \frac{4!}{(4-2)!} = 12$$

S has (12) elts. $= \left\{ \begin{matrix} (1,2), (2,1), (3,1), (4,1) \\ (1,3), (2,3), (3,2), (4,2) \\ (1,4), (2,4), (3,4)(4,3) \end{matrix} \right\}$

$$A = \{(2,1),(2,3),(2,4)\} \Rightarrow P(A) = \frac{3}{12} = \frac{1}{4}$$

$$B = \{(2,1),(3,1),(4,1)\} \Rightarrow P(B) = \frac{3}{12} = \frac{1}{4}$$

$$AB = \{(2,1)\} \Rightarrow P(AB) = \frac{1}{12}$$

$$P(A) \times P(B) = \frac{1}{16} \neq \frac{1}{12} = P(AB)$$

A and B are dependent

2. With replace.

$$(n^{K}) = 4^{2} = 16$$

$\therefore$ S has (16) samples

$$A = \{(2,1),(2,2),(2,3),(2,4)\} \Rightarrow P(A) = \frac{4}{16} = \frac{1}{4}$$

$$B = \{(1,1),(2,1),(3,1),(4,1)\} \Rightarrow P(B) = \frac{4}{16} = \frac{1}{4}$$

$$AB = \{(2,1)\} \Rightarrow P(AB) = \frac{1}{16}$$

$$P(A) \times P(B) = \frac{1}{16} = P(AB)$$

$\therefore$ A and B are indep.

Theorem 8 : If A and B are independent event such that $A \neq \Phi, B \neq \Phi$ then A and B are Joint events .

Proof :- $\because$ A *and* B ind $\Rightarrow P(A),P(B) = P(AB)$

T.P/ A and B are Joint

ie/ T.P/ $AB \neq \Phi$

$\because A \neq \Phi \Rightarrow P(A) \neq 0$

$B \neq \Phi \Rightarrow P(B) \neq 0$

$P(A) \times P(B) \neq 0$

$P(AB) \neq 0$ By hyp

$\therefore AB \neq \Phi$

ملاحظة : العكس من النظرية اعلاه غير صحيح بمعنى اذا كانت A و B حادثتين متصلتين (Joint) فانه ليس من الضروري ان تكون الحادثتين A و B مستقلتين (independent) مثال على ذلك المثال السابق في حالة بدون ارجاع (Without repel).

Theorem 9 : If A and B are disjoint events , such that $A \neq \Phi, B \neq \Phi$ then A and B are dependent .

Proof :- $\therefore$ A and B are disjoint

$\therefore AB = \Phi \Rightarrow P(AB) = 0 ......(1)$

$\because A \neq \Phi \Rightarrow P(A) \neq 0$

$B \neq \Phi \Rightarrow P(B) \neq 0$

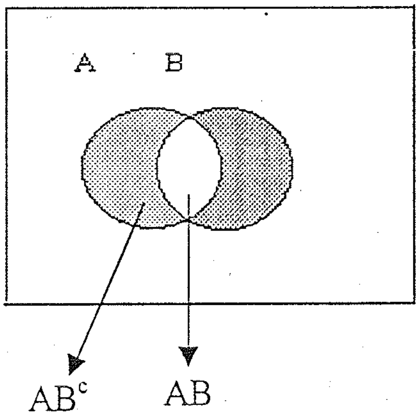
$P(A) \times P(B) \neq 0 ....(2)$

$P(A)P(B) \neq P(AB)$

A , B are dependent

Theorem 10 : If A and B are independent events , then .

1. A and B^c are independent .
2. B and A^c are independent .
3. A^c and B^c are independent .



Proof (1) :- Tp. A&B^c are ind.

ie/ Tp. $P(A)P(B^c) = P(AB^c)$

$A = AB^c \cup AB$

$P(A) = P(AB^c) + P(AB) \, by \quad AX.3$

$P(AB^c) = P(A) - P(AB)$

$= P(A) - P(A).P(B)$ by hyp.

$= P(A)[1 - P(B)]$

$= P(A)P(B^c)$

Independence of Three Events :

Def :- If A , B and C are events , then A,B and C are independent events iff

1. a. $P(A).P(B) = P(AB)$ (A, B are ind .)

b. $P(A).P(C) = P(AC)$(A, C are ind .)

c. $P(B).P(C)=P(BC)$(B, C are ind.)

2. $P(A).P(B).P(C)=P(ABC)$

Note :- If satisfy only condition (I) , then A,B and C are said to be pairwise independent .

ex/ Given $S=\{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$

A: 1^{st} coordinate is (1)

B: 2^{nd} coordinate is (1)

C= 3^{rd} coordinate is (1)

Are A,B and C indep ? why ?

Conditional Probability الاحتمال الشرطي

Def :- Let A,B are events if event A happens first , then event B happens .

Or event A given then event B happens denoted by (B | A) .

Were (B\A) is called condition events .

Also P(B\A) is called conditional probability .

Where $P(B\setminus A)=\dfrac{P(AB)}{P(A)}, P(A)\neq 0$ (يعني A يجب ان تحدث [illegible] $A\neq\emptyset$)

Note :- 1. If A and B are indep. Events

$$P\ (B\setminus A)=\frac{P(A)P(B)}{P(A)}=P(B)$$

2. From def. of cond. pr.

علاقة ضربي $P(AB)=P(A).P(B\setminus A)$ multiplication rule (معنى احتمال B [illegible] A .)

ex/ Toss a dice twice اما علاقة الجمع $P(A\cup B)=P(A)+P(B)-P(AB)$

Find the pr. That $d_1+d_2\leq 6$

Given that $d_1 + d_2 = \text{odd}$

بالتعويض $P(AB)$

$A \backslash B$ indep.

$P(AB) = P(A) \cdot P(B)$

$A \backslash B$ dep.

$P(AB) = P(A) \cdot P(B|A)$

Sol/ $$P(B \setminus A) = \frac{P(AB)}{P(A)}$$

$$A = \begin{Bmatrix} (1,2), (2,1), (3,2), (4,1), (5,2), (6,1) \\ (1,4)\ (2,3)\ (3,4)\ (4,3)\ (5,4)\ (6,3) \\ (1,6)\ (2,5)\ (3,6)\ (4,5)\ (5,6)\ 6,5 \end{Bmatrix}$$

$$B = \begin{Bmatrix} (1,1), (2,1), (3,1), (4,1), (5,1) \\ (1,2)(2,2)(3,2)(4,2) \\ (1,3)\ (2,3)\ (3,3) \\ (1,4)(2,4) \\ (1,5) \end{Bmatrix}$$

$$P = (A) = \frac{18}{36}$$

$$P(B) = \frac{15}{36}$$

$$AB = \begin{Bmatrix} (1,2), (2,1), (3,2) \\ (1,4), (2,3), (2,4) \end{Bmatrix}$$

$$\therefore P(AB) = \frac{6}{36}$$

$$P(B \setminus A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{6}{36}}{\frac{18}{36}} = \frac{6}{18} = \frac{1}{3} \in [0,1]$$

$$P(A \setminus B) = \frac{P(AB)}{P(B)}$$

$$= \frac{6}{15}$$

Theorem 11 :- let $A_1, A_2, \ldots, A_n$ *be* an events where

$A i \neq \Phi \qquad i = 1,2,\ldots n$ then

$$P(A_1, A_2, \ldots, A_n) = P(A_1).P(A_2 \setminus A_1).P(A_3 \setminus A_1A_2)P(A_4 \setminus A_1A_2A_3) \ldots P(A_n \setminus A_1A_2A_3 \ldots A_{n-1})$$

Proof :- right side

$$= P(A_1)\frac{P(A_1A_2)}{P(A_1)}.\frac{P(A_1A_2A_3)}{P(A_1A_2)}.\frac{P(A_1A_2A_3A_4)}{P(A_1A_2A_3)}\cdots\cdots\frac{P(A_1A_2A_3\cdots A_n)}{P(A_1A_2A_3\cdots A_{n-1})}$$

$$= P(A_1A_2A_3 \ldots A_n)$$

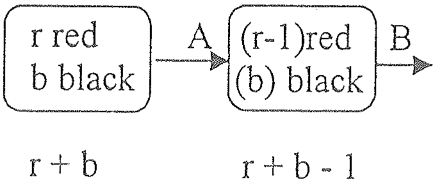
= left side

ex/ A box has (r) red balls and (b) black balls . Choose (2) balls one by one without replac.

1. If the first chosen ball is red find the pr. that both balls has different colour .
2. Find the pr. that the 1st &2nd chosen ball are red .
3. Find the pr. That at most one , ball is red

Sol/ Let A: be the 1st red ball

B: be 1st chosen black ball



([illegible])

$$P(B \setminus A) = \frac{b}{r+b-1}, \quad P(A) = \frac{r}{r+b}$$

2. C: be 2nd chosen ball is red

$$P(AC) = P(A).P(C \setminus A)$$

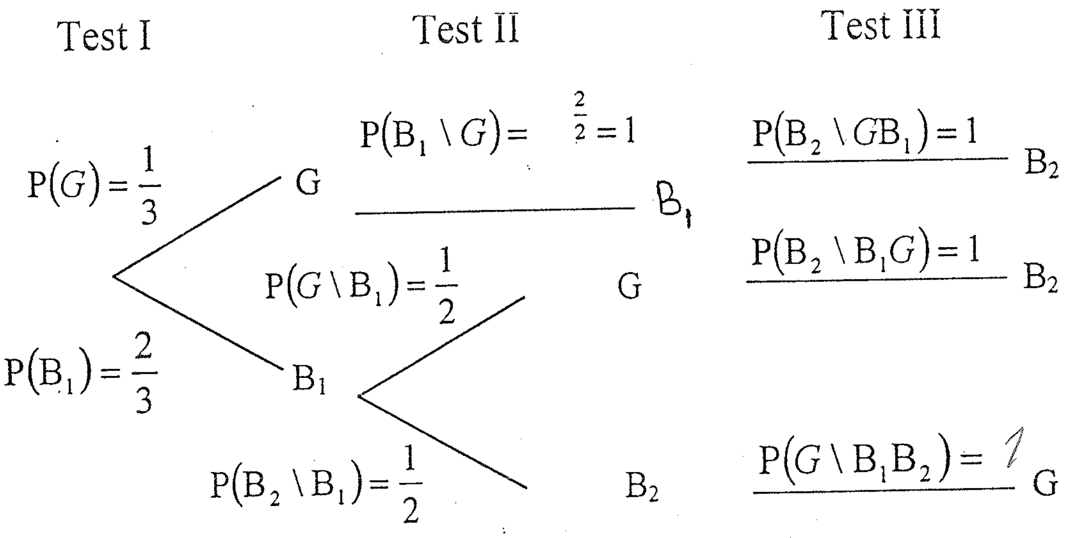
$$= \frac{r}{r+b}.\frac{r-1}{r+b-1}$$

3.

$$
\begin{aligned}
&P(A \cup B) - P(AB) \\
&= 1 - P(A)P(B \backslash A) \\
&= 1 - \frac{r}{r+b} \cdot \frac{b}{r+b-1}
\end{aligned}
$$

ex. 2/ Given (2) bad tubes and (1) good tube . Take the tube one by one until both bad tubes are founds . find the pr. that 2nd bad tube is found on

1. test 1
2. test 2
3. test 3



1. $P(B_2 \text{ on test I}) = P(\Phi) = 0$

2. $P(B_2 \text{ on test II}) = P(B_1B_2)$

$$= P(B_1).P(B_2 \setminus B_1)$$

$$= \frac{2}{3}.\frac{1}{2} = \frac{1}{3}$$

3. $P(B_2 \quad on \quad test \quad III) = P(GB_1B_2 \cup B_1GB_2)$

$$= P(GB_1B_2) + P(B_1GB_2)$$

$$= P(G)P(B_1 \setminus G)P(B_2 \setminus GB_1) + P(B_1)P(G \setminus B_1)P(B_2 \setminus B_1G)$$

$$= \frac{1}{3}.1:1 + \frac{2}{3}.\frac{1}{2}.1$$

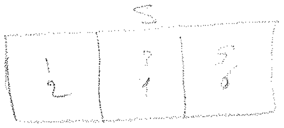
$$= \frac{1}{3} + \frac{1}{3}$$

$$= \frac{2}{3}$$

مثلاً في تجزئة الفراغ/

roll a dice one

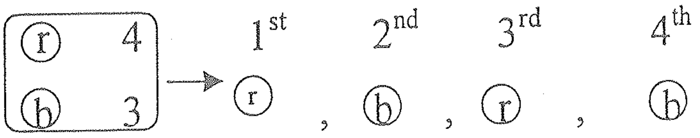
$A_1 = \{1,2\}$, $A_2 = \{3,4\}$ و $A_3 = \{5,6\}$



$A_1 \cup A_2 \cup A_3 = S$ و $\cap A_i = \emptyset$
$i=1,2,3$

ex/ A box has (4) red and (3) black balls choose a sample of (4) balls one by one without repl . Find the pr. to get a sample r,b,r,b

Sol/



N = 4+3=7

Let R_1 : to Choose 1st red ball

$B_1 \backslash R_1$: to choose 1st black ball , given 1st red ball

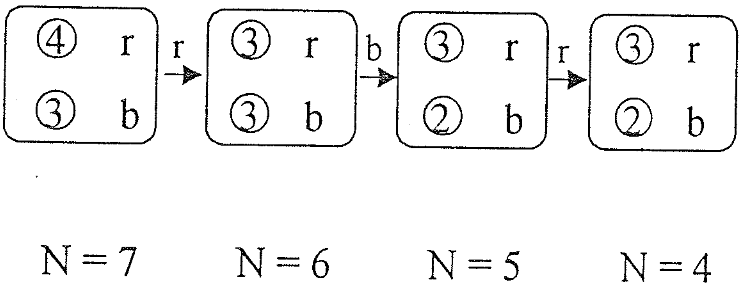
$R_2 \backslash R_1 B_1$: to choose 2nd red ball , given 1st red , 1st black balls

$B_2 \backslash R_1 B_1 R_2$: to choose 2nd black ball , given 1st red , 1st black and 2nd red balls

$$P(R_1 B_1 R_2 B_2) = P(R_1).P(R_1 \backslash B_1)P(R_2 \backslash R_1 B_1).P(B_2 \backslash R_1 B_1 R_2)$$

$$= \frac{4}{7}.\frac{3}{6}.\frac{3}{5}.\frac{2}{4}.$$

$$= \frac{3}{35}$$



Partition Of a Sample Space

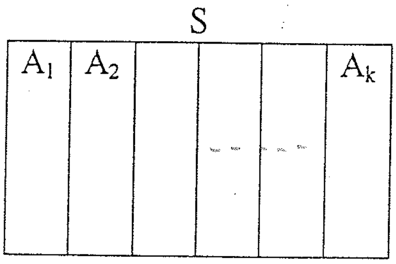
(Ex. Rolling a die once)

Def :- A Finite sequence of event $A_1, A_2, \cdots A_K$ Form partition of (S) iff

حوادث متنافية

1. $A_1, A_2, \ldots, A_K$ are disjoint

i.e/ $\bigcap_{i=1}^{K} Ai = \Phi$



2. $\bigcup_{i=1}^{K} Ai = S$

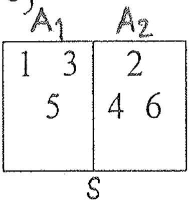
مثل: مراحل الكلية (١، ٢، ٣، ٤) هي حوادث متنافية لأنه طالب الصف ١ لا يداوم في الصف ٢ في نفس الوقت

$\emptyset = \cap 1,2$

ex/ Toss a dice once $S=\{1,2,3,4,5,6\}$

$A_1=\{1,3,5\}$

$A_2=\{2,4,6\}$



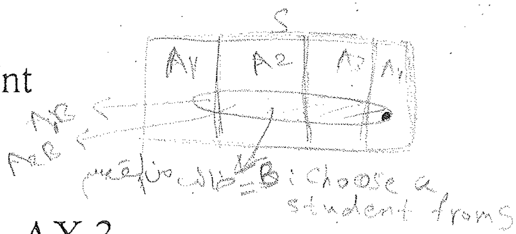
$$\left.\begin{array}{l} A_1A_2=\Phi \\ A_1\cup A_2=S \end{array}\right\} \Rightarrow A_1, A_2 \ form \quad apartition \quad of \quad S$$

نظرية تجميعية [illegible]

Theorem 12:- If Ai $(i=1,2,\ldots,K), Ai\neq\Phi$ from a partition of S. If

$\Phi\neq B\subset S, \quad then \quad P(B)=\sum_{i=1}^{K}P(Ai)P(B\setminus Ai)$

مثال قسم الرياضيات ع اربع مراحل ما احتمال اختيار طالب واحد منه



Proof :- $A_1B, A_2B, A_3B, \ldots A_KB$ are disjoint

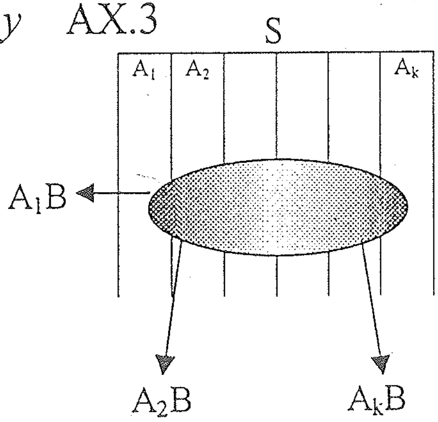
$B=A_1B\cup A_2B\cup A_3B\cup\ldots\ldots\cup A_KB$

$P(B)=P(A_1B)+P(A_2B)+\ldots\ldots+P(A_KB) by \quad AX.3$

$=\sum_{i=1}^{K}P(A_iB)$

لأنها (حوادث معتمدة) dep. Ai على B تعتمد

$=\sum_{i=1}^{K}P(Ai)P(B\setminus Ai) \quad by \quad theorem \ 11$



Theorem 13 :- (Bayes Theorem)

If $Aj(j=1,2,\ldots K)$ From a partitions of S , where $Aj\neq\Phi$, and if $\Phi\neq B\subset S$ then

$$P(A\text{[illegible]})=\frac{P(Ai)P(B\setminus Ai)}{\sum_{j=1}^{K}P(Aj)P(B\setminus Aj)}$$

من نظريات (prior information) تحتاج معلومات مسبقة

EX: قسم الرياضيات S ومراحله

if P(B) ... ما احتمال ان تكون [illegible] ؟

P(B)g ما احتمال ان تكون [illegible]

	A1	A2	A3	A4
نسبة [illegible] في كل مرحلة	.25	.30	.20	.25
[illegible] في كل مرحلة	.70 g	.10 g	.25 g	.20 g
	.30 b	.40 b	.75 b	.80 b

$\cup A_i = S$

Bayes نستخدم نظرية

$P(B)g=\sum_{i=1}^{4}P(Ai)\,P(B|Ai)_b$

في EX السابق (قسم الرياضيات)
ما احتمالية طالب من الذكور علماً شرط انه يكون من A4
فانها نظرية Bayes .
اي عندما يعطي شرط نستخدم نظرية Bayes .

where (i) is a one value of (j)

Proof :- by theorem 12

$$P(B)=\sum_{i=1}^{K} P(A_i)P(B\setminus A_i)$$

$$P(A_i\setminus B)=\frac{P(A_iB)}{P(B)} \quad \text{by def of cond.pr.}$$

$$=\frac{P(A_i)P(B\setminus A_i)}{\sum_{j=1}^{K} P(A_i)P(B\setminus A_j)} \quad \text{by theorem II and theorem 12}$$

Note :

1. $P(A_i\setminus B)$ is called the posterior pr. (it is the pr. Of an event wich is the source when a result is given) .

2. $P(A_i)$ is called prior pr.

ex/ Given the following boxes :-

box 1 has (3) red and (5) white balls

box 2 has (2) red and (4) white balls

choose a box , then choose a ball from the chosen box

a. Find the pr. that a white ball is chosen .

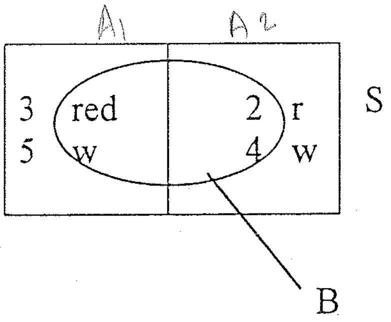
b. If a red ball is chosen , Find the pr. that it is from box 2 .

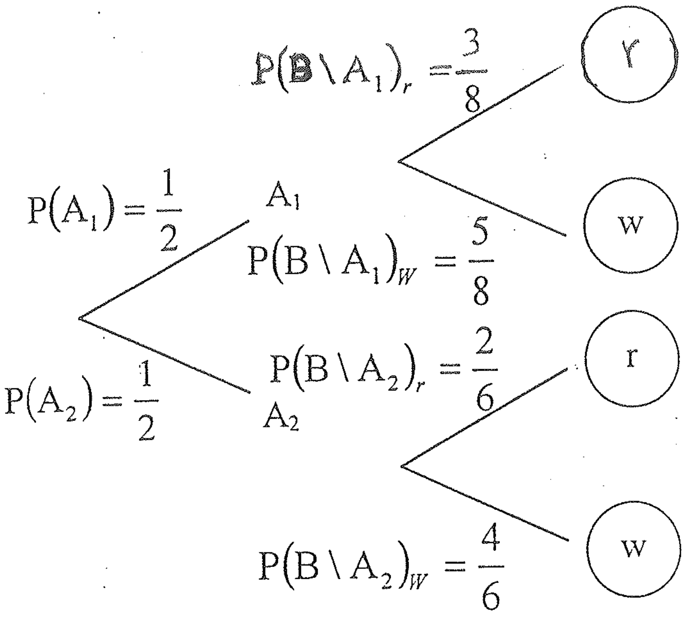
Sol/

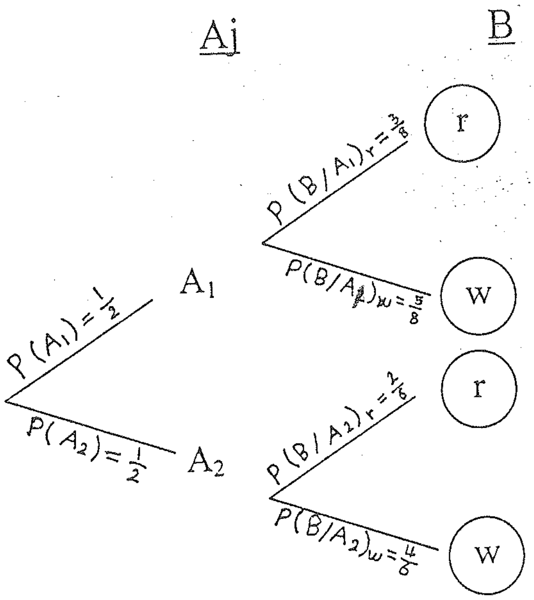
Let A_1 : choose box 1

A_2 : choose box 2

B : choose a ball







a. $P(B)_W = \sum_{j=1}^{2} P(Aj).P(B\setminus Aj)_W$

$$= P(A1)P(B\setminus A\ )_W + P(A2)P(B\setminus A_2)_W$$

$$= \left(\frac{1}{2}\right)\left(\frac{5}{8}\right) + \left(\frac{1}{2}\right)\left(\frac{4}{6}\right) = ?$$

b. $P(A_2\setminus B)_r = \dfrac{P(A_2)P(B\setminus A_2)_r}{\sum_{j=1}^{2} P(Aj)P(B\setminus Aj)_r}$

$$= \frac{\left(\frac{1}{2}\right).\left(\frac{2}{6}\right)}{\left(\frac{1}{2}\right)\left(\frac{3}{8}\right) + \left(\frac{1}{2}\right)\left(\frac{2}{6}\right)}$$

H.W. Find p (B) $_r$, $P(A\setminus B)_W$

ex/ Three Machines M_1 , M_2 and M_3 produce glasses

M_1 Produce 20% of glasses

M_2 Produce 30% of glasses

M_3 Produce 50% of glasses

Also

1% of glass produced by M_1 is defective

2% of glass produced by M_2 is defective

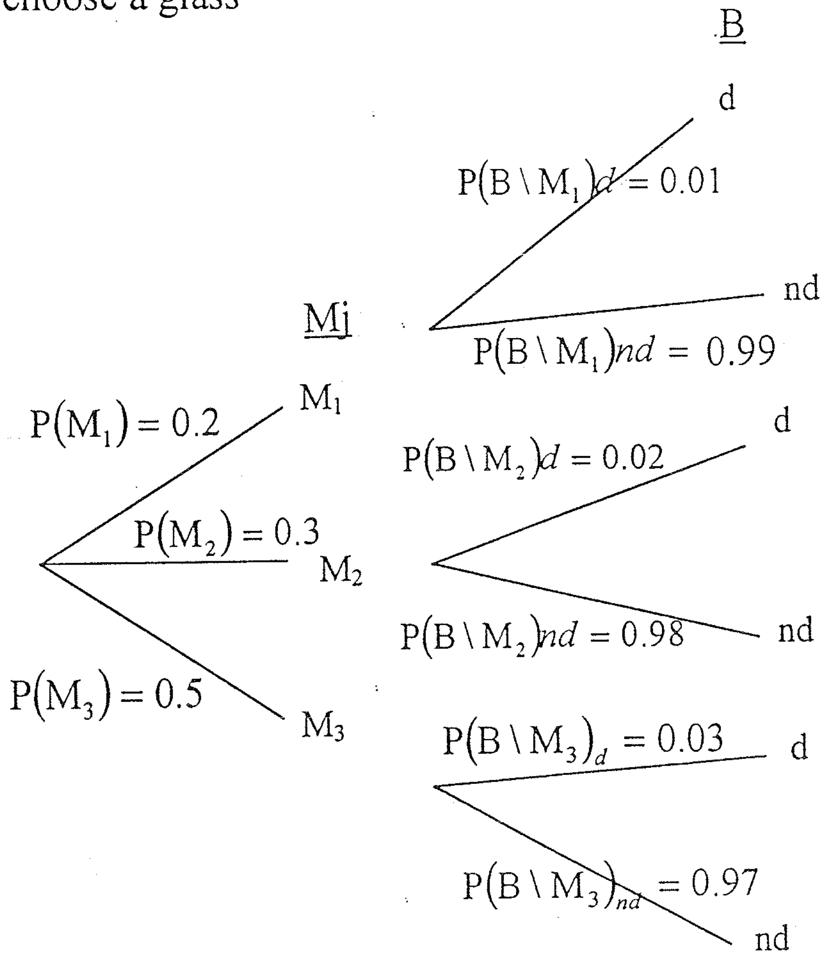
3% of glass produced by M_3 is defective

Choose a glass ,then

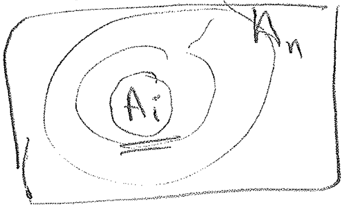
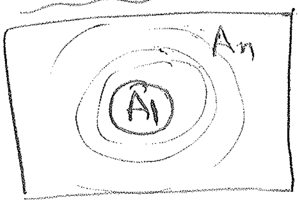
a. Find the pr. That the glass is produced by M_3 if it is defe.

b. Find the pr. That the glass is def.

Let B choose a glass



أخطاء مطبعية في ملزمة ch.2 / مادة الاحتمالية
المرحلة الثالثة / قسم الرياضيات

رقم الصفحة	الخطأ	التوضيح
١٥		
١٧	$A=\{(a,b),(b,c),(d,d)\}$	$A=\{(a,b),(b,c),(b,d)\}$
١٨	$b.\ \frac{3b}{3g}\longrightarrow\frac{2b}{1g}$	$b.\ \frac{3b}{4g}\longrightarrow\frac{2b}{1g}$
٢٤	(Defi- 5 - Find)	(Defi- σ - field)
٣٠	$P(B,A)=\frac{P(AB)}{P(B)}$	$P(B\|A)=\frac{P(AB)}{P(A)}$
٣٤	$=P(R_1)P(R_1\|R_1)P(R_2\|R_1B_1)+\cdots$	$=P(R_1)P(B_1\|R_1)P(R_2\|R_1B_1)$
٣٥	Th.13 $P(A_i)=\frac{P(A_i)P(B\|A_i)}{\sum_{j=1}^{k}P(A_j)P(B\|A_j)}$	$P(A_i\|B)=\frac{P(A_i)P(B\|A_i)}{\sum_{i=1}^{k}P(A_j)P(B\|A_j)}$
٣٦	$P(B)=\sum_{j=1}^{k}P(A_i)P(B\|A_i)$ $P(A_i\|B)=\frac{P(A_i)P(B\|A_i)}{\sum_{j=1}^{k}P(A_j)P(B\|A_j)}$	$P(B)=\sum_{j=1}^{k}P(A_j)P(B\|A_j)$ $=\frac{P(A_i)P(B\|A_i)}{\sum_{j=1}^{k}P(A_j)P(B\|A_j)}$
٤٠	2 Coin 3 Coin 4 Coin	(2) Coins (3) " (4) "

a.

$$P(M3 \setminus B)d = \frac{P(M_3)P(B \setminus M_3)_d}{\sum_{j=1}^{3} P(Mj)P(B \setminus Mj)_d}$$

$$= \frac{(0.5)(0.03)}{(0.5)(0.03)+(0.3+0.02)+(0.2)(0.01)}$$

b.

$$P(B)_d = \sum_{j=1}^{3} P(Mj)P(B \setminus Mi)d$$

$$= P(M_1)P(B \setminus M_1)_d + P(M_2)P(B \setminus M_2)_d + P(M_3)P(B \setminus M_3)d$$

$$= \left(\frac{2}{10}\right)\left(\frac{1}{100}\right) + \left(\frac{3}{10}\right)\left(\frac{2}{100}\right) + \left(\frac{5}{10}\right)\left(\frac{3}{100}\right) = \frac{23}{1000}$$

H.w/ C. $P(B)_{nd}$ d. $P(M_1 \setminus B)nd$

ex/ Box 1 has (3) red and (1) white and (2) black balls

box 2 has (1) red and (3) black balls

Toss a dice if (1) or (6) appears choose box 1 , otherwise choose box 2 . Find the pr. that .

a. The ball is chosen from box 2 . If the ball is red .

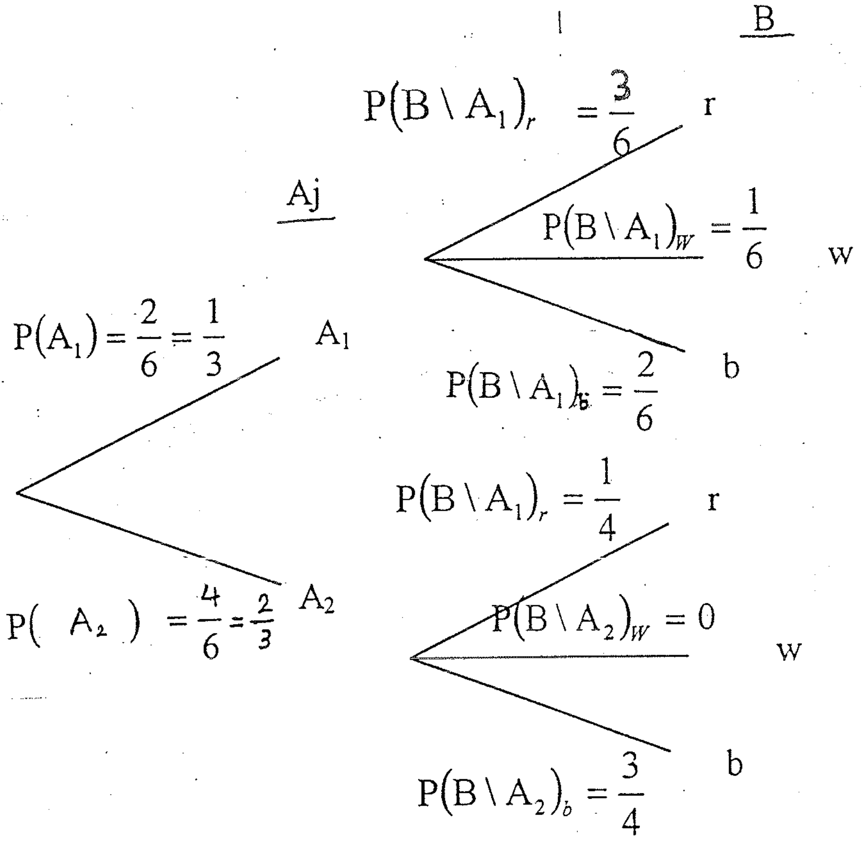
b. A white ball is chosen .

Sol/

A_1 : chose box 1 .

A_2 : chose box 2 .

B : chose a ball .



a.

$$P(A_2 \setminus B)_r = \frac{P(A_2)P(B \setminus A_2)_r}{P(A_1)P(B \setminus A_1)_r + P(A_2)P(B \setminus A_2)_r}$$

$$= \frac{\left(\frac{4}{6}\right)\left(\frac{1}{4}\right)}{\left(\frac{2}{6}\right)\left(\frac{3}{6}\right) + \left(\frac{4}{6}\right)\left(\frac{1}{4}\right)}$$

b.

$$P(B)_W = \sum_{j=1}^{2} P(Aj)r(B \setminus Aj)w$$

$$= \left(\frac{2}{6}\right)\left(\frac{1}{6}\right) + \left(\frac{4}{6}\right)(0)$$

$$= \frac{2}{36} = \frac{1}{18}$$

(Some Questions About Chapter two)

سلّة

Q1: Urn has (8) cards which have the number (1, 2, ..., 8); Choose one Card then find the pr. that Chosen Card have a number which divided by 3 or 4.

Sol.

$A = \{3, 6\} \Rightarrow P(A) = \frac{2}{8} = \frac{1}{4}$

$B = \{4, 8\} \Rightarrow P(B) = \frac{2}{8} = \frac{1}{4}$

Since $AB = \emptyset \Rightarrow A, B$ are disjoint events.

$\therefore P(A \underset{or}{\cup} B) = P(A) + P(B)$ (By Ax.3)

$= \frac{1}{4} + \frac{1}{4}$

$= \frac{1}{2} \in [0, 1]$

Are A and B independent events?

$P(A) \cdot P(B) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{2}$

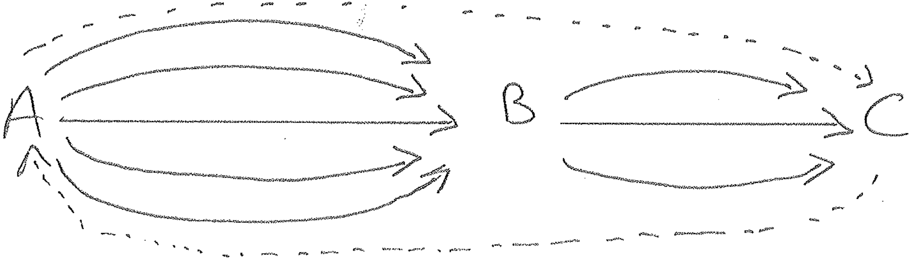
$AB = \emptyset \Rightarrow P(AB) = 0$

$\Big\} \; P(AB) \neq P(A) \cdot P(B)$ $\therefore$ A, B are independent.

OR By Th. (Since A, B are disjoint then A, B are depen.

Q2 If there are five roods from A to B and there are three roods from B to C, then how many ways can one mak roods from A to C Crossing B and return from C to A.

Sol.



$A \longrightarrow C \quad \times \quad A \longleftarrow C$

$15 \qquad\qquad 15$

$P^5_1 \cdot P^3_1 \qquad\qquad P^3_1 \cdot P^5_1$

$5 \times 3 \quad \times \quad 3 \times 5$

$15 \quad \times \quad 15 = (\underline{\underline{225}}$ ways from A)

Q3: How many arrangement can be made of the letter of words (MISSISSIPPI) taken all together?

Sol.

الحرف	التكرار
M	1
I	4
S	4
P	2
	$\Sigma = 11$

$$P^{11}_{1,4,4,2} = \frac{11!}{1!\,4!\,4!\,2!} = 34650$$

①

Q4: Choose a Card from playing Cards, find:

(1) The pr. that a card will be diamond (♦) (A).
(2) The pr. that a card will be a picture (B).
(3) The pr. of getting a Jach (C).
(4) The pr. of getting a queen (D).

Sol.

(1) $P(A) = \frac{C^{13}_1}{C^{52}_1} = \frac{13}{52}$

(2) $P(B) = \frac{C^{12}_1}{C^{52}_1} = \frac{12}{52}$

(3) $P(C) = \frac{C^{4}_1}{C^{52}_1} = \frac{4}{52}$

(4) $P(D) = \frac{C^{4}_1}{C^{52}_1} = \frac{4}{52}$

Q5: Three Cards are drawn at random from deck of (52) Cards, let the events:

A: 2 Cards of diamond
B: one number Card; one Jack and the queen of harts.

Sol. $P(A) = C^{13}_2 \cdot C^{39}_1 / C^{52}_3$

→ follows

Q9: في مدينة ما ، 40% من المواطنين لهم شعر بني اللون ، 25% منهم لهم عيون بنية اللون و 15% لهم شعر بني وعيون بنية اللون ، اختير مواطن بطريقة عشوائية من المدينة .

(i) اذا كان شعره بني فما هو احتمال ان تكون ايضاً عيناه بنيتان ؟

(ii) اذا كانت عينه بنية ، فما هو احتمال ان يكون شعره بنياً ؟

(iii) ماهو احتمال ان لايكون شعره بنياً ولا ان تكون عينه بنية ؟

Sol.

$P(A) = .40$ ، $P(B) = .25$ و $P(AB) = .15$

(i) $P(B|A) = \dfrac{P(AB)}{P(A)} = \dfrac{.15}{.40} = \dfrac{3}{8} \in [0,1]$

(ii) $P(B|A) = \dfrac{P(AB)}{P(B)} = \dfrac{.15}{.25} = \dfrac{3}{5} \in [0,1]$

(iii) $P(A^cB^c) = P((A\cup B)^c) = 1 - P(A\cup B)$ (By Demo. Law & Th. 2)

$P(A\cup B) = P(A) + P(B) - P(AB)$ (A, B are joint).

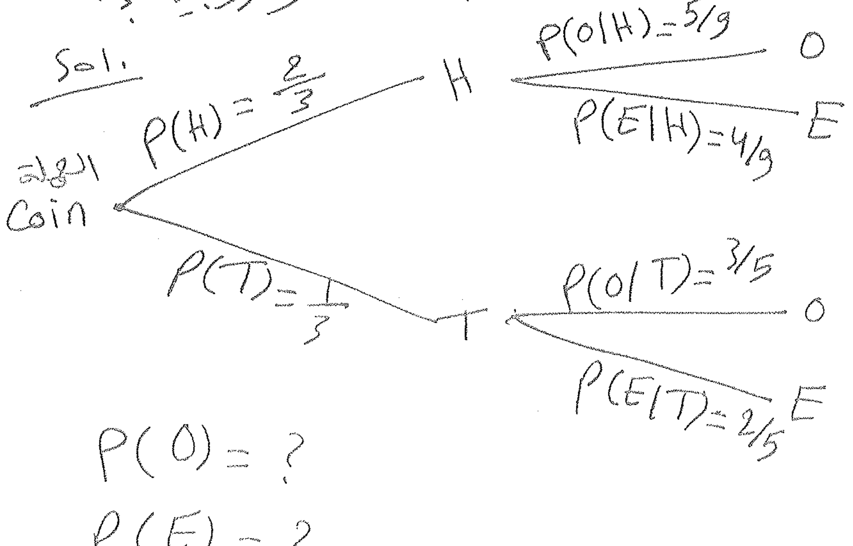
$= .40 + .25 - .15$

$= .50 \Rightarrow 50\%$

$P(A^cB^c) = 1 - P(A\cup B) = 1 - .50 = .50$

Q10: صنعت قطعة نقود بحيث يكون احتمال ظهور الصورة H (2/3) والكتابة T (1/3) ، ألقيت هذه القطعة مرة واحدة . نختار بعد ذلك عدداً عشوائياً من 1 الى 9 اذا ظهرت الصورة . اما اذا ظهرت الكتابة نختار بطريقة عشوائية عدداً من 1 الى 5 . ماهو احتمال ان يكون العدد المختار زوجية ؟

Sol.



$O = \{1, 3, 5, 7, 9\}$

$E = \{2, 4, 6, 8\}$

$O = \{1, 3, 5\}$

$E = \{2, 4\}$

$P(O) = ?$

$P(E) = ?$

$$P(B) = \frac{C_1^{40} \cdot C_1^{4} \cdot C_1^{1} \cdot C_0^{7}}{C_3^{52}}$$

Q6: One card are drawn at random from deck of (52) cards let the events:

A: to get 10.
B: to get diamoned ($\diamondsuit$).
C: to get no. card.

$52 = 13 \times 4$ = العدد الكلي
$12 = 4 \times 3$ = عدد الصور
$40 = 4 \times 10$ = عدد الارقام

sol.

$$P(A) = P(10) = \frac{C_1^{4}}{C_1^{52}}$$

$$P(B) = P(\diamondsuit) = \frac{C_1^{13}}{C_1^{52}}$$

$$P(C) = P(no.) = \frac{C_1^{40}}{C_1^{52}}$$

Q7: Find pr. that:

1. $P(\text{red card}) = \frac{26}{52}$
2. $P(2) = \frac{4}{52}$
3. $P(K) = \frac{4}{12}$
4. $P(\underset{\underset{26}{\downarrow}}{\text{red}}, 2) = \frac{2}{26}$
5. $P(\text{pic.}) = \frac{12}{52}$
6. $P(\underset{\underset{13}{\downarrow}}{\heartsuit}, 7) = \frac{1}{13}$

Q8 اخذ رجل (5) ورقات من اوراق اللعب واحدة بعد الاخرى، ماهو احتمال ان تكون جميع الاوراق من نوع ($\heartsuit$) ؟

$$P(\heartsuit\heartsuit\heartsuit\heartsuit\heartsuit) = \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \cdot \frac{10}{49} \cdot \frac{9}{48}$$

(By Th. 11)

$$P(O) = \frac{2}{3} \cdot \frac{5}{9} + \frac{1}{3} \cdot \frac{3}{5} = \frac{77}{135} \in [0,1]$$

(باستخدام نظرية بيز التمهيدية)

$$P(E) = \frac{2}{3} \cdot \frac{4}{9} + \frac{1}{3} \cdot \frac{2}{5} = \frac{58}{135} \in [0,1]$$

---

⑨11 في احدى الكليات رسب 25٪ من الطلبة في امتحان الرياضيات ورسب : 15٪ من الطلبة في امتحان الكيمياء ورسب 10٪ في امتحان الرياضيات والكيمياء.
اختير أحد الطلبة بطريقة عشوائية :
(i) اذا كان راسباً في الكيمياء . فما هو احتمال ان يكون راسباً في الرياضيات ؟
(ii) اذا كان راسباً في الرياضيات . فما هو احتمال ان يكون راسباً في الكيمياء ؟
(iii) ماهو احتمال ان يكون راسباً في الرياضيات أو الكيمياء ؟

Sol.

$M = \{$ الطلبة الراسبون في الرياضيات $\}$, $C = \{$ الطلبة الراسبون في الكيمياء $\}$

$MC = \{$ الطلبة الراسبون في الرياضيات والكيمياء $\}$

$$P(M) = .25 \text{ , } P(C) = .15 \text{ , } P(MC) = .10$$

$$(i)\quad P(M|C) = \frac{P(MC)}{P(C)} = \frac{.10}{.15} = \frac{2}{3} \in [0,1]$$

$$(ii)\quad P(C|M) = \frac{P(MC)}{P(M)} = \frac{.10}{.25} = \frac{2}{5} \in [0,1]$$

$$(iii)\quad P(M \cup C) = P(M) + P(C) - P(MC) = \frac{3}{10}$$

$= .30$

Q12: حادثتان A, B بحيث ان $\left(P(A|B) = \frac{1}{4}\right)$ وان $\left(P(B|A) = \frac{1}{2}\right)$ و $P(A)$ $= \frac{1}{4}$ ، اذكر هل العبارات التالية صحيحة ؟

① A, B are independent ?

② IS $A \subset B$?

Sol. $$P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(AB)}{1/4} = \frac{1}{2}$$

$$\Rightarrow \frac{1}{4} \cdot \frac{1}{2} = P(AB) \Rightarrow \boxed{P(AB) = \frac{1}{8}}$$

①

$\Rightarrow$ fallows

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{1/8}{P(B)} = \frac{1}{4}$$

$$\Rightarrow \frac{\frac{1}{8}}{1/4} = P(B) \Rightarrow P(B) = \frac{1}{\cancel{8}} \cdot \cancel{4} = \frac{1}{2}$$

$$\therefore \boxed{P(B) = \frac{1}{2}}$$

(لا يمكن باستخدام النظرية)

(1) $P(AB) = P(A)P(B)$, $\frac{1}{8} = \frac{1}{4} \cdot \frac{1}{2}$ $\Big\} \Rightarrow$ A, B are independent events.

(2) By Th. 4 If $A \subseteq B \Rightarrow P(A) \leq P(B)$

$$P(A) = \frac{1}{4} \quad , P(B) = \frac{1}{2}$$

$$P(A) \leq P(B)$$

$$\frac{1}{4} \leq \frac{1}{2}$$

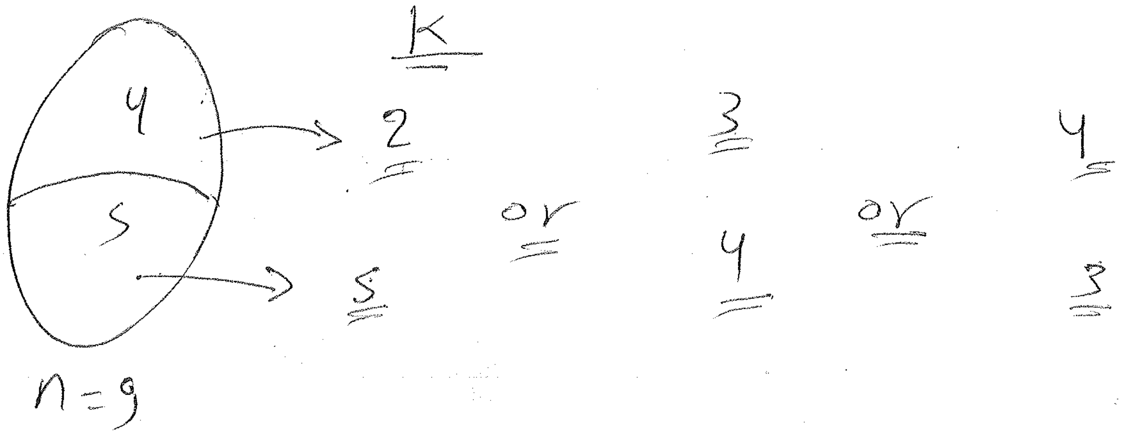
$$\Rightarrow A \subseteq B$$

Q13: A student is to answer (7) out of (9) questions on an exam. Find the pr. that he must answer at least (2) of the first (4) questions.

Sol. ①

Let W: be a student answer at least (2) of the first (4) questions.

$$P(W) = \frac{C_2^4 C_5^5 + C_3^4 C_4^5 + C_4^4 C_3^5}{C_7^9} \in [0,1]$$



Chapter Two
Introduction to Probability

- Exercises -

(PP.14)

For any events A and B, Show that:

1. $P(AB) \leqslant P(A) \leqslant P(A \cup B) \leqslant P(A) + P(B)$

Proof:

$\because AB \subset A$ (By Fact)

$\therefore P(AB) \leqslant P(A)$ (By Th.4) ... ①

$\because A \subset A \cup B$ (By Fact)

$\therefore P(A) \leqslant P(A \cup B)$ (By Th.4) ... ②

$\because$ A, B are joint events (since $P(AB) \neq \emptyset$)

$\Rightarrow P(A \cup B) = P(A) + P(B) - P(AB)$ (By Th.3)

$\Rightarrow P(AB) = P(A) + P(B) - P(A \cup B)$

But $0 \leqslant P(AB) \leqslant 1$ (By Axiom 1)

$\Longrightarrow 0 \leqslant P(A) + P(B) - P(A \cup B) \leqslant 1$

$\Longrightarrow P(A) + P(B) - P(A \cup B) \geqslant 0$

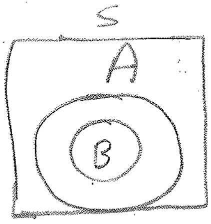
$\Longrightarrow P(A) + P(B) \geqslant P(A \cup B)$

i.e. $P(A \cup B) \leqslant P(A) + P(B)$ ... ③

By ① + ② + ③ we get:

$P(AB) \leqslant P(A) \leqslant P(A \cup B) \leqslant P(A) + P(B)$

. If A and B are joint events, when $P(A) = 0.8$, $P(B) = 0.5$. Find the Conditions and the Value of Max $P(AB)$ and Min $P(AB)$.



Sol. $\because$ A and B are joint.

$\Longrightarrow \quad AB \neq \emptyset$

Case ① If $B \subseteq A$ (Since $P(B) \leq P(A)$ / By Th. 4)

$B \subseteq A \Longrightarrow AB = B$

$\Longrightarrow P(AB) = P(B) = 0.5$

Case ② If $B \not\subseteq A$ & A, B are joint events;

$P(A \cup B) = P(A) + P(B) - P(AB)$ (By Th. 3)

$= 0.8 + 0.5 - P(AB)$

$P(AB) = 1.3 - P(A \cup B) \geq 0$ (By Ax. 1)

$0 \leq P(A \cup B) \leq 1$ (By Axiom 1)

$0 \geq -P(A \cup B) \geq -1$

$1.3 \geq 1.3 - P(A \cup B) \geq -1 + 1.3$

$1.3 \geq 1.3 - P(A \cup B) \geq 0.3$

$1.3 \geq P(AB) \geq 0.3 \Longrightarrow P(AB) \geq 0.3$

$\therefore$ If $A \subset B \Longrightarrow$ Max $P(AB) = 0.5$

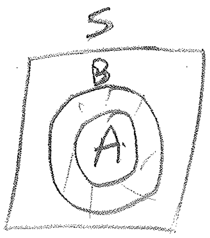
If $A \not\subset B \Longrightarrow$ Min $P(AB) \geq 0.3$

3. If $P(A) = \frac{1}{3}$ & $P(B) = \frac{1}{2}$. Find the Value of $P(BA^c)$ When:

a. A & B are disjoint events

b. $A \subseteq B$

c. $P(AB) = \frac{1}{8}$

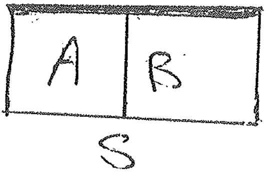


$A \subseteq B$

Sol.

a. If A and B are disjoint.

$\therefore$ A, B are disjoint

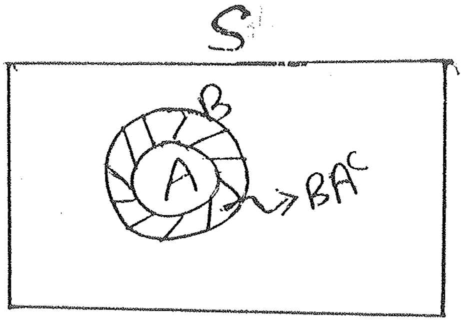


$P(BA^c) = P(B) = \frac{1}{2}$

b. If $A \subseteq B$

$B = A \cup BA^c$

where A & BA^c are disjoint events.



$P(B) = P(A) + P(BA^c)$ (By AX. 3)

$\frac{1}{2} = \frac{1}{3} + P(BA^c)$

$P(BA^c) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

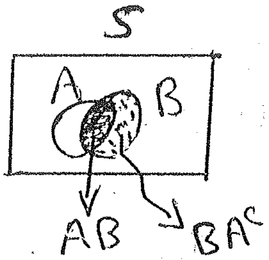
c. $P(AB) = \frac{1}{8} \Rightarrow P(AB) \neq 0 \Rightarrow AB \neq \emptyset$ (By Th. 1).

$B = AB \cup BA^c$ (when AB, BA^c are disjoint events)

$P(B) = P(AB) + P(BA^c)$ (By Ax. 3)

$\frac{1}{2} = \frac{1}{8} + P(BA^c)$

$\Rightarrow P(BA^c) = \frac{3}{8} \in [0,1]$



4. If A, B and C are disjoint events, find:

1. $P[(A \cup B) \cap C] = P[AC \cup BC] = P(\emptyset \cup \emptyset)$ (since $AB = \emptyset$, $AC = \emptyset$, $BC = \emptyset$, A, B, C are disj.) (By Th. 1)

$= P(\emptyset)$

$= 0$

2. $P(A^c \cup B^c) = P[(AB)^c] = P(\emptyset)^c = P(S) = 1$

(By Demo. Lows) By $AB = \emptyset$ (A, B are disj. events) (By AX. 2)

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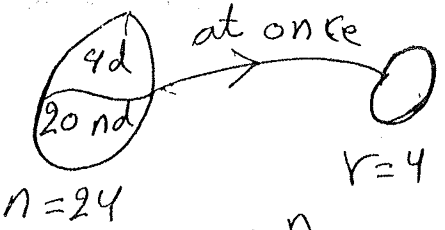
- Exercises -

1. A box has (24) bulbs of which (4) are defective. Choose (4) bulbs, find the pr. that they are defective.

Sol.



$n(S) = C^{n}_{r} = C^{24}_{4}$ (choose at once)

Let E : be defective bulbs.

$n(E) = \overset{=n}{C^{4}_{4}} \cdot \underset{nd=r}{C^{20}_{0}} = 1 . 1 = 1$ (Since $C^{n}_{n} = 1$, $C^{n}_{0} = 1$)

$P(E) = \frac{n(E)}{n(S)} = \frac{C^{4}_{4} \cdot C^{20}_{0}}{C^{24}_{4}} = \frac{1}{C^{24}_{4}} \in [0,1]$.

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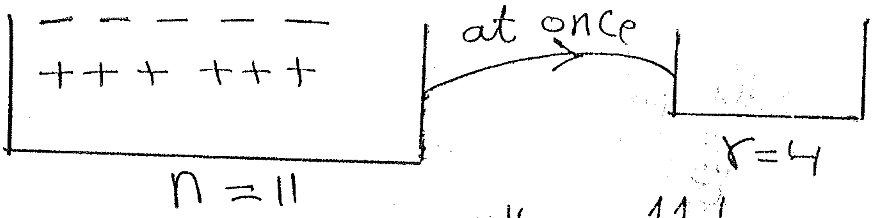
2. A set of (11) integers; (5) of them are negative and the others are positive. Choose a sample of (4) integers and multiply them, then find the pr. that the product is:

a. negative b. positive.

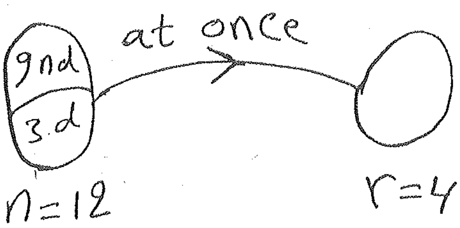
Sol.



$n(S) = C^{n}_{r} = C^{11}_{4} = \frac{11!}{4!\,7!} = 330$

∴ S has (330) samples.

a. Let A be the sample that the product multible of (4) chosen integers is positive.

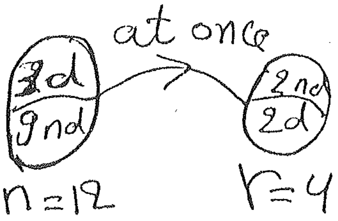


$$n(S) = C^{12}_{4} = \frac{12!}{4!\,(12-4)!}$$
$$= 495 \text{ samples}$$

ⓐ Let A: be a sample has 2d (2-defective) and 2nd (2-not defective).

$$n(A) = C^{3}_{2}\, C^{9}_{2}$$

$$\therefore P(A) = \frac{C^{3}_{2}\, C^{9}_{2}}{C^{12}_{4}}$$



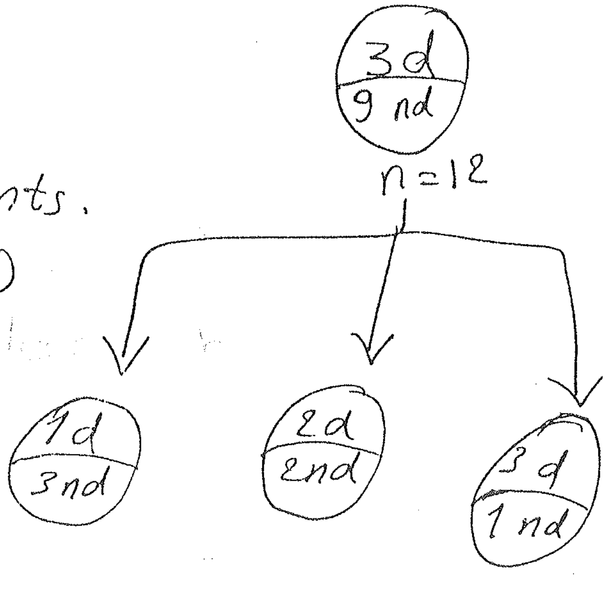
ⓑ Let B: be the sample that has at least one defective than.

$$B = B_1 \cup B_2 \cup B_3$$

B_1, B_2, B_3 are disjoint events.

$$\therefore P(B) = P(B_1) + P(B_2) + P(B_3)$$

$$P(B_1) = C^{3}_{1}\, C^{9}_{3} / C^{12}_{4}$$
$$P(B_2) = C^{3}_{2}\, C^{9}_{2} / C^{12}_{4}$$
$$P(B_3) = C^{3}_{3}\, C^{9}_{1} / C^{12}_{4}$$



$$P(B) = \frac{C^{3}_{1} C^{9}_{3} + C^{3}_{2} C^{9}_{2} + C^{3}_{3} C^{9}_{1}}{C^{12}_{4}} \in [0,1]$$

PP.(42) — Exercises —

① Given (9) Coins. (2) Coins have (H) on one side and (T) on the other, (3) Coins have (H) on both side and (4) Coins have (T) on both side. Choose a Coin and find the pr. that H happears.

i.e $+\cdot+\cdot+\cdot+ = + \rightarrow A_1$
$+\cdot-\cdot- = + \rightarrow A_2$
$\cdot-\cdot-\cdot- = + \rightarrow A_3$

$A = A_1 \cup A_2 \cup A_3$ (3– Cases)
& A_1, A_2 and A_3 are disjoint events.

$\therefore P(A) = P(A_1) + P(A_2) + P(A_3)$

$$P(A_1) = \frac{{}^{+}C^{6}_{4}\; C^{5}_{0}{}^{-}}{C^{11}_{4}} = \frac{15}{330} \in [0,1]$$

$$P(A_2) = \frac{{}^{+}C^{6}_{2}\; C^{5}_{2}{}^{-}}{C^{11}_{4}} = \frac{150}{330} \in [0,1]$$

$$P(A_3) = \frac{{}^{+}C^{6}_{0}\; C^{5}_{4}{}^{-}}{C^{11}_{4}} = \frac{5}{330} \in [0,1]$$

$$\therefore P(A) = \frac{15+150+5}{330} = \frac{170}{330} = \frac{17}{33} \in [0,1]$$

b. Let B: be the sample that the Product multiple of (4) Chosen integer is negative.

i.e. $+\cdot+\cdot+\cdot- = - \rightarrow B_1$
$+\cdot-\cdot-\cdot- = - \rightarrow B_2$

$B = B_1 \cup B_2$
B_1, B_2 are disjoint events

$\therefore P(B) = P(B_1) + P(B_2)$

OR since the product is positive or negative, then;

$$P(B) = P(A^c) = 1 - P(A) \quad \text{(By Th.2)}$$
$$= 1 - \frac{17}{33}$$
$$= \frac{16}{33}$$

S
$\boxed{+}\boxed{-}$
· = product

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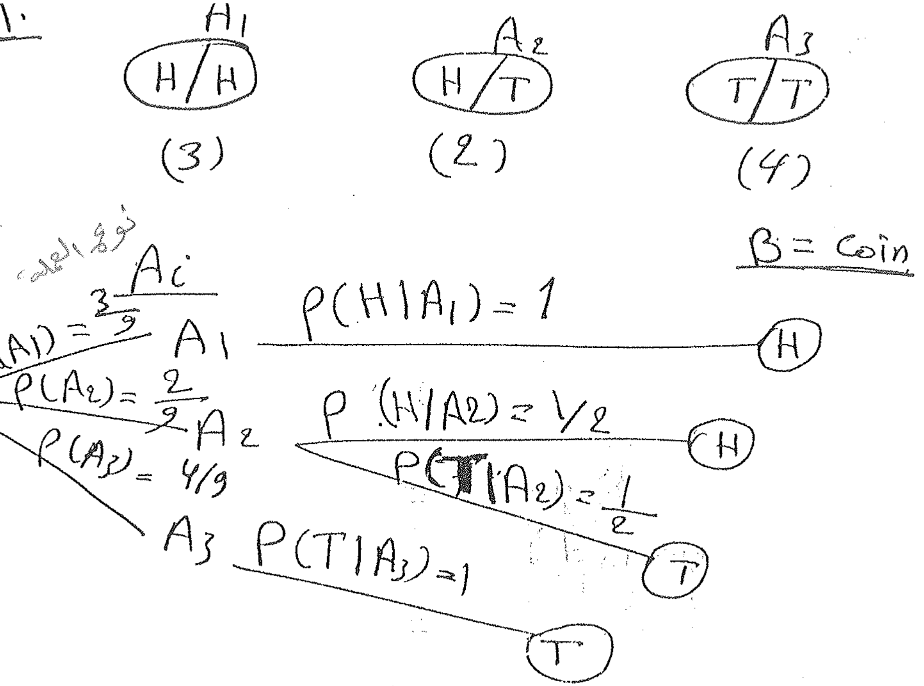
③ Given a set of (12) transistors of which (3) are defective. Choose a sample of (4) transistors, then find the pr. that:

a. Two transistors are defective
b. At least one transistors is defective.

①

Sol.

Sol.



Let B: be a coin to get (H)

$$P(\text{to get } H) = P[(A_1 H) \cup (A_2 H)]$$
$$= P(A_1 H) + P(A_2 H)$$
$$= P(A_1) \cdot P(H|A_1) + P(A_2) \cdot P(H|A_2)$$
$$= \left(\frac{3}{9}\right) \cdot (1) + \left(\frac{2}{9}\right)\left(\frac{1}{2}\right)$$
$$= \frac{4}{9} \in [0,1]$$

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② Choose a sample of (5) balls from (6) white and (10) black balls one by one with out replace. Find the Pr. to get a sample (w, w, b, w, b).

Sol.
$$\underset{n=16}{\boxed{\begin{matrix} 6 & w \\ 10 & b \end{matrix}}} \xrightarrow[\text{with rep.}]{(1-1)} \underset{r=5}{\boxed{\quad}}$$

$$P(w_1 w_2 b_1 w_3 b_2) = ? \quad (\text{By multip. rule})$$

$$P(w_1 w_2 b_1 w_3 b_2) = P(w_1) \cdot P(w_2|w_1) \cdot P(b_1|w_1 w_2) \cdot P(w_3|w_1 w_2 b_1) \cdot P(b_2|w_1 w_2 b_1 w_3) \cdot$$
$$= \frac{6}{16} \cdot \frac{5}{15} \cdot \frac{10}{14} \cdot \frac{4}{13} \cdot \frac{9}{12} \in [0,1]$$

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①

Given (one) blue card and (4) red cards which are named A, B, C and D, choose (2) cards one by one without replace. Find the pr. that:

a- both cards are red, given that card A is chosen.

b. both cards are red, given one red card is chosen.

Sol.

$$\underset{n_1}{\boxed{\text{blue}}} + \underset{n_2}{\boxed{A|B|C|D}} \longrightarrow \underset{n}{\boxed{n_1+n_2}}$$

$$n(S) = P_2^5 = \frac{5!}{3!} = \frac{5\times4\times\cancel{3!}}{\cancel{3!}} = 5\times4 = 20 = n(S)$$

a- $\underbrace{\text{Both cards are red}}_{E}$, given that $\underbrace{\text{card A is chosen}}_{F}$.

$$E = \left\{\begin{array}{llll}(A,B), & (B,A), & (C,A), & (D,A) \\ (A,C), & (B,C), & (C,B), & (D,B) \\ (A,D), & (B,D), & (C,D), & (D,C)\end{array}\right\} \to n(E) = 12$$

$$P(E) = \frac{12}{20} \in [0,1]$$

$$F = \left\{\begin{array}{ll}(A,b), & (b,A) \\ (A,B), & (B,A) \\ (A,C), & (C,A) \\ (A,D), & (D,A)\end{array}\right\} \to n(F) = 8$$

$$P(F) = \frac{8}{20} \in [0,1]$$

$$EF = \left\{\begin{array}{ll}(A,B), & (B,A) \\ (A,C), & (C,A) \\ (A,D), & (D,A)\end{array}\right\} \to n(EF) = 6$$

$$P(EF) = \frac{6}{20} \in [0,1]$$

→

L. Side $\binom{n+1}{r} = \frac{(n+1)!}{r!((n+1)-r)!}$

*

$$= \frac{(n+1)\,n!}{r!((n+1)-r)!}$$

$$= \frac{n!\,(r+n-r+1)}{r!\,(n-r+1)!}$$

$$= \frac{rn! + n!(n-r+1)}{r!\,(n+1-r)!}$$

$$= \frac{rn!}{r!(n+1-r)!} + \frac{n!(n-r+1)}{r!(n+1-r)!}$$

$$= \frac{\cancel{r}\,n!}{\cancel{r}(r-1)!(n+1-r)!} + \frac{\cancel{(n-r+1)}\,n!}{r!\cancel{((n+1)-r)}(n-r)!}$$

$$= \frac{n!}{(r-1)!\,(n-(r-1))!} + \frac{n!}{r!(n-r)!}$$

$$= \binom{n}{r-1} + \binom{n}{r}$$

$$\therefore \quad \binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{\frac{6}{20}}{\frac{8}{20}} = \frac{6}{8} = \frac{3}{4} \in [0,1]$$

b- $\underbrace{\text{Both cards are red}}_{E}$, given $\underbrace{\text{one red card is chosen}}_{G}$.

$$G = \left\{ \begin{matrix} (A,b), & (b,A) \\ (B,b), & (b,B) \\ (C,b), & (b,C) \\ (D,b), & (b,D) \end{matrix} \right\} \rightarrow n(G) = 8$$

$$P(G) = \frac{8}{20} \in [0,1]$$

$$EG = \emptyset \rightarrow P(EG) = P(\emptyset) = 0$$

$$P(E|G) = \frac{P(EG)}{P(G)} = \frac{0}{8/20} = 0 \in [0,1]$$

Ex. Prove that: $\underline{PP. 10}$

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

$\underline{\text{sol.}}$

$\underline{\text{R. Side}}$ $\binom{n}{r-1} + \binom{n}{r} =$

$$= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}$$

$$= \frac{n!}{r(r-1)!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)(n-r)!}$$

$$= n! \left[\frac{(n-\cancel{r}+1)+\cancel{r}}{r(r-1)!(n-r+1)(n-r)!} \right]$$

$$= \frac{(n+1)n!}{r!(n-r+1)!} = \frac{(n+1)!}{r!(n-r+1)!} = \frac{(n+1)!}{r!((n+1)-r)!}$$

$$= \binom{n+1}{r} = \underline{\underline{L.S}}$$

$$P(B)_W = \sum_{j=1}^{2} P(Aj)P(B \setminus Aj)_w$$

$$= \left(\frac{X}{X+Y}\right)\left(\frac{V}{U+V+1}\right) + \left(\frac{Y}{X+Y}\right)\left(\frac{V+1}{V+U+1}\right)$$

b.

$$P(B)_r = \sum_{i=1}^{2} P(Aj)P(B \setminus Aj)_r = \left(\frac{X}{X+Y}\right)\left(\frac{U+1}{U+V+1}\right) + \left(\frac{Y}{Y+X}\right)\left(\frac{U}{U+V+1}\right)$$

Exercises :

1. Given (9) coins .

2. coins have (H) on one side and (T) on the other .

3. coins have (H) on both side .

4. coins have (T) on both side .

Choose a coin and find the pr. That H appears .

2. Choose a sample of (5) balls from (6) white and (10) black balls one by one with out replace . Find the pr. To get a sample (w , w , b , w , b) ([illegible])

3. Given (one) blue card and (4) red cards which are named A, B , C , D choose (2) cards one by one without replace .find the pr. that .

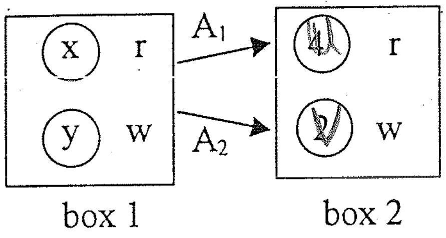
a. both cards are red , given that card A is chosen .

b. both cars are red given one red card is chosen .

ex/ A box 1 has (x) red and (y) white balls and box 2 has (u) red and (V) white balls .Where $(x, y, u, V \geq 2)$ choose a ball from box 1 and put it into box 2 , then choose a ball from box 2 .

Find the pr. That the chosen ball from box 2 is a. white b. red

Sol.



EX: [illegible]

Roll adice / [illegible] box [illegible]

اذا ظهر [illegible] 1,2,3,4 فاسحب الصندوق 1

5, 6 فاختار الصندوق

box I box II

$p(A_1) = \frac{4}{6}$ $p(A_2) = \frac{2}{6}$

A1	A2
5r 4b	4r 3b

$n_1 = 9, n_2 = 7$

$p(B)_r = \sum p(A_i) p(B/A_i)$

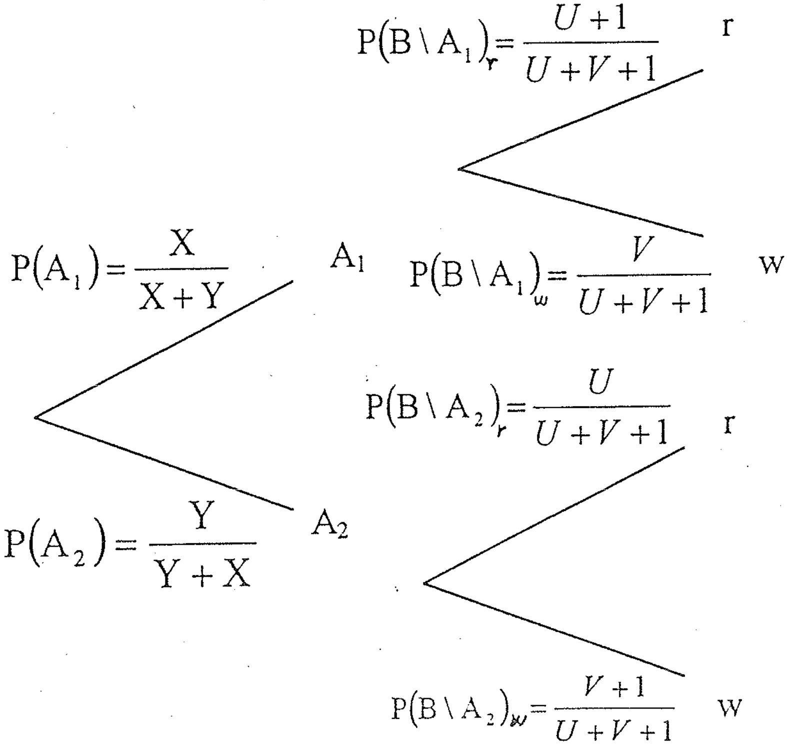
$= \left(\frac{4}{6}\right)\left(\frac{5}{6}\right) + \left(\frac{2}{6}\right)\left(\frac{4}{7}\right)$

Let

A_1 : choose red ball from box 1

A_2 : choose white ball from box 1

B : choose a ball from box 2



$$P(A_1) = \frac{X}{X+Y}$$

$$P(A_2) = \frac{Y}{Y+X}$$

$$P(B \setminus A_1)_r = \frac{U+1}{U+V+1}$$

$$P(B \setminus A_1)_w = \frac{V}{U+V+1}$$

$$P(B \setminus A_2)_r = \frac{U}{U+V+1}$$

$$P(B \setminus A_2)_w = \frac{V+1}{U+V+1}$$

" In Playing Cards "

Example (1) : How many 5 – Cards will have 3 aces and 2 kin

sol. $n = C_3^4 \cdot C_2^4$

aces(A) ↗ King ↗

How many 5-cards will have 3 hearts and 2 spaces ?

$n = C_3^{13} \cdot C_2^{13}$

hearts(♡) ↗ spaces (♠) ↗

Note : Ace (A) , Heart (♡) , Club (♣) , Diamond(◇) , Spade(

Jack (J) , Queen (Q) , King (K) = the face Cards (Pic.)

These are standard deck of (52) cards has four suits (

♡, ♣, ◇, ♠ with (13) cards in each suit .

(◇, ♡) → red cards (26) & (♣, ♠) → black cards (2

Example ②:

Let E = the drawn card is a spade (♠) .

F = the drawn card is a face card . (صور)

Let G = the drawn card is a heart (♡) .

H = the drawn card is a club (♣) .

① Are E & F independent events ?

② Are G & H independent events ?

sol. ① $P(E) = \left(\frac{13}{52}\right)$, $P(F) = \left(\frac{12}{52}\right)$

$P(E \cap F) = P(J♠, Q♠, K♠) = \frac{3}{52}$

$P(E \cap F) = \frac{3}{52} = P(E)\,P(F) = \left(\frac{13}{52}\right) \cdot \left(\frac{12}{52}\right) = \frac{3}{52}$

Then E & F are indep. events .

② $P(G) = \frac{13}{52}$, $P(H) = \frac{13}{52}$, $G \cap H = \phi$

$\therefore P(G \cap H) = P(\phi) = 0$ (by Th.1)

$$P(G)\,P(H) = \left(\frac{13}{52}\right)\left(\frac{13}{52}\right) = \frac{1}{16} \neq P(GH) = 0$$

Example (3): A single card is drawn from a standard 52-card deck. Test the following events for independence:

Ⓐ E = the drawn card is a red card.
F = the drawn card's number is divisible by 5.

Ⓑ G = the drawn card is a king.
H = the drawn card is a queen.

① Are E & F indep. events?
② Are G & H indep. events?

H.W.

Example (4): In a single card is drawn from a standard 52-card deck. Find the pr. of each of the following events.

① $P(\text{ace}) = \frac{4}{52}$
$\downarrow$
$\{♡A, ◇A, ♣A, ♠A\}$

② $P(\text{face card}) = \frac{12}{52}$

③ $P(\text{spade}) = \frac{13}{52}$

④ $P(\text{spade or heart}) = \frac{13}{52} + \frac{13}{52} = \frac{26}{52}$

⑤ $P(\text{red card}) = \frac{26}{52}$,
$\downarrow$
(♡ & ◇)
13 13

⑥ $P(\text{red or face}) = \frac{26}{52} + \frac{12}{52} - \frac{6}{52}$
↗ (face ∩ Red) تقاطع
(J, K, Q (2 red) = 6)

Example (5): Find the pr. of randomly drawing two aces from an ordinary deck of 52 playing cards; if we sample: (a) without replacement. (b) with replacement.

Sol. (a) $P = \frac{4}{52} \cdot \frac{3}{51} = P(AA)$ (by multiplication rule)

اول كارد ↗ ثاني كارد ↗

(b) $P = \frac{4}{52} \cdot \frac{4}{52} = P(AA)$ (" " ").

(c) at once $\longrightarrow$ $P = \frac{C_2^4}{C_2^{52}} \underset{\underline{\underline{2}}}{=} P(2A)$

Example (1): If a coin is tossing twice.

$S = \{HH, HT, TH, TT\}$

and Let:

A = a head on the first toss $= \{HH, HT\}$

B = a head on the second toss $= \{HH, TH\}$

Are A & B independent events?

Sol. $P(A) = \frac{2}{4} = \frac{1}{2}$, $P(B) = \frac{2}{4} = \frac{1}{2}$

$AB = \{TH\} \rightarrow P(AB) = \frac{1}{4}$

$P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = P(A \cup B)$

$\therefore$ A & B are indep. events.

Example (2): Roll a die once.

$S = \{1, 2, 3, 4, 5, 6\}$

Find the pr. that:

H.W.

E_1 : The die shows an event number.

E_2 : The die shows a (1).

E_3 : The die shows a مضاعفات multiple of (3) $= \{3, 6\}$

E_4 : The die shows a number less than (5) $= \{1, 2, 3, 4\}$

E_5 : The die shows a number (7) $= \phi$

E_6 : The die shows a number less than (10).
$= \{1, 2, 3, 4, 5, 6\} = S$

Example (3): Rolling a die twice. Find the pr. that the مجموع sum of the numbers rolled is greater than (3).

Sol. $P(\text{sum} > 3) = ?$

$$P(\text{sum} \leq 3) = P(\text{sum is } 2) + P(\text{sum is } 3)$$

$$= \left(\underset{\{(1,1)\}}{\frac{1}{36}} + \underset{\{(2,1),(1,2)\}}{\frac{2}{26}}\right) = \frac{3}{36} = \frac{1}{12}$$

$$\therefore P(\text{sum} > 3) = 1 - P(\text{sum} \leq 3) \qquad (\text{by Th. } P(A) = 1 - P(\bar{A}))$$

$$= 1 - \frac{1}{12} = \frac{11}{12}.$$

Example (4): Suppose that two dice are rolled.

(A) what is the pr. that a sum of (7) or (11) turns up? (تظهر)

(B) what is the pr. that both dice turn up the same or that a sum less than (5) turns up?.

Sol.

Ⓐ $A = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\} \rightarrow n(A) = 6$

$B = \{(5,6), (6,5)\} \rightarrow n(B) = 2$

$n(S) = 6^2 = 36$, $AB = \varphi$ (A & B are disjoint).

$$\therefore P(A \cup B) = P(A) + P(B)$$

$$= \frac{6}{36} + \frac{2}{36} = \frac{8}{36} = \frac{2}{9}$$

Ⓑ $A = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$

$B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (3,1)\}$

$AB = \{(1,1)(2,2)\} \longrightarrow$ A & B are joint events.

$$P(A \cup B) = P(A) + P(B) - P(AB)$$

$$= \frac{6}{36} + \frac{6}{36} - \frac{2}{36} = \frac{10}{36} = \frac{5}{18}$$

Now; Ⓒ what is the pr. that a sum of (2) or (3) turns up? .

Ⓓ what is the pr. that both dice turn up the same or that a sum greater than (8) turns up? (H.w.)

Example (5): What is the pr. that a number selected at random from the first (500) positive integers is (exactly) divided by (3) or (4)?

$A: \frac{500}{3} = 166$, $B: \frac{500}{4} = 125$

AB : the largest integer less than or equal to $\frac{500}{12} = 41$

$$\therefore P(A \cup B) = P(A) + P(B) - P(AB)$$

$$= \frac{166}{500} + \frac{125}{500} - \frac{41}{500} = .5$$

Example (6): what is the pr. that a number selected at random from the first (140) positive integer is (exactly) divedid by (4) or (6)? (H.w.)

كلية معينة راسبون في الرياضيات

Example (7): In a certain College, 25% of the students failed mathematics, 15% of the students failed Chemistry, and 10% of the students failed both mathematics and Chemistry. A student is selected at random.

والرسوب في الكيمياء

① If the student failed chemistry, what is the pr. that he failed mathematics?

(ii) If he failed mathematics, what is the pr. that he failed chem.?

(iii) What is the pr. that he failed math. or Chemistry?

Sol. Let $M = \{\text{students who failed math.}\}$
$C = \{\text{students who failed chem.}\}$

$P(M) = .25$, $P(C) = .15$, $P(MC) = .10$

① $P(M|C) = \dfrac{P(MC)}{P(C)} = \dfrac{.10}{.15} = \dfrac{2}{3}$

(ii) $P(C/M) = \dfrac{P(MC)}{P(M)} = \dfrac{.10}{.25} = \dfrac{2}{5}$

(iii) $P(M \cup C) = P(M) + P(C) - P(MC)$
$= .25 + .15 - .10 = .30 = \dfrac{3}{10} = 30\%$

Example (8): Rolling a die twice. Find the pr. that:

(a) the first die shows a (2) or the sum of the results is (6) or (7).

Sol. $P(A) = \dfrac{6}{36}$, $P(B) = \dfrac{11}{36}$, $P(AB) = \dfrac{2}{36}$ s.t.

$A = \{(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)\}$
$B = \{(5,1), (5,2), (6,1), (4,2), (4,3), (3,3), (3,4), (2,4), (2,5), (1,5), (1,6)\}$.

$\therefore P(A \cup B) = P(A) + P(B) - P(AB)$ (by Th. 3)
$= \dfrac{6}{36} + \dfrac{11}{36} - \dfrac{2}{36}$ (Since A & B are joint events.)
$= \dfrac{15}{36} = \dfrac{5}{12} \in [0,1]$

(b) The sum of the results is 11, or the second die shows a (5).

Sol. $P(\text{sum is } 11) = \dfrac{2}{36}$, $P(2^{nd} \text{ die shows a}(5)) = \dfrac{6}{36}$.

يتبع

P(sum is 11 <u>and</u> 2^{nd} die shows a (5)) $= \frac{1}{36}$

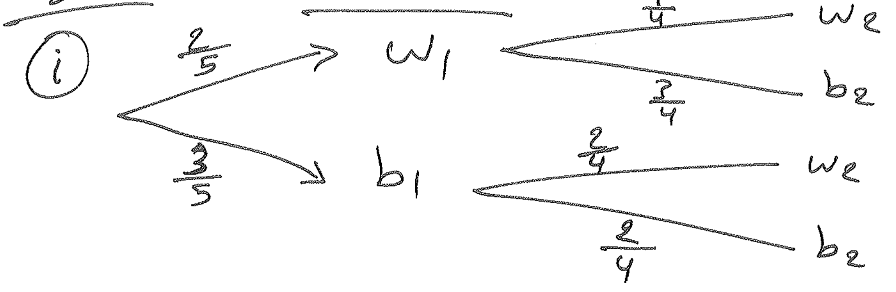
P(sum is 11 <u>or</u> 2^{nd} die shows a (5)) $= \frac{2}{36} + \frac{6}{36} - \frac{1}{36}$

$= \frac{7}{36}$

Example (9): Two balls are drawn without replacement, from a box containing (3) blue and (2) white balls. What is the pr. of drawing a white ball on the 2^{nd} draw?

<u>Sol.</u>

(i) <u>First draw</u>



sum =1 <u>Second draw</u>

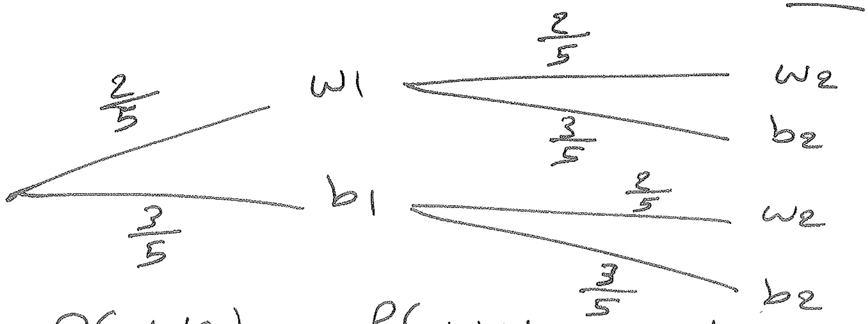
sum =1

$P(w_2) = P(w_1 w_2) + P(b_1 w_2)$ ← by (نظرية بيز التمهيدية) (Th. 12)

$= (\frac{2}{5})(\frac{1}{4}) + (\frac{3}{5})(\frac{2}{4})$

$= \frac{1}{10} + \frac{3}{10} = \frac{4}{10} = \frac{2}{5} \in [0,1]$

(ii) If two balls are drawn <u>with</u> replacement?



$P(w_2) = P(w_1 w_2) + P(b_1, w_2)$ (by Th. 12) (نظرية بيز التمهيدية)

$= (\frac{2}{5})(\frac{2}{5}) + (\frac{3}{5})(\frac{2}{5})$

$= (.16) + (.24)$

$= .40$

<u>المطلوب</u> : ان تكون الكرة بيضاء (w) ومن السحبة (2) : w_2 ←

#

س 1 : اذا كان احتمال ان يعيش رجل (١٠) سنوات اخرى هو $(\frac{1}{4})$.
واحتمال ان تعيش زوجته (١٠) سنوات اخرى هو $(\frac{1}{3})$. جد احتمال :

① ان يعيش الاثنان (١٠) سنوات اخرى .
② ان يعيش احدهما على الاقل (١٠) سنوات اخرى .
③ ان يموت الاثنان خلال السنوات العشر
④ ان تعيش الزوجة (١٠) سنوات (ويموت الرجل) .
⑤ ان يعيش احدهما على الاكثر (١٠) سنوات اخرى .

Sol.

A : ان يعيش الرجل (١٠) سنوات اخرى .
B : ان تعيش زوجته (١٠) سنوات اخرى .
(الحوادث مستقلة indep.)

$P(A) = \frac{1}{4}$ ، $P(B) = \frac{1}{3}$

① $P(AB) = P(A)P(B)$ (by def. of independent events)
$= (\frac{1}{4})(\frac{1}{3}) = \frac{1}{12}$

② $P(A\cup B) = P(A) + P(B) - \underline{P(AB)}$ يعيشان كلاهما
$= (\frac{1}{4} + \frac{1}{3} - \frac{1}{12})$

③ $P(A^cB^c) = 1 - P(A\cup B)$ by ($A^cB^c = (A\cup B)^c$ Demorgan law)
$= 1 -$

④ $P(A^cB) = P(A^c) \cdot P(B)$ by Th. (Th. 10)
$= \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4}$ ($P(A^c) = 1 - P(A) = 1 - \frac{1}{4} = \frac{3}{4}$)

⑤ $P = P(A^cB) \underset{\text{أو}}{+} P(AB^c) \underset{\text{أو}}{+} P(A^cB^c)$

تعيش المرأة ويموت الرجل
يعيش الرجل وتموت المرأة
يموت الاثنان

$P(B^c) = 1 - P(B) = 1 - \frac{1}{3} = \frac{2}{3}$

Complete ---

مثال :اختيرت ورقة من اوراق العب البالغة (٥٢) ورقة جد

أ) احتمال الورقة التي تحمل الرقم ١٠

ب)احتمال ان تكون الورقة نوع ◊

ج)اذا اختيرت (٤) اوراق , ما احتمال ان تكون احدهم J .

د) اذا اختيرت (٣) ورقات ما احتمال ان تكون احدهم 3 والاخرى صورة والثالثة أي ورقة اخرى من اوراق اللعب .

الحل :- أ)

$$\boxed{P(10)=\frac{\binom{4}{1}}{\binom{52}{1}}=\frac{4}{52}=\frac{1}{13}}$$

ب)

$$\boxed{P(\Diamond)=\frac{\binom{13}{1}}{\binom{52}{1}}=\frac{13}{52}=\frac{1}{4}}$$

ج)

$$\boxed{P(J,\underbrace{C,C,C}_{\text{كارتات ازعل}})=\frac{\binom{4}{1}\binom{48}{3}}{\binom{52}{4}}}$$

د)

$$\boxed{P(3,\underset{\text{صورة}}{\underset{\uparrow}{P}},c)=\frac{\binom{4}{1}\binom{12}{1}\binom{36}{1}}{\binom{52}{3}}=\frac{4*12*36}{\binom{52}{3}}}$$

مثال / البيانات التالية تمثل مجموعة من الطلبة مصنفين حسب القسم والجنس

	قسم الفيزياء B_1	قسم علوم الحياة B_2
A1 ذكور (٣٩٠)	٢١٠	١٨٠
A2 اناث (٢٤٠)	١١٥	١٢٥
العدد الكلي (٦٣٠)	٣٢٥	٣٠٥

اذا اختير طالب واحد من العينة ما احتمال أ)ان يكون الطالب من قسم الفيزياء ب) ان يكون ذكر ج) ان تكون انثى ومن علوم الحياة د) ان يكون ذكر او من الفيزياء

الحل :- أ)

$$\boxed{P(B_1)=\frac{\binom{325}{1}}{\binom{630}{1}}=\frac{325}{630}=\frac{65}{120}}$$

ب)

$$\boxed{P(A_1)=\frac{\binom{390}{1}}{\binom{630}{1}}=\frac{390}{630}=\frac{13}{21}}$$

ج)

$$\boxed{P(A_2B_2)=\frac{\binom{125}{1}}{\binom{360}{1}}=\frac{125}{630}=\frac{25}{126}}$$

د)

$$\boxed{\begin{aligned}P(A_1\cup B_1)&=P(A_1)+p(B_1)-P(A_1B_1)\\&=\frac{\binom{390}{1}}{\binom{630}{1}}+\frac{\binom{325}{1}}{\binom{630}{1}}-\frac{\binom{210}{1}}{\binom{630}{1}}\\&=\frac{390+325-210}{630}\\&=\frac{505}{630}=\frac{101}{126}\end{aligned}}$$

مثال : مجتمع يضم (١٠٠) شخص تم اجراء فحص لمعرفة صنف الدم لكل منهم وكانت النتائج كالتالي

صنف الدم	A	B	AB	O	
العدد	٢٢	١١	٦٠	٧	المجموع (١٠٠)شخص

ما احتمال أ) شخص يحمل صنف الدم AB

ت) شخص لا يحمل صنف الدم O

ث) ثلاث اشخاص احدهم يحمل صنف الدم o

ج) شخصين احدهم يحمل صنف الدم A والاخر B

أ)

$$\boxed{\begin{aligned} P(AB) &= \frac{\binom{60}{1}}{\binom{100}{1}} \\ &= \frac{60}{100} = 0.6 \end{aligned}}$$

ب)

يحمل صنف O

$$\boxed{\begin{aligned} P(O) &= \frac{\binom{7}{1}}{\binom{100}{1}} \\ &= \frac{7}{100} = 0.07 \end{aligned}}$$

$$P(O^c) = 1 - P(O) \longrightarrow P(O^c) = 1 - 0.07 = 0.93$$

لا يحمل صنف o

ح) F : تمثل ثلاث اشخاص احدهم يحمل الصنف O

$$\boxed{P(F) = \frac{\binom{7}{1}\binom{93}{2}}{\binom{100}{3}}}$$

د) E: تمثل القيمة المكونة من شخصين احدهم يحمل الصنف A والاخر الصنف B

$$\boxed{\begin{aligned} &P(A,B) = P(E) \\ &= \frac{\binom{22}{1}\binom{11}{1}}{\binom{100}{2}} \\ &= \frac{22*11}{\frac{100!}{2!(98)!}} \end{aligned}}$$

مثال / البيانات التالية تمثل عدد الاشخاص الذين هم بحاجة الى طبيب عام او اسنان من نوع الدعم المالي

الحل :-

	مصدر الدعم/الاختصاص	طبيب عام B1	طبيب اسنان B2	
A1	حكومي	٢٨٠	٤٧٠	٧٥٠
A2	شخصي	١٤٠	١١٠	٢٥٠
		٤٢٠	٥٨٠	١٠٠٠

اذا اختير شخص وبشكل عشوائي فما احتمال ان

أ) ان يكون الدعم حكومي ب) يراجع طبيب اسنان

ج)ذو دعم شخصي او يراجع طبيب عام

د) ذو دعم حكومي ويراجع طبيب اسنان

الحل :- أ)

$$\boxed{P(A1) = \frac{\binom{750}{1}}{\binom{1000}{1}}}$$

ب)

$$\boxed{P(B2) = \frac{\binom{580}{1}}{\binom{1000}{1}}}$$

ج)

$$P(A_2 \cup B1) = P(A2) + P(B1) - P(A2B1)$$

$$P(A2 \cup B_1) = \frac{\binom{250}{1}}{\binom{1000}{1}} + \frac{\binom{420}{1}}{\binom{1000}{1}} - \frac{\binom{140}{1}}{\binom{1000}{1}}$$

$$= \frac{53}{100}$$

د)

$$P(A1B2) = \frac{\binom{470}{1}}{\binom{1000}{1}}$$

$$= \frac{470}{1000} = 0.47$$

اسئلــــــة عـــامـــــة

١) اختيرت ورقة واحدة من اوراق اللعب البالغة ٥٢ فما احتمال

١) ان يكون نوع ٥   ٣) ان يكون K او Q

٢) ان تكون صورة   ٤) ليست اس A

٢) اذا اختيرت ثلاث ورقات فما احتمال   توضيح

١) ان تكون جميعها حمراء   حمراء 26

٢) احتوائها على الاقل A (اس) (١ على الاكثر)   سوداء 26

٣) احتوائها على الاكثر رقم ١٠

٤) اثنان صور   الكلي 52

الحل :- ١)

$$P(RRR) = \frac{\binom{26}{3}}{\binom{52}{3}}$$

$$= \frac{\frac{26!}{3!23!}}{\frac{52!}{3!99!}} = \frac{24}{204}$$

٢) على الاقل A ⬅ AXX ، AAX ، AAA المكملة هي XXX (المكملة لايوجد فيها A)

$$\boxed{P(A^{C})=\frac{\binom{48}{3}}{\binom{52}{3}}=} \longrightarrow \boxed{P(A)=1-P(A^{C})=1-\frac{\binom{48}{3}}{\binom{52}{3}}}$$

٣)

$$\boxed{P(10)=\frac{\binom{4}{1}\binom{48}{2}}{\binom{52}{3}}+\frac{\binom{48}{3}}{\binom{52}{3}}3}$$

٤)

$$P\,(\text{صورة وصورة}, C) = \boxed{\frac{\binom{12}{2}\binom{40}{1}}{\binom{52}{3}}}$$

الحوادث المستقلة *"Independent Events "*

يقال للحادثين B,A في الفضاء العيني لتجربة عشوائية معينة انها مستقلان اذا لم يؤثر احدهما على وقوع الاخر .

وهذا يعني ان B,A حادثين مستقلين اذا وفقط اذا تحققت العلاقة التالية

$$\boxed{P(A)X\,P(B)=P(AB)}$$

ملاحظة : كل حادثين مستقلين في Ω فانهما يجب ان يكون متصلين والعكس غير صحيح

مثال :- اذا كان B,A حادثين مستقلين في Ω بحيث ان

$P(A) = 0.4\ ,\ P(B) = 0.2$

جد :- $P(A\cup B)$

الحل :-

$P(A\cup B) = P(A) + P(B) - P(AB)$

$= P(A) + P(B) - P(A) \cdot P(B)$

$= 0.4 + 0.2 - (0.4)(0.2)$

$= 0.6 - 0.08$

$= 0.52$

نظرية ١

نظرية (١) اذا كان كل من A و B حادثين مستقلين في تجربة عشوائية معينة فان :-

١) A , B^C حادثين مستقلين ايضا [indep. Events] $\xleftarrow{\text{معناه}}$ $P(A).P(B^c) = P(AB^c)$

٢) B, A^C حادثين مستقلين ايضا [indep. Events] ← $P(A^c).P(B) = P(A^cB)$

٣) B^C, A^C حادثين مستقلين ايضا [indep. Events] ← $P(A^c)P(B^c) = P(A^cB^c)$

مثال :- اذا كان احتمال نجاح احمد في امتحان معين هو (٠,٨) واحتمال نجاح سعيد في نفس الامتحان هو (٠,٧) جد :-

١) احتمال نجاح احمد وعدم نجاح سعيد

٢) احتمال نجاح احدهما على الأكثر

الحل :-

نفرض ان نجاح احمد هو $A \leftarrow P(A) = 0.8$

نفرض ان نجاح سعيد هو $B \leftarrow P(B) = 0.7$

نلاحظ استقلال الحادثين

١)

$$\boxed{P(AB^C) = P(A)P(B^C) = (0.8)[1-0.7] = 0.24}$$

٢)

$$\boxed{P(AB^C) + P(A^CB) + P(A^CB^C)}$$

$$\boxed{\begin{aligned} &= P(A)P(B^C) + P(A^c).P(B) + P(A^C)P(B^C) \\ &= (0.8)(0.3) + (0.2)(0.7) + (0.2)(0.3) \\ &= 0.44 \end{aligned}}$$

نظرية (٢) اذا كان كل من A,B حادثين منفصلين (disjoint events) بحيث ان $A \neq \emptyset, B \neq \emptyset$ فان A,B حادثين معتمدين (غير مستقلين) (dependent)

الحوادث المعتمدة "Dependent Events"

يقال ان A , B حادثين معتمدين (غير مستقلين) اذا وفقط اذا $P(A).P(B) \neq P(AB)$

مثال :-سحب عنصرين من مجموعة مكونة من اربعة عناصر {1,2,3,4} عنصر عنصر بدون ارجاع (ومع الارجاع) فاذا كانت الحادثين B,A كما يلي :-

A : العنصر الاول فيها هو (٢).

B : العنصر الثاني بها هو (١) .

هل ان A,B حادثين مستقلين ؟

الحل :- ١) طريقة السحب الاولي (بدون ارجاع)

$$P_2^4 = \frac{4!}{2!} = 12$$

$$\Omega = \left\{\begin{matrix} (1,2) & (2,1) & (3,1) & (4,1) \\ (1,3) & (2,3) & (3,2) & (4,2) \\ (1,4) & (2,4) & (3,4) & (4,3) \end{matrix}\right\}$$

$$A = \{(2,1),(2,3),(2,4)] \subset S \Rightarrow P(A)\frac{3}{12} = \frac{1}{4}$$

$$B = \{(2,1),(3,1),(4,1)\} \subset S \Rightarrow P(B) = \frac{3}{12} = \frac{1}{4}$$

$$AB = \{(2,1)\} \subset S \Rightarrow P(AB) = \frac{1}{12}$$

نلاحظ الآن

$$P(A)xP(B) = \frac{1}{4} * \frac{1}{4} = \frac{1}{16}$$

$$P(AB) = \frac{1}{12}$$

$$\therefore P(A)xP(B) \neq P(AB)$$

∴ B,A غير مستقلين

الحادثين A,B غير مستقلين

٢)طريقة السحب الثانية (مع الارجاع)

عدد عناصر هو $(4^2)=16$

$$\Omega = \begin{Bmatrix} (1,1) & (2,1) & (1,11) & (4,11) \\ (1,2) & (2,2) & (3,2) & (4,2) \\ (1,3) & (2,3) & (3,3) & (3,3) \\ (1,4) & (2,4) & (3,4) & (4,4) \end{Bmatrix}$$

$$A = \{(2,1),(2,2),(2,3),(2,4)\} \Rightarrow P(A) = \frac{4}{16} = \frac{1}{4}$$

$$B = \{(1,1),(2,1),(3,1),(4,1)\} \Rightarrow P(B) = \frac{4}{16} = \frac{1}{4}$$

$$AB = \{(2,1)\} \Rightarrow P(AB) = \frac{1}{16}$$

$$P(A)xP(B) = \frac{1}{4}x\frac{1}{4} = \frac{1}{16}$$

$$\therefore P(AB) = P(A)xP(B)$$

$\therefore$ A,B حادثين مستقلين

اسئلة وتوضيح حول ورق اللعب :

احمر	اسود	احمر	اسود
♥	♠	♦	♣
A	A	A	A
2	2	2	2
3	3	3	3
4	4	4	4
5	5	5	5
6	6	6	6
7	7	7	7
8	8	8	8
9	9	9	9
10	10	10	10
J	J	J	J
Q	Q	Q	Q
K	K	K	K

العدد الكلي 4 × 13=52

عدد الصور 3 × 4 =12

اسئلة : اختيرت ورقة من الاوراق البالغة (52) جد :

أ. احتمال ان تكون الورقة تحمل رقم (10) .

ب. احتمال ان تكون الورقة من نوع ◇ .

جـ. اذا اختيرت (4) ما هو احتمال ان تكون احدهم A .

د. اذا اختيرت (5) ورقات فما هو احتمال ان تكون احدهم A والثانية صورة والثالثة اي ورقة اخرى من اوراق اللعب .

الحل :-

Let B = to get 10

$$P(B)=\frac{\binom{4}{1}}{\binom{52}{1}}=\frac{4}{52}=\frac{1}{13}$$

ب.

Let C = to get ◊ (◊ diomoned)

$$\therefore P(C)=\frac{\binom{13}{1}}{\binom{52}{1}}=\frac{13}{52}=\frac{1}{4}$$

جـ

D = To get 4 cards one of them is A

52 – 4 = 48

$$\therefore P(D)=\frac{\binom{4}{1}\binom{48}{3}}{\binom{52}{1}}$$

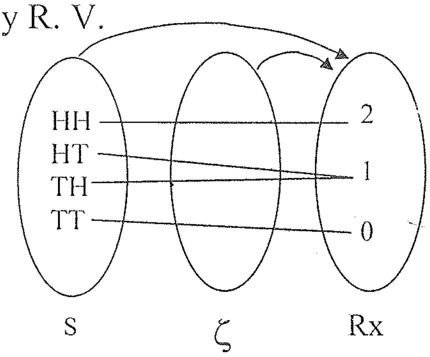
Let E = to get (3) cards one of then A and the second is a picture and the third any other card

$$\therefore P(E)=\frac{\binom{4}{1}\binom{12}{1}\binom{36}{1}}{\binom{52}{3}}$$

Chapter Three

"Random variables and Probability Distribution"

Def.: A random variable X is a function that mapps all elements s $\in$ S (all event in ζ) to a real numbers (Rx) denoted by R. V.



Ex 1.: Toss a coin twice.

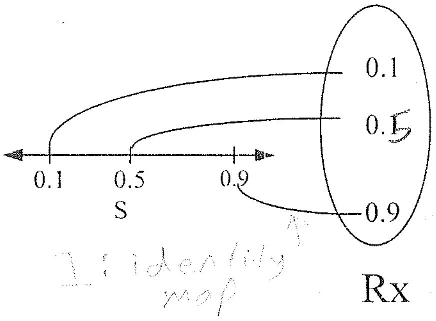
Let X = number of H show that X is a R. V.

S = {HH, HT, TH, TT}

x(HH) = 2, x(HT) = 1, x(TH) = 1, x(TT) = 0

Rx = {x; x = 0 , 1, 2} countable

Note: We shall use X to denoted of R. V.X and x to denote of value of R. V. X, x $\in$ X: 0, 1, 2 (in ex. "1").



Ex2.: Choose a point from interval (0, 1).

Let X be the chosen point, to show X is a R. V.

S consist all point in (0, 1)

$\therefore$ S has infinite number of points

The points in S are mapped to a real numbers.

then X is a R. V.

Rx = {x; 0 < x < 1} uncountable.

Def.: A random variable X is a say to be discret r. v. if Rx is countable.

See ex. "1" above, denoted by d. r. v.

Def.: A random variable x is say to be continuous r. v. if Rx is uncountable denoted by c.r.v.

See ex, "2" above.

Ex.3: Toss a coin until first H appears.

Let x: number of tosses. show that x is d.r.v.

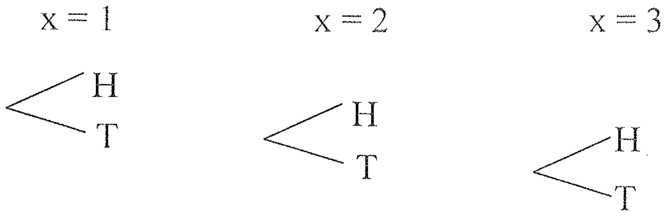
الاحتمالية

الاحتمالية

الفصل الثالث

أ. عباس نجم سلمان

Sol.:



$x \in X$: 1, 2, 3, 4, ….

$\therefore R_x = \{x; x \in N\}$ countable

$\therefore$ x is d.r.v.

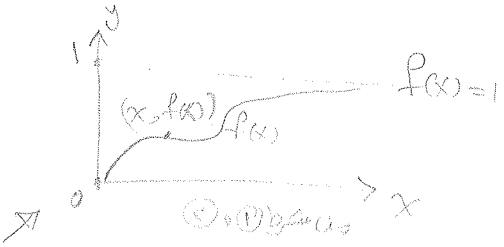
__Def.: Probability Mass function (P. M. f).__

Let X be a d.r.v. (القيمة الاحتمالية للمتغير X عند القيمة x)

A function f is a p.m.f. of X if $f(x) = p(X = x)$, and satisfy the following conditions:-

1. $f(x) \geq 0, \forall x \in X$, 2. $\sum_{\forall x \in X} f(x) = 1$

الدالة موجبة — الدالة مجموعها الواحد



Note: 1. condition (1) shows the graph of f(x) above of the x – axis.

2. Also, if $A \subset S$ then $P(x \in A) = \sum_{x \in A} f(x) = 1$

التوزيع الاحتمالي :
هو جدول الازواج المرتبة
$(x, f(x))$ الناتجة من تجربة (ظاهرة)
معينة ..

Ex.: Given $f(x) = \begin{cases} \frac{x}{10}, & \text{for } x = 0, 1, 2, 3, 4 \\ 0, & \text{otherwise} \end{cases}$

Show that f(x) is a p.M.f.

$X = d_1 + d_2$ واجب نفس السؤال اعلاه لكن مجموع

p.d.f. = $\{(0, \frac{6}{36}), (1, \frac{10}{36}), (2, \frac{8}{36}), \ldots\}$ التوزيع الاحتمالي

ex. Two dices are thrown.
let $X = |d_1 - d_2|$ الفرق المطلق للعدد

Find the p.m.f. f(x) and Sketch f(x).

Sol. $S = \{(d_1, d_2); 1 \leq d_i \leq 6, i = 1, 2\}$
$n(S) = 36 = 6^2$

توضيح

Rx

6 6 X=0 الفرق صفر

1 2 X=1 الفرق=1

وهكذا

R.X = {0, 1, 2, 3, 4, 5} = discrete

(6,1) و (1,6) (بين) 5 الفرق 5

$f(0) = P(X=0) = \frac{6}{36} > 0$

$f(1) = P(X=1) = \frac{10}{36} > 0$

$f(2) = P(X=2) = \frac{8}{36}$

$f(3) = P(X=3) = \frac{6}{36} > 0$

$f(4) = P(X=4) = \frac{4}{36} > 0$

$f(5) = P(X=5) = \frac{2}{36} > 0$

$\sum_{i=1}^{6} f(x_i) = 1$

cond. "1": T. p. $f(x) \ge 0 \ \forall x \in X$

$$f(0)=0,\ f(1)=\frac{1}{10},\ f(2)=\frac{2}{10},\ f(3)=\frac{3}{10},\ f(4)=\frac{4}{10}$$

$\therefore f(x) \ge 0 \ \forall x \in X$

$\therefore$ cond (1) satisfied.

cond "2": T.p $\sum_{x=0}^{4} f(x) = 1$

$$\sum_{x=0}^{4} f(x) = 0 + \frac{1}{10} + \frac{2}{10} + \frac{3}{10} + \frac{4}{10} = \frac{10}{10} = 1 .$$

$\therefore$ f(x) is a p. M.f

$* \ p(x=1) = f(1) = \frac{1}{10}$

$* \ p(x=8) = f(8) = 0$

$* \ p(x \ge 3) = p[(x=3) \cup (x=4)] = p(x=3) + p(x=4)$

$$= f(3) + f(4) = \frac{3}{10} + \frac{4}{10} = \frac{7}{10}$$

or by note 2 $p(x \ge 3) = \sum_{x=3}^{4} f(x) = f(3) + f(4) = \frac{7}{10}$

$* \ p(x \le 2) = p[(x=2) \cup (x=1)] = p(x=2) + p(x=1)$

$$= f(2) + f(1) = \frac{2}{10} + \frac{1}{10} = \frac{3}{10}$$

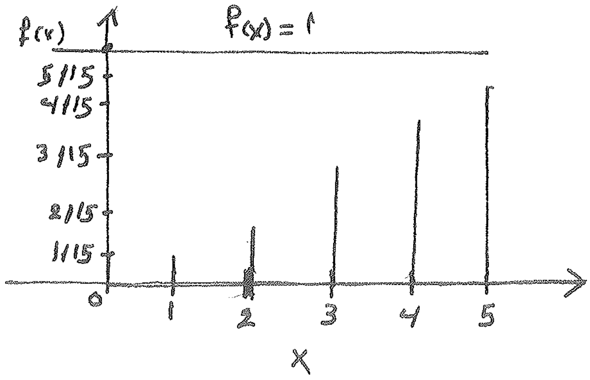
كيف تشتق دالة صريحة لل p.m.f ؟

▷ بالورقة (الأخيرة)

Ex.: Given a p. m . f. $f(x) = \begin{cases} \frac{x}{k}, & \text{for } x = 1,2,3,4,5 \\ 0, & \text{otherwise} \end{cases}$

find the value of k and sketch f(x).

Sol.: $\because$ f(x) is a p. m .f.



$\therefore$ by cond "2" we get $\sum_{x=1}^{5} f(x) = 1$

$f(1) + f(2) + f(3) + f(4) + f(5) = 1$

$$\frac{1}{k} + \frac{2}{k} + \frac{3}{k} + \frac{4}{k} + \frac{5}{k} = 1 \Rightarrow \ \therefore k = 15$$

$$\therefore f(x) = \begin{cases} \frac{x}{15}, & \text{for } x = 1,2,3,4,5 \\ 0, & \text{otherwise} \end{cases}$$

Ex.: Toss a coin 3-times. Let x = number of H. find the p. M.f. of x and sketch it's graph.

Sol.: S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

$\therefore$ S has (8) elements

x = no. of H, $x \in X$; $x = 0, 1, 2, 3$ → x_1, x_2, x_3, x_4

فما هي موجودة ؟
اي ما احتمالية
← ؟ $P(X = x_1)$

Rx = {x; x = 0, 1, 2, 3}, Rx is a countable

x is d. r. v.

Event (X = x): to get xH, x = 0, 1, 2, 3 when toss a coin 3- times

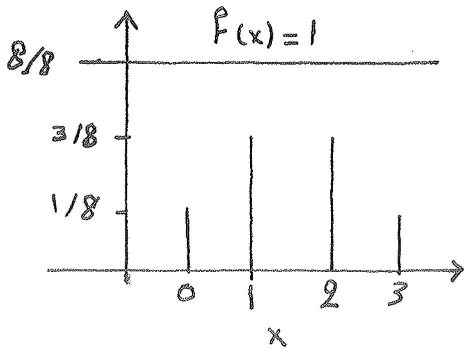
$\binom{3}{x}$: number of samples in event (X = x) when toss a coin 3-times

$$f(x) = p(X = x) = \begin{cases} \dfrac{\binom{3}{x}}{8} & \text{for } x = 0, 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

$pr.d.f. = \{(x_i, f(x_i)) ; \forall x \in X\}$

x	$f(x) = \dfrac{\binom{3}{x}}{8}$
0	1/8
1	3/8
2	3/8
3	1/8

$$\sum_{x=0}^{3} f(x) = 1$$



Def.: "probability distribution"

A probability dist. of a r.v.X is a set of all ordered pair of x and f(x), $\forall x \in X$.

i. e.: pr. dist.of x = {(xi, f(xi)); $\forall$xi $\in$X} of X.

In ex. above: pr. dist. of x $=\left\{(0,\frac{1}{8}),\ (1,\frac{3}{8}),(2,\frac{3}{8}),(3,\frac{1}{8})\right\}$

Ex.: Given (3) red and (5) white balls choose a sample of any (4) balls, let x = number of white ball in a sample.

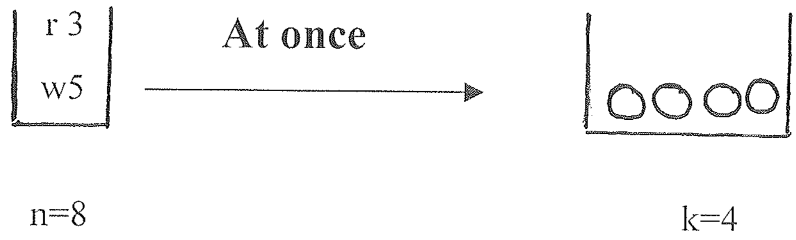
a. Find the P. m. f of x

b. Find the pr. that a sample has (2) white balls.

c. Find the pr. that a sample has at least (3) white balls.

Sol.:

a.



$\binom{8}{4}$: no. of sample in S

x = no. of w ball in a sample, x = 1, 2, 3, 4

Event (X = x): to get x w ball and (4-x) r ball in a sample.

$\therefore$ Number of samples $\in$ (X = x) is $\binom{5}{x}\binom{3}{4-x}$

$$f(x)=P(X=x)=\begin{cases}\dfrac{\binom{5}{x}\binom{3}{4-x}}{\binom{8}{4}}, & for\ x=1,2,3,4\\ 0 & otherwis.e\end{cases}$$

b. $p(x=2)=f(2)=\dfrac{\binom{5}{2}\binom{3}{2}}{\binom{8}{4}}$

c. $p(x\geq 3)=\sum_{x=3}^{4} f(x)=f(3)+f(4)$

Uniform Distribution of discreet random variable:

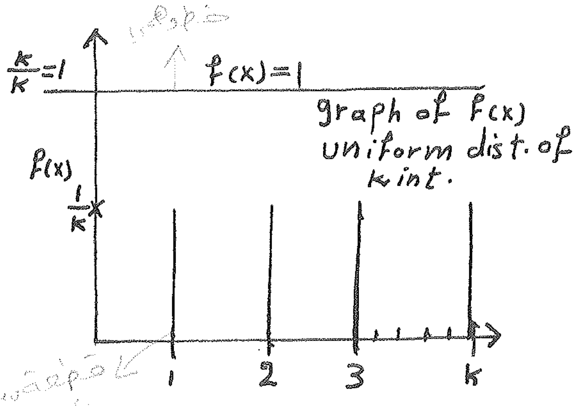
Def.: Given k integers: 1, 2, 3, …., k choose one integer

Let x = the chosen int., S = {1, 2, 3, …., k}

$$\therefore f(x) = p(X = x) = \begin{cases} \frac{1}{k}, & for\ x = 1, 2, 3, \ldots\ldots. k \\ 0 & otherwise \end{cases}$$

is called uniform dist. of k integers.

To show f(x) is a p. m. f. ?



cond. "1": T. p. f(x) ≥ 0

$$\because k > 0 \Rightarrow \frac{1}{k} > 0$$

$$\therefore f(x) = \frac{1}{k} > 0 \qquad for\ x = 1,2,3,\ldots.,k$$

$$= 0 \qquad otherwise$$

$$\therefore f(x) \geq 0$$

cond "2": T. p $\sum_{x=1}^{k} f(x) = 1$

$$\sum_{x=1}^{k} f(x) = \underbrace{\frac{1}{k} + \frac{1}{k} + \ldots. + \frac{1}{k}}_{k-times} = \frac{k}{k} = 1$$

$\therefore f(x)$ *is a p.m.f.*

Exercise: 1

1. A r.v.x. has a discret dist. with p.m.f., $f(x) = \begin{cases} cx, & x = 1, 2, 3, 4, 5 \\ 0, & o.w. \end{cases}$
Find the pr. dist. of X.

2. x has a uniform dist. on six integers: 2, 3, 4, 5, 6, 7 find the p.m .f. of x.

3. Given a set of integers 2, 3,…., 15 choose one integer which divisable by 3.

Let x = the chosen int. find the p.m.f. of x.

4. Given a set of integers {1, 2, …., 10} choose an integer and determine it's divisors.

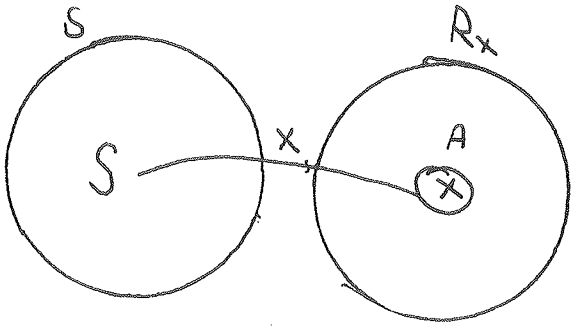
Let x = number of divisors. find the p.M.f. of x.

Probability density function (P.d.f):

Let x be a c.r.v.

A function f(x) is a p.d.f of x if for any interval $A \subset Rx$ $p(x \in A) = \int_A f(x)dx$. and satisfy the following conditions:

1. $f(x) \geq 0 \ \forall x \in R$, 2. $\int_{-\infty}^{\infty} f(x)dx = 1$



$X(s) = x \in Rx$, Rx is uncountable $A \subset Rx$

A is a set of real no.

Suppose that A = {x; a < x< b}.

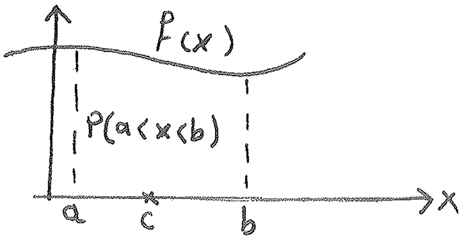
$$p(x \in A) = p(a < x < b) = \int_a^b f(x)dx$$

= Area under curve from a to b.

f(x) is cont. over (a, b)

let $c \in (a, b)$

f(x) is cont. at (c)



$$p(x = c) = \text{no. Area} = 0 \Rightarrow \int_c^c f(x)dx = 0$$

$$p(a \leq x \leq b) = p\ (x = a) + p(a < x < b) + p(x = b)$$
$$= p(a < x < b)$$

Note: When x is a d. r. v.

1. $p(a \leq x \leq b)$ not necessary equal to $p(a < x < b)$.

2. $p(x = c) = f(c)$ not necessary equal to zero.

Ex "1": Given a p.d.f, $f(x)=\begin{cases} kx & \text{for } 0<x<2 \\ k & \text{for } 2\le x<4 \\ 0 & \text{otherwise} \end{cases}$

a. find the value of k and sketch f(x).

b. find $p(x>1)$, $p(x<3)$, $p(\frac{3}{2}<x<\frac{5}{2})$, $p(x<1 \mid \frac{1}{2}<x<\frac{3}{2})$

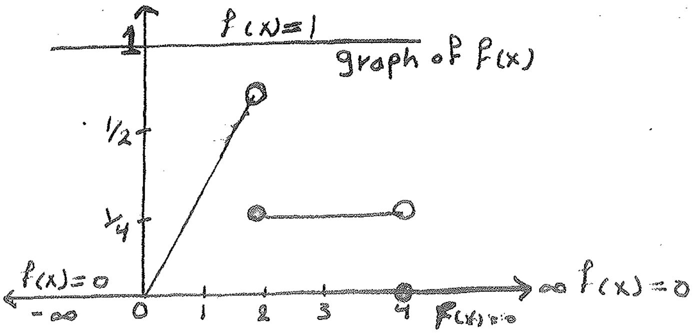
Sol.: a. by cond. "2" of p.d.f. $\Rightarrow \int\limits_{-\infty}^{\infty} f(x)dx=1$

$$1=\int\limits_{0}^{2} kxdx+\int\limits_{2}^{4} kdx \;\Rightarrow\; 1=\frac{k}{2}x^2\Big|_0^2+kx\Big|_2^4$$

$$\Rightarrow\; 1=\frac{k}{2}[4-0]+k[4-2] \;\Rightarrow 1=2k+2k$$

$$\Rightarrow 1=4k \;\Rightarrow\; k=\frac{1}{4}$$

$$f(x)=\begin{cases} \frac{x}{4} & \text{for } 0<x<2 \\ \frac{1}{4} & \text{for } 2\le x<4 \\ 0 & \text{o.w.} \end{cases}$$



MATHEMATICS

$$\text{b. } p(x>1)=\int_1^2 \frac{x}{4}dx+\int_2^4 \frac{1}{4}dx$$

$$=\frac{1}{8}x^2 \Big|_1^2+\frac{1}{4}x \Big|_2^4=\frac{1}{8}(4-1)+\frac{1}{4}(4-2)=\frac{3}{8}+\frac{1}{2}=\frac{7}{8}$$

$$p(x<3)=\int_0^2 \frac{x}{4}dx+\int_2^3 \frac{1}{4}dx$$

$$=\frac{1}{8}x^2 \Big|_0^2+\frac{1}{4}x \Big|_2^3=\frac{1}{8}(4-0)+\frac{1}{4}(3-2)=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$$

$$p(\frac{3}{2}<x<\frac{5}{2})=\int_{\frac{3}{2}}^{2} \frac{x}{4}dx+\int_2^{\frac{5}{2}} \frac{1}{4}dx$$

$$=\frac{1}{8}x^2 \Big|_{\frac{3}{2}}^{2}+\frac{1}{4}x \Big|_2^{\frac{5}{2}}=\frac{1}{8}(4-\frac{9}{4})+\frac{1}{4}(\frac{5}{2}-2)$$

$$=\frac{1}{8}(\frac{7}{4})+\frac{1}{4}(\frac{1}{2})=\frac{7}{32}+\frac{1}{8}=\frac{11}{32}$$

$$P(x<1|\frac{1}{2}<x<\frac{3}{2})=P(A|B)=\frac{P(AB)}{P(B)}$$

$A=\{x;\ x<1\}=\{x;\ 0<x<1\}$, $B=\{x;\ \frac{1}{2}<x<\frac{3}{2}\ \}$

$AB=\{x,\ \frac{1}{2}<x<1\}$

$$P(B)=\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{x}{4}dx=\frac{1}{8}x^2 \Big|_{\frac{1}{2}}^{\frac{3}{2}}=\frac{1}{8}(\frac{9}{4}-\frac{1}{4})=\frac{1}{4}$$

$$P(AB)=\int_{\frac{1}{2}}^{1} \frac{x}{4}dx=\frac{1}{8}x^2 \Big|_{\frac{1}{2}}^{1}=\frac{1}{8}(1-\frac{1}{4})=\frac{1}{8}(\frac{3}{4})=\frac{3}{32}$$

$$P(x<1|\frac{1}{2}<x<\frac{3}{2})=\frac{3/32}{1/4}=\frac{3}{8}$$

7

Ex “2”: Given a p. d. f $f(x)=\begin{cases} ke^{-x} & \text{for } x>0 \\ 0 & \text{for o.w} \end{cases}$

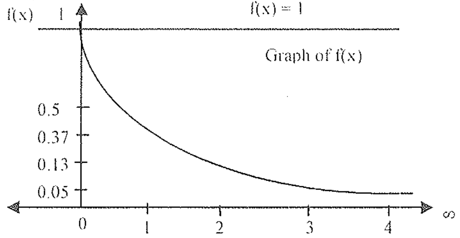
a. find the value of k and sketch f(x)

b. find p(x < 2).

Sol.: a. by cond. “2” of p.d.f $\int_{-\infty}^{\infty} f(x)dx = 1$

$$1 = k\int_{-\infty}^{\infty} e^{-x}dx \Rightarrow 1 = -ke^{-x}\,|_0^{\infty} \Rightarrow 1 = -k(e^{-\infty} - e^{0}) \Rightarrow 1 = k$$

$$f(x)=\begin{cases} e^{-x} & \text{for } x>0 \\ 0 & \text{for o.w} \end{cases}$$

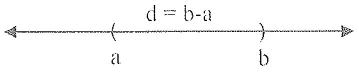


x	$f(x) = e^{-x}$
0	$e^{0} = 1$
1	$e^{-1} = \frac{1}{e} = \frac{1}{2.7} = 0.37$
2	$e^{-2} = 0.13$
3	$e^{-3} = 0.05$
.	
∞	$e^{-\infty} = 0$

$$b.\ \ p(x<2) = \int_{-\infty}^{2} f(x)dx = \int_{-\infty}^{0} o dx + \int_{0}^{2} e^{-x}dx = -e^{-x}\,|_0^2$$

$$= -\left[e^{-2} - e^{0}\right] = 1 - e^{-2} = 1 - 0.13 = 0.87$$

Uniform distribution on interval (a, b):



Given an interval (a, b), b > a

Choose a point x from (a, b) then a < x < b

A function $f(x)=\begin{cases} \frac{1}{b-a} & \text{for } a<x<b \\ 0 & \text{o.w} \end{cases}$

Is called uniform dist. on (a, b)

To show that f(x) a p.d.f

Cond. "1": T.p $f(x) \geq 0$

$$\because b - a > 0 \text{ since } a > b$$

$$\therefore \frac{1}{b-a} > 0 \Rightarrow f(x) = \frac{1}{b-a} > 0 \quad \text{for } a < x < b$$

$f(x) \geq 0$ $\qquad = 0 \qquad$ o.w

Cond "2": T.p $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} 0dx + \int_{a}^{b} \frac{1}{b-a}dx + \int_{b}^{\infty} 0dx$$

$$= \int_{a}^{b} \frac{1}{b-a}dx = \frac{1}{b-a} x|_{a}^{b}$$

$$= \frac{1}{b-a}(b-a) = 1$$

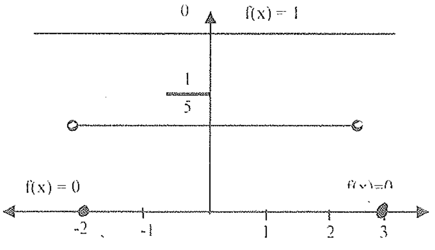
$\therefore$ f(x) is a p.d.f

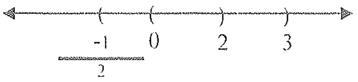
Ex "1": If x has a uniform dist. on (-2,3) [x ~ unif. (-2,3)]

a. find a p.d.f of x and sketch f(x).

b. Find $f(x > 0 | -\frac{1}{2} < x < 2)$

Sol.: a. $$f(x) = \begin{cases} \frac{1}{3-(-2)} = \frac{1}{5} & \text{for } -2 < x < 3 \\ 0 & \text{o.w} \end{cases}$$





b. $$p(x > 0 | -\frac{1}{2} < x < 2) = \frac{p(AB)}{P(B)}$$

$$A = \{x;\ 0 < x < 3\}$$

$$B = \left\{x; -\frac{1}{2} < x < 2\right\}$$

$$AB = \{x;\ 0 < x < 2\}$$

$$P(B) = \int_{-\frac{1}{2}}^{2} \frac{1}{5}dx = \frac{1}{5} x|_{-\frac{1}{2}}^{2} = \frac{1}{5}(2 + \frac{1}{2}) = \frac{1}{5}(\frac{5}{2}) = \frac{1}{2}$$

$$P(AB) = \int_{0}^{2} \frac{1}{5}dx = \frac{1}{5} x|_{0}^{2} = \frac{1}{5}(2 - 0) = \frac{2}{5}$$

$$\therefore p(x < 0 | -\frac{1}{2} < x < 2) = \frac{2/5}{1/2} = \frac{4}{5}$$

H.W.: 2

1. x ~ unif. (-1, 3) find p.d.f of x and find $p(0 \leq x < 2)$, $p(x < 1|-1 < x< 2)$.

2. Given $f(x) = \dfrac{1}{\pi(1+x^2)}$ for $-\infty < x < \infty$. Show that f(x) is a p.d.f.

Note: f(x) above is called cauchy dist.

Comulative Distribution Function (c. d. f)

(Distribution Function (d. f)) or (Comulative Distribution fun.) c. d. F

Def.: Let x be a r. v. either d. r.v. or c.r.v

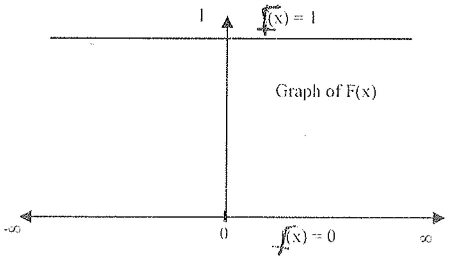
A function F(x) is a c.d.f of x iff

$F(x) = p(X \leq x); -\infty < x< \infty$

Event $(X \leq x)$ = all point in $(-\infty, x]$, $0 \leq p (X \leq x) \leq 1 \Rightarrow 0 \leq F(x) \leq 1$

That is mean F(x) is bounded by zero and one.

i.e.: The graph of F(x) lies between the line F(x) = 0 and the line F(x) = 1



Note: We shall use F(x) to denote of a c.d.f of x and f(x) to denote of p.m.f or p.d.f .

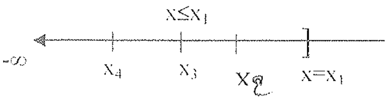
To find F(x) when x is d.r.v.

Let x be a d.r.v. with p.m.f f(x)

$Rx = \{xj \in I, j = 1, 2, 3, \ldots.\}$ countable

$F(x) = p(X \leq x)$ by def.

Choose x and let $x = x_1$ $(x_1 \in I)$



Event $(X \leq x) = \{xj \leq xi, i,j = 1,2,\ldots\}$

$(X\leq x) = \{(X=x_1) \cup (X=x_2) \cup (X=x_3) \cup \ldots \cup (X=xj) \cup \ldots\}$

$$P(X\le x) = P(X=x_1)+P(X=x_2)+P(X=x_3)+\ldots+P(X=xj)+\ldots$$
$$= f(x1) + f(x2) + f(x3) + \ldots.$$

$$p(X\le x)=\sum_{xj\le x} f(xj)$$

$$F(x)=\sum_{xj\le x} f(xj)$$ c.d.f of x when it's d.r.v.

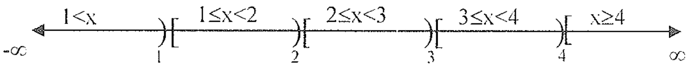
Ex. "1": Given a p.m.f
$$f(x)=\begin{cases}\frac{x}{10} & \text{for } x=1,2,3,4\\ 0 & \text{o.w}\end{cases}$$

a. Find the c.d.f F(x) and sketch it's graph.

b. Find $p(x\le 2)$, $p(x>3)$, $p(x<2)$, $p(1<x\le\frac{3}{2})$, $p(1\le x<\frac{3}{2})$

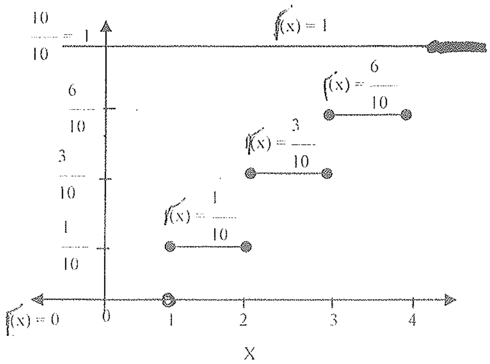
Sol.: a. Rx = {x, x = 1, 2, 3, 4} count.

$$F(x)=\sum_{xj\le x} f(xj)$$



Interval of x	xj ∈ I	$f(x)=\frac{x}{10}$	$F(x)=\sum_{xj\le x} f(xj)$
$x<1$	0	0	$F(x)=0$
$1\le x<2$	1	$\frac{1}{10}$	$F(x)=0+\frac{1}{10}=\frac{1}{10}$
$2\le x<3$	2	$\frac{2}{10}$	$F(x)=0+\frac{1}{10}+\frac{2}{10}=\frac{3}{10}$
$3\le x<4$	3	$\frac{3}{10}$	$F(x)=0+\frac{1}{10}+\frac{2}{10}+\frac{3}{10}=\frac{6}{10}$
$x\ge 4$	4	$\frac{4}{10}$	$F(x)=1$

$$F(x)=\begin{cases}0 & for\, x<1\\ \frac{1}{10} & for\, 1\le x<2\\ \frac{3}{10} & for 2\le x<3\\ \frac{6}{10} & fo3\le x<4\\ 1 & for x\ge 4\end{cases}$$



NOTE: 1- F(X) discont. at x=1,2,3,4.

2- F(X) is cont. to the right for each interval of continiouity.

b. $p(x \leq 2) = F(2) = \frac{3}{10}$

$(F(a) = p(x \leq a))$

or $p(x \leq 2) = \sum_{X=1}^{2} f(x) = f(1) + f(2) = \frac{1}{10} + \frac{2}{10} = \frac{3}{10}$

$$P(X>3) = 1 - P(X \leq 3) = 1 - F(3) = 1 - \frac{6}{10} = \frac{4}{10}$$

Or $p(x>3) = p(x=4) = f(4) = \frac{4}{10}$

$P(x<2) \neq F(2)$ $[p(x \leq x) = F(x)]$

$P(x<2) = p(x=1) = F(1) = \frac{1}{10}$

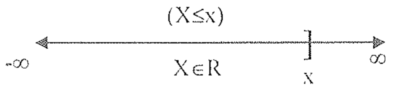
$P(1<x\leq \frac{3}{2}) = p(\phi) = 0$

Since $\nexists$ integer $\in (1, \frac{3}{2}]$

$P(1 \leq x < \frac{3}{2}) = p(x=1) = f(1) = \frac{1}{10}$

H.W: Given an integers ,5,6,7,.....,15 Choose one even integer. Let x be the chosen integer. Find F(x) and sketch it's → graph.

To find F(x) when x is c.r.v.



Let x be a c.r.v. with p.d.f f(x)

$F(x) = p(X \leq x)$ by def.

To find $p(X \leq x)$

$\because$ x has a p.d.f $f(x) \Rightarrow p(X \le x) = \int_{-\infty}^{x} f(t)dt$

$\therefore F(x) = \int_{-\infty}^{x} f(t)dt$ [c.d.f of x when x is c.r.v]

f(x) is cont. $\forall x \in (-\infty, x)$

Ex. "1": Given a p.d.f $f(x) = \begin{cases} 3(1-x)^2 & \text{for } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$

a. Find F(x) and sketch it's graph.

b. Find $p(x \le \frac{1}{3}),\ p(x < \frac{1}{3}),\ p(x > 1),\ p(x \le -\frac{1}{2}),\ p(-\frac{1}{3} < x < \frac{1}{4})$

Sol.: a. $F(x) = \int_{-\infty}^{x} f(t)dt$

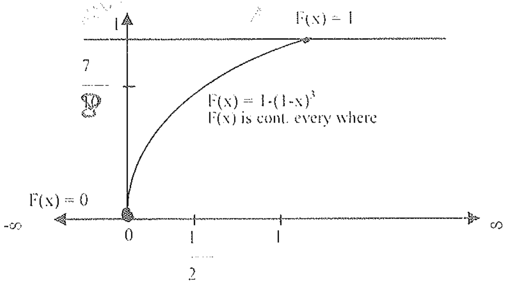
$$= \int_{-\infty}^{0} 0dt + \int_{0}^{x} 3(1-t)^2 dt = 3\int_{0}^{x} (1-t)^2 dt$$

$$= -3\frac{(1-t)^3}{3}\Big|_0^x = -[(1-x)^3 - 1] = 1-(1-x)^3$$

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1-(1-x)^3 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

[illegible]

x	F(x) = 1-(1-x)^3
0	0
$\frac{1}{2}$	$\frac{7}{8}$
1	1



b. $p(x \le \frac{1}{3}) = F(\frac{1}{3}) = 1-(1-\frac{1}{3})^3 = 1-\frac{8}{27} = \frac{19}{27}$

$p(x < \frac{1}{3}) \ne F(\frac{1}{3})$

$$p(x < \frac{1}{3}) = 3\int_0^{\frac{1}{3}}(1-x)^2\,dx = -3\frac{(1-x)^3}{3}\Big|_0^{\frac{1}{3}} = -(1-x)^3\Big|_0^{\frac{1}{3}}$$

$$= -[(1-\frac{1}{3})^3-1] = -[(\frac{2}{3})^3-1] = 1-\frac{8}{27} = \frac{19}{27}$$

$p(x > 1) = 1 - p(x \le 1) = 1 - F(1) = 1 - 1 = 0$

$p(x \le -\frac{1}{2}) = F(-\frac{1}{2}) = 0$

$$p(-\frac{1}{3} < x < \frac{1}{4}) = \int_{-\frac{1}{3}}^{\frac{1}{4}} f(x)dx = \int_{-\frac{1}{3}}^{0} o dx + \int_0^{\frac{1}{4}} 3(1-x)^2\,dx$$

$$= -(1-x)^3\Big|_0^{\frac{1}{4}} = -[(1-\frac{1}{4})^3-1] = 1-(\frac{3}{4})^3$$

$$= 1-\frac{27}{64} = \frac{37}{64}$$

Note: If F is conts. at $x = a$, then $P(X \le a) = P(X < a) = F(a)$ only when X is conv

H.W4

1. Given a p.d.f $f(x) = \begin{cases} \frac{1}{x^2} & \text{for } x > 1 \\ 0 & \text{o.w} \end{cases}$

a. Find c.d.f of x

b. Find p(x > 4.2) and p (x < 3.5) and p (x ≤ -1)

2. Given a p.d.f $f(x) = \begin{cases} Ce^{-x} & \text{for } x > 0 \\ 0 & \text{o.w} \end{cases}$

a. Find the value of c and sketch F(x).

b. Find F(x) and sketch it's graph.

c. p(x ≤ 3), p (x < 3).

Chapter Three

« Random Variables and Probability Distribution »

Exercise/P.6

Q_1: A r.v. X has a discret distribution with p.m.f.

P.6

$$f(x) = \begin{cases} cx & \text{for } x = 1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$$

Find the pr. dist. of X.

Sol.

∵ f(x) is a p.m.f.

∴ Condition ② is satisfied $\sum_{x=1}^{5} f(x) = 1$

$$\sum_{x=1}^{5} f(x) = f(1) + f(2) + f(3) + f(4) + f(5) = 1$$

$$= c[1+2+3+4+5] = 1$$

$$= c\left[\frac{5(5+1)}{2}\right] = 1$$

$$= 15c = 1 \implies c = \frac{1}{15}$$

$$\therefore f(x) = \begin{cases} \frac{x}{15} & \text{for } x = 1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$$

$$\text{pr.dist. of } X = \{(x_i, f(x_i)), \ \forall x_i \in X\}$$

$$= \{(1, \tfrac{1}{15}), (2, \tfrac{2}{15}), (3, \tfrac{3}{15}), (4, \tfrac{4}{15}), (5, \tfrac{5}{15})\}$$

Q_2: X has a uniform dist. on six integers: 2, 3, 4, 5, 6, 7. Find the p.m.f. of X.

Sol. $X = 2, 3, 4, 5, 6, 7 \implies$ 6-integers

$X \sim \text{uniform}(6)$

$$\therefore f(x) = \begin{cases} \frac{1}{K} & \text{for } x = 1, 2, \dots, K \\ 0 & \text{o.w} \end{cases} = \begin{cases} \frac{1}{6} & \text{for } x = 2, 3, 4, 5, 6, 7 \\ 0 & \text{o.w.} \end{cases}$$

Q3: Given a set of integers 2, 3, ---, 15 Choose one integer which divisable by 3.

Let X = the chosen int. find the P.m.f. of X.

Sol. $X = \{3, 6, 9, 12, 15\} = $ d.r.v.

$R_x = \{3, 6, 9, 12, 15\}$ is Countable.

$X \sim$ uniform (5)

$$f(x) = \begin{cases} \frac{1}{5} & \text{for } x = 3, 6, 9, 12, 15 \\ 0 & \text{o.w.} \end{cases}$$

↑

P.m.f.

Q4: Given a set of integers {1, 2, ..., 10}, choose an integer and determine it's divisors.

Let X = no. of divisors.

Find the P.m.f. of X.

Sol. X = no. of divisors

X = R.v.

integer	divisor	no. of divisor	$f(x)$ = P.m.f. = $\frac{X \text{ تكرار}}{10}$
1	1	1	$\underline{\underline{X}}$
2	1, 2	2	$x = 1 \xrightarrow{\text{تكرار}} 1 \Rightarrow \frac{1}{10}$
3	1, 3	2	$x = 2 \xrightarrow{\text{تكرار}} 4 \Rightarrow \frac{4}{10}$
4	1, 2, 4	3	$x = 3 \xrightarrow{\text{تكرار}} 2 \Rightarrow \frac{2}{10}$
5	1, 5	2	$x = 4 \xrightarrow{\text{تكرار}} 3 \Rightarrow \frac{3}{10}$
6	1, 2, 3, 6	4	
7	1, 7	2	
8	1, 2, 4, 8	4	
9	1, 3, 9	3	
10	1, 2, 5, 10	4	

$$\therefore f(x) = \begin{cases} \frac{1}{10} & \text{for } x = 1 \\ \frac{4}{10} & \text{for } x = 2 \\ \frac{2}{10} & \text{for } x = 3 \\ \frac{3}{10} & \text{for } x = 4 \\ 0 & \text{o.w} \end{cases}$$

↑

P.m.f.

Exercise / P.12

Q1: $X \sim$ unif. $(-1,3)$. Find p.d.f. of X and find $P(0\leq X<2)$, $P(X<1/-1<X<2)$

given

Sol.

$X \sim$ unif. $(-1,3)$

$$\therefore f(X)=\begin{cases}\frac{1}{3-(-1)} & \text{for } -1<X<3\\ 0 & \text{o.w.}\end{cases}$$

$$=\begin{cases}\frac{1}{4} & \text{for } -1<X<3\\ 0 & \text{o.w.}\end{cases}$$

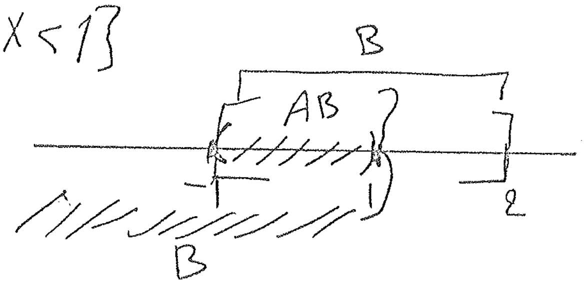
$$P(0\leq X<2)=\int_0^2 f(x)\,dx=\int_0^2\frac{1}{4}\,dx=\frac{1}{4}X\Big|_0^2=\frac{1}{4}[2-0]=\frac{1}{2}$$

$$P(\underbrace{X<1}_{A}/\underbrace{-1<X<2}_{B})=P(A|B)=\frac{P(AB)}{P(B)}$$

$A=\{X: X<1\}=\{X: -1<X<1\}$

$B=\{X: -1<X<2\}$

$AB=\{X: -1<X<1\}$



$$P(B)=\int_{-1}^{2}\frac{1}{4}\,dX=\frac{1}{4}X\Big|_{-1}^{2}=\frac{1}{4}[2+1]=\frac{3}{4}\in[0,1]$$

$$P(AB)=\int_{-1}^{1}\frac{1}{4}\,dx=\frac{1}{4}X\Big|_{-1}^{1}=\frac{1}{4}(1+1)=\frac{1}{2}\in[0,1]$$

$$P(X<1/-1<X<2)=\frac{1/2}{3/4}=\frac{2}{3}\in[0,1]$$

✓Q2: Given $f(x)=\frac{1}{\pi(1+X^2)}$ for $-\infty<X<\infty$. Show that $f(x)$ is a p.d.f.

Note: $f(x)$ is called cauchy dist.

Sol: I.P. cond.① $f(x)\geq 0 \quad \forall X\in R_X=(-\infty,\infty)$

$x^2 \geq 0 \Rightarrow \pi(1+x^2) \geq 0 = \frac{1}{\pi(1+x^2)} \geq 0$

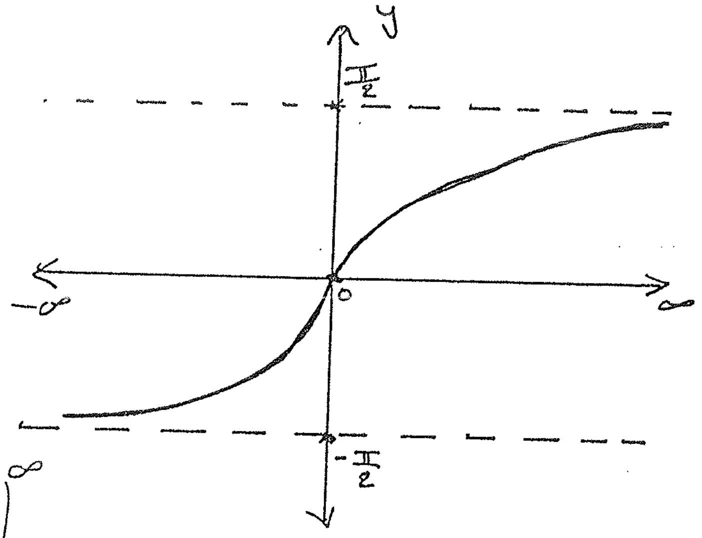
$= f(x) \geq 0$

$X \in (-\infty, \infty)$

if $x < \infty \longrightarrow f(x) \geq 0$

if $x > -\infty \longrightarrow f(x) \geq 0$

$\therefore f(x) \geq 0 \quad \forall x \in (-\infty, \infty)$



T.P. Cond. ② is satisfied

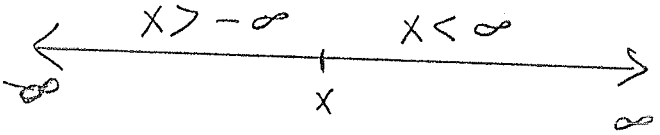
i.e. T.P. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx = \frac{1}{\pi}\tan^{-1}x \Big|_{-\infty}^{\infty}$$

$$= \frac{1}{\pi}\left[\tan^{-1}(\infty) - \tan^{-1}(-\infty)\right] = \frac{1}{\pi}\left[\frac{\pi}{2} - \left[-\frac{\pi}{2}\right]\right] = \frac{1}{\cancel{\pi}}(\cancel{\pi}) = 1$$

$\therefore$ Cond. ② is satisfied.

$\therefore f(x)$ is a p.d.f.



H.W. P.14 Given an integer 5, 6, 7, ⋯, 15. Choose one even integer. Let X be the chosen integer. Find F(x) and sketch it's graph.

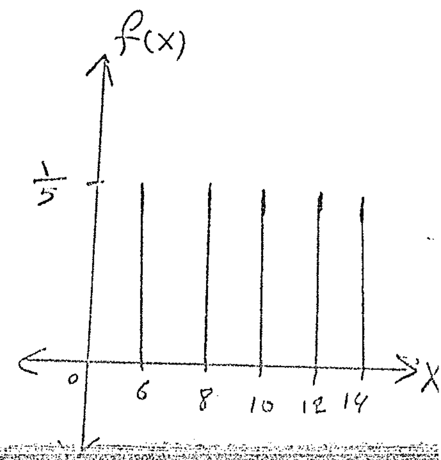
Sol. X = The even integer

$\therefore X = 6, 8, 10, 12, 14$

$R_X = \{x : x = 6, 8, 10, 12, 14\}$ = d.r.v.

$X \sim$ unif.(5)

$$\therefore f(x) = \begin{cases} \frac{1}{5} & \text{for } x = 6, 8, 10, 12, 14 \\ 0 & \text{o.w.} \end{cases}$$

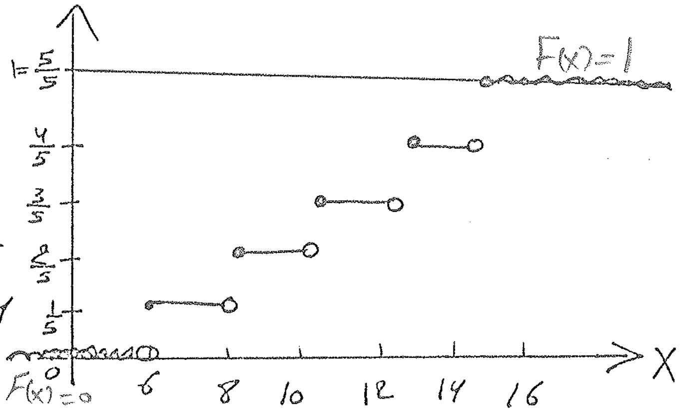


interval of X	X integer	p.m.f $f(x) = \frac{1}{5}$	$F(x) = $ c.d.f.
$X < 6$	no. int.	0	$0 \leftarrow$ تبدأ بالصفر
$6 \leq X < 8$	6	$\frac{1}{5}$	$0 + \frac{1}{5} = \frac{1}{5}$
$8 \leq X < 10$	8	$\frac{1}{5}$	$0 + \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$
$10 \leq X < 12$	10	$\frac{1}{5}$	$0 + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{3}{5}$
$12 \leq X < 14$	12	$\frac{1}{5}$	$0 + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{4}{5}$
$X \geq 14$	14	$\frac{1}{5}$	$0 + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{5}{5} = 1$

تنتهي بالواحد ↗

$$F(x) = \begin{cases} 0 & \text{for } X < 6 \\ 1/5 & \text{for } 6 \leq X < 8 \\ 2/5 & \text{for } 8 \leq X < 10 \\ 3/5 & \text{for } 10 \leq X < 12 \\ 4/5 & \text{for } 12 \leq X < 14 \\ 1 & \text{for } X \geq 14 \end{cases}$$

(d.f.)



$0 \leq F(x) \leq 1$

Exercise / P.16

Q1: Given a p.d.f.

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{for } x > 1 \\ 0 & \text{o.w.} \end{cases}$$

a. Find c.d.f. of X

b. Find $P(X > 4.2)$ and $P(X < 3.5)$ and $P(X \leq -1)$

Sol: $F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{1}^{x} \frac{1}{t^2}\,dt = \frac{-1}{t}\Big|_1^x = -\left[\frac{1}{x} - 1\right] = 1 - \frac{1}{x}$

$$\therefore F(x) = \begin{cases} 1 - \frac{1}{x} & \text{for} \quad x \geq 1 \\ 0 & \\ & \text{o.w.} \end{cases}$$

Note: $F(x)$ is conts. at $x = 1$

since $\lim_{x \to 1^+} F(x) = 1 - \frac{1}{1} = 0 = F(1^+)$

$\lim_{x \to 1^-} F(x) = 0 = F(1^-)$

Now, $P(X > 4.2) = 1 - P(X \leq 4.2)$

$= 1 - F(4.2)$

$= 1 - \left(1 - \frac{1}{4.2}\right) = 0.238$

$P(X < 3.5) = \int_1^{3.5} \frac{1}{x^2} dx = \frac{-1}{x}\Big|_1^{3.5} = -\left[\frac{1}{3.5} - 1\right] = 0.714$

$P(X \leq -1) = F(-1) = 0$

Q2: Given a p.d.f.

$$f(x) = \begin{cases} ce^{-x} & \text{for} \quad x > 0 \\ 0 & \text{o.w.} \end{cases}$$

a. Find the value of c and sketch $F(x)$.

b. Find $F(x)$ and sketch it's graph.

c. $P(X \leqslant 3)$, $P(X < 3)$.

Sol.

a. $\because$ $f(x)$ is p.d.f.

$\therefore$ cond. ② is satisfied : $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\int_0^{\infty} f(x)\,dx = c\int_0^{\infty} e^{-x}\,dx = -ce^{-x}\Big|_0^{\infty} = -c[e^{-\infty} - e^{0}] = -c(-1) = c = 1$

$\therefore \boxed{c = 1}$

$F(1^-) = \lim\limits_{x \to 1^-} F(x) = 0$

$\therefore\ F(1^+) \neq F(1^-) \Rightarrow F(x)$ is not conts. at $x=1$

• H.W / P. 20 Given a c.d.f.

$$F(x) = \begin{cases} 1 - \frac{4}{x^2} & \text{for } x > 2 \\ 0 & \text{for } x \leq 2 \end{cases}$$

1. Find $f(x)$ and sketch $F(x)$ & $f(x)$ (H.w.)
2. Find $P(X \leq 4)$, $P(X > 8)$, $P(2 < X \leq 8)$, $P(X=2)$.

Sol: $F(X)$ is conts at $x=2$

<u>Since</u> $F(2^+) = F(2^-)$

$F(2^+) = \lim\limits_{x \to 2^+} F(x) = 1 - \frac{4}{4} = 0$

$F(2^-) = \lim\limits_{x \to 2^-} F(x) = 0$

$$f(x) = F'(x) = \begin{cases} \frac{8}{x^3} & \text{for } x > 2 \\ 0 & \text{for } x \leq 2 \end{cases}$$

$P(X \leq 4) = F(4) = 1 - \frac{4}{16} = 1 - \frac{1}{4} = \frac{3}{4}$ (By def. of $F(x)$)

$P(X > 8) = 1 - P(X \leq 8)$

$= 1 - F(8)$

$= 1 - (1 - \frac{1}{16}) = 1 - \frac{15}{16} = \frac{1}{16}$

$P(2 < X \leq 8) = F(8) - F(2)$ By <u>Th.</u> $P(a < X \leq b) = F(b) - F(a)$

$= (1 - \frac{4}{64}) - 0$

$= 1 - \frac{1}{16} = \frac{15}{16}$

$P(X=2) = F(2^+) - F(2^-)$ By <u>Th.</u> $P(X=x) = F(x^+) - F(x^-)$

$= 0$

<u>Since</u> $F(x)$ is conts. at x [illegible]

$$\therefore\ f(x) = \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{o.w.} \end{cases}$$

b. $F(x) = \int_{-\infty}^{x} f(t)\,dt$

$$= \int_{0}^{x} e^{-t}\,dt = -e^{-t}\Big|_{0}^{x} = -[e^{-x} - \underbrace{e^{0}}_{=1}] = 1 - e^{-x}$$

$$F(x) = \begin{cases} 1 - e^{-x} & \text{for } x \geq 0 \\ 0 & \text{o.w.} \end{cases} \longleftarrow$$ F(x) is conts at $x = 0$ since

$$\left\{ \begin{array}{l} F(0^{+}) = F(0^{-}) \\ F(0^{+}) = 1 - e^{0} = 1 - 1 = 0 \\ F(0^{-}) = 0 \end{array} \right\}$$

c. $P(X \leq 3) = F(3) = 1 - e^{-3}$

$$P(X < 3) = \int_{0}^{3} e^{-x}\,dx = -e^{-x}\Big|_{0}^{3} = -[e^{-3} - e^{0}]$$

$$= 0.9502$$

or $P(X < 3) = P(X \leq 3) = F(3)$

since (c.r.v.) & F is conts. at $x = 3$

H.w / P.17 Given a c.d.f. F(x):

$$F(x) = \begin{cases} 1 - e^{-x} & \text{for } x \geq 1 \\ 0 & \text{o.w.} \end{cases}$$

Find the p.d.f. f(x).

Sol. F(x) is conts. function. $\Rightarrow$ F(x) is diff.

$$f(x) = \frac{dF(x)}{dx} = F'(x)$$

$$f(x) = F'(x) = \begin{cases} e^{-x} & \text{for } x > 1 \\ 0 & \text{o.w.} \end{cases}$$

But is F(x) conts. at $x = 1$?

$$F(1^{+}) = \lim_{x \to 1^{+}} F(x) = 1 - e^{-1} \neq 0$$

Q_1: Toss a Coin 3-times. If X be the number of head (H). Find the p.f. of X.

Sol. $X \equiv$ number of H.

$$x = 0, 1, 2, 3$$

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ TTT & HTT & HHT & HHH \end{array}$$

$R_X = \{ x : x = 0, 1, 2, 3 \} =$ Countable

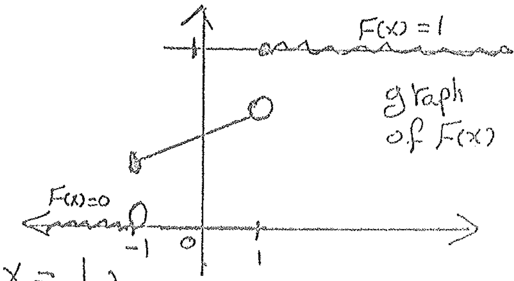
$\therefore$ X is d.r.v.

$\binom{3}{x} \equiv$ number of sample in event $(X = x)$ when toss a Coin 3-times.

$$P(X=x) = \underset{\nearrow \text{ p.f.}}{f(x)} = \begin{cases} \dfrac{\binom{3}{x}}{8} & \text{for } x = 0, 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$$

Q_2: Given a d.f.

$$F(x) = \begin{cases} 0 & \text{for } x < -1 \\ \dfrac{x+2}{4} & \text{for } -1 \leq x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$



① Sketch $F(x)$ and find $P(X=-1)$ and $P(X=\frac{1}{2})$.

② If $F(x)$ conts. at $x = \frac{1}{2}$? why?

Sol. $P(X=-1) = F(-1^+) - F(-1^-)$ By Th. $P(X=x) = F(x^+) - F(x^-)$

$$= \frac{-1+2}{4} - 0 = \frac{1}{4}$$

$$P(X=\tfrac{1}{2}) = \frac{\frac{1}{2}+2}{4} - \frac{\frac{1}{2}+2}{4} = 0$$

$F(x)$ is conts. at $x = \frac{1}{2}$ since $P(X=\frac{1}{2}) = 0$

Q3: Given a p.d.f. $f(x)$:

$$f(x) = \begin{cases} 1-|x| & \text{for } |x| < 1 \\ 0 & \text{o.w.} \end{cases}$$

Find the d.f. $F(x)$ and find $P(X>0)$ & $P(|X| > \frac{1}{2})$.

sol:

$$|x| < 1 \Rightarrow -1 < X < 1$$

$$|x| = -x \Rightarrow -1 < X < 0$$

$$|x| = x \Rightarrow 0 \leq X < 1$$

$$f(x) = \begin{cases} 1+X & \text{for } -1 < X < 0 \\ 1-X & \text{for } 0 \leq X < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$F(X) = \begin{cases} \int_{-1}^{x} (1+t)\,dt & \text{for } -1 \leq X < 0 \\ \int_{0}^{x} (1-t)\,dt & \text{for } 0 \leq X \leq 1 \\ 0 & \text{for } X < -1 \\ 1 & \text{for } X \geq 1 \end{cases}$$

Find $P(X>0) = 1 - P(X \leq 0)$

$$= 1 - F(0) = 1 - \int_0^x (1-t)\,dt = \ldots$$

$$P(|X| > \tfrac{1}{2}) = 1 - P(|X| \leq \tfrac{1}{2})$$

$$= 1 - P(-\tfrac{1}{2} \leq X \leq \tfrac{1}{2})$$

$$= 1 - \left[\int_{-\frac{1}{2}}^{0} f(x)\,dx + \int_{0}^{\frac{1}{2}} f(x)\,dx \right]$$

$$= 1 - \left[\int_{-\frac{1}{2}}^{0} (1+x)\,dx + \int_{0}^{\frac{1}{2}} (1-x)\,dx \right]$$

$$= \ldots$$

Q4: Two dice are thrown once, let X be the absolute difference between the two numbers shown by the dice. Find the p.m.f. and c.d.f. of X.

Sol. $S = \{ (d_1, d_2) ; 1 \le d_i \le 6 ; i = 1, 2 \} = 6^2 = 36$ ✓

$$\underset{R.V.}{X} = \left\{ \begin{array}{lll} |1-1| = 0 , & |2-1| = 1 , & \ldots , |6-1| = 5 \\ |1-2| = 1 & |2-2| = 0 & |6-2| = 4 \\ |1-3| = 2 & |2-3| = 1 & |6-3| = 3 \\ \vdots & \vdots & \vdots \\ |1-6| = 5 & |2-6| = 4 & |6-6| = 0 \end{array} \right\}$$

$\therefore X = 0, 1, 2, 3, 4, 5$

Event $(X=0) = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\} = (6)$ elements

$f(x) = P(X=x)$

$f(0) = P(X=0) = \frac{6}{36}$

Event $(X=1) = \{(2,1), (1,2), (2,3), (3,2), (4,3), (3,4), (4,5), (5,4), (5,6), (6,5)\} = (10)$ elements

$f(1) = P(X=1) = \frac{10}{36}$

Event $(X=2) = \{(1,3), (3,1), (2,4), (4,2), (3,5), (5,3), (4,6), (6,4)\} = (8)$ elements.

$f(2) = P(X=2) = \frac{8}{36}$

Event $(X=3) = \{(1,4), (4,1), (2,5), (5,2), (3,6), (6,3)\} = (6)$ elements.

$f(3) = P(X=3) = \frac{6}{36}$

Event $(X=4) = \{(1,5), (5,1), (2,6), (6,2)\} = (4)$ elements.

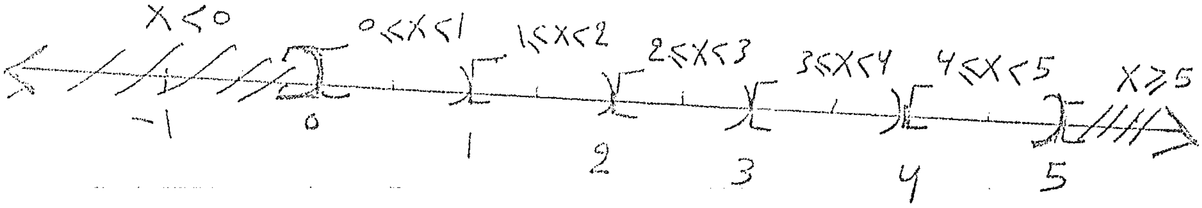
$f(4) = P(X=4) = \frac{4}{36}$

Event $(X=5) = \{(1,6), (6,1)\} = (2)$ elements

$f(5) = P(X=5) = \frac{2}{36}$

$$\therefore f(x) = \begin{cases} \frac{6}{36} & \text{for} \quad x=0 \\ \frac{10}{36} & \text{for} \quad x=1 \\ \frac{8}{36} & \text{for} \quad x=2 \\ \frac{6}{36} & \text{for} \quad x=3 \\ \frac{4}{36} & \text{for} \quad x=4 \\ \frac{2}{36} & \text{for} \quad x=5 \\ 0 & \text{o.w.} \end{cases}$$

$$\underset{(\text{d.f.})}{F(x)} = \begin{cases} 0 & \text{for} \quad X<0 \\ \frac{6}{36} & \text{for} \quad 0 \leq X < 1 \\ \frac{16}{36} & \text{for} \quad 1 \leq X < 2 \\ \frac{24}{36} & \text{for} \quad 2 \leq X < 3 \\ \frac{30}{36} & \text{for} \quad 3 \leq X < 4 \\ \frac{34}{36} & \text{for} \quad 4 \leq X < 5 \\ 1 & \text{for} \quad X \geq 5 \end{cases}$$



Q5: Given a p.f.

$$f(x) = \begin{cases} K\left(\frac{2}{3}\right)^x & \text{for } x = 1, 2, 3, \dots \\ 0 & \text{o.w.} \end{cases}$$

Find the value of K.

Sol: Since f(x) is p.m.f. then Cond. ② $\sum_{\forall x \in R_x} f(x) = 1$ is satisfied.

By Cond. ② we get: $\sum_{x=1}^{\infty} K\left(\frac{2}{3}\right)^x = 1$

$$1 = K \sum_{x=1}^{\infty} \left(\frac{2}{3}\right)^x$$

$$1 = K\left[\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \cdots + \left(\frac{2}{3}\right)^n + \cdots\right]$$

$$1 = \left(\frac{2}{3}\right)K\left[\underbrace{1 + \left(\frac{2}{3}\right) + \cdots + \left(\frac{2}{3}\right)^{n-1} + \cdots}_{\text{Geometric Series}}\right]$$

لأن $r = \left(\frac{2}{3}\right) < 1 \Rightarrow$ the series is convergent.

$$1 = \left(\frac{2}{3}\right)K\left[\frac{1}{1-\left(\frac{2}{3}\right)}\right] = \left(\frac{2}{3}\right)K\left[\frac{1}{\frac{1}{3}}\right] = \left(\frac{2}{\cancel{3}}\right)K\,(\cancel{3}) = 2K$$

مجموع متوالية هندسية حدها الأول واحد وأساسها $\left(\frac{2}{3}\right)$

$$\therefore 2K = 1 \Rightarrow \boxed{K = \frac{1}{2}}$$

$$\therefore f(x) = \begin{cases} \frac{1}{2}\cdot\left(\frac{2}{3}\right)^x & \text{for } x = 1, 2, 3, \dots \\ 0 & \text{o.w.} \end{cases}$$

Q6: Given a p.f.:

$$f(x) = \begin{cases} \frac{e^{-1}}{x!} & \text{for } x = 0, 1, 2, 3, \dots \\ 0 & \text{o.w.} \end{cases}$$

Show that f(x) is a p.m.f.

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Sol. T.P. Cond. ① : $f(x) \geq 0 \quad \forall x \in R_x$

$\frac{1}{e} > 0 \quad , \frac{1}{x!} > 0 \qquad \forall\, x = 0, 1, 2, \ldots$

since if $x = 0 \Rightarrow \frac{1}{e} > 0 \; , \; \frac{1}{\underset{=1}{0!}} = 1 > 0$

& $x = 1 \Rightarrow \frac{1}{e} > 0 \; , \; \frac{1}{1!} = 1 > 0$

$\vdots$

$\Rightarrow \frac{1}{e\, x!} > 0 \Rightarrow$ Cond. ① is satisfied.

T.P. Cond. ② : $\sum_{x=0}^{\infty} \frac{1}{e\; x!} = 1$

$$\sum_{x=0}^{\infty} \frac{1}{e\; x!} = \frac{1}{e} \sum_{x=0}^{\infty} \frac{1}{x!} = \cancel{\frac{1}{e}} \cdot \cancel{e} = 1$$

∴ Cond. ② is satisfied. $\Rightarrow$ ∴ $f(x)$ is p.m.f.

Q7 : Given a p.d.f.

$$f(x) = \begin{cases} \frac{1}{8}\, x & \text{for } 0 \leq x \leq 4 \\ 0 & \text{o.w.} \end{cases}$$

(a) Find the Value (t) s.t. $P(X \leq t) = \frac{1}{4}$

(b) Find the Value (t) s.t. $P(X \leq t) = \frac{1}{2}$

Sol.

(a) $P(X \leq t) = \frac{1}{4} \quad \ldots ①$

$\frac{1}{8} \int_0^t x\, dx = \frac{1}{16} x^2 \Big|_0^t = \frac{1}{16} t^2 \Rightarrow P(X \leq t) = \frac{1}{16} t^2 = \frac{1}{4}$ (by ①)

$\frac{1}{4} t^2 = 1 \Rightarrow t^2 = 4 \Rightarrow t = \sqrt{4} = \pm 2 \Rightarrow \boxed{t = 2}$ only

since $\{0 \leq x \leq 4\} = R_x$ is positive.

(b) $P(X \leq t) = \frac{1}{2}$

$P(X \leq t) = \frac{1}{8} \int_0^t x\, dx = \frac{1}{16} x^2 \Big|_0^t = \frac{1}{16} t^2 \Rightarrow \frac{1}{16} t^2 = \frac{1}{2} \rightarrow$ يتبع

$\frac{1}{8}t^2 = 1 \Rightarrow t^2 = 8 \Rightarrow t = \pm 2\sqrt{2} \Rightarrow t = +2\sqrt{2}$ (only) since $R_X = \{X : 0 \leq X \leq 4\}$ is positive.

Now; Find $P(X \leq 2\sqrt{2})$?

$$P(X \leq 2\sqrt{2}) = \frac{1}{8}\int_0^{2\sqrt{2}} x\,dx = \frac{1}{16}x^2\Big|_0^{2\sqrt{2}} = \frac{1}{16}(2\sqrt{2})^2 = \frac{1}{16}(8) = \underline{\underline{\frac{1}{2}}}$$

(c) Find the c.d.f. of X.

$$F(x) = \int_0^x f(t)\,dt = \int_0^x \frac{1}{8}t\,dt = \frac{x^2}{16}$$

$$\therefore F(x) = \begin{cases} 0 & \text{for } X < 0 \\ \frac{X^2}{16} & \text{for } 0 \leq X < 4 \\ 1 & \text{for } X \geq 4 \end{cases}$$

(الدالة f غير
مستمرة عند $X=4$)

Note $F(X)$ is conts. at $X=0$ since $F(0^+) = \frac{0^2}{16} = 0 = F(0^-)$.
$F(x)$ is conts. at $x=4$

Q8: Suppose that the c.d.f. of a r.v. X is as follows:

$$F(x) = \begin{cases} 0 & \text{for } X \leq 0 \\ \frac{X^2}{9} & \text{for } 0 < X \leq 3 \\ 1 & \text{for } X > 3 \end{cases}$$

Find and sketch (H.w.) the p.d.f. of X.

Sol. $f(x) = \frac{dF(x)}{dx} = F'(x)$ and $F(x)$ is conts. and diff.

$$f(x) = F'(x) = \frac{2}{9}X$$

$$\therefore f(x) = \begin{cases} \frac{2X}{9} & \text{for } 0 < X \leq 3 \\ 0 & \text{o.w.} \end{cases}$$

Note $F(3^+) = 1$
$F(3^-) = \frac{3^2}{9} = 1$
$\therefore F(3^+) = F(3^-)$
$\therefore F(x)$ is conts. at $x=3$

Q9: Answer by (true) or (False) and given the reason

① If X is c.r.v., then $P(a \leq X \leq b) \neq P(a < X < b)$.

(False) since $P(a \leq X \leq b) = P(a < X < b)$ if X is c.r.v., because $P(X=a)=0$ & $P(X=b)=0$ when X is c.r.v.

② If X is d.r.v. with c.d.f. $F(x)$, then $f(x) = \frac{d}{dx} F(x)$.

(False), only when X is c.r.v., $f(x) = \frac{d}{dx} F(x)$.

③ If X is a r.v. with c.d.f. $F(x)$, then $P(a < X < b) = F(b) - F(a)$.

(False), since by Th. $P(a < X \leq b) = F(b) - F(a)$

④ If a c.d.f. $F(x)$ conts. at $x=a$, then $P(X=a)=0$.

(True), since if $F(x)$ conts. at $x=a$, then $F(a^+) = F(a^-)$ and $P(X=a) = F(a^+) - F(a^-)$ (by Theorem).

⑤ $F(\infty) = 0$ & $F(-\infty) = 1$

(False), $F(\infty) = 1$ and $F(-\infty) = 0$ (By Theorem).

⑥ state (Cauchy dist.):

$$f(x) = \begin{cases} \frac{1}{\pi(1+x^2)} & \text{for } -\infty < x < \infty \\ 0 & \text{o.w.} \end{cases}$$

and p.m.f.

(False), Cauchy dist. is p.d.f. since $R_X = \{x; -\infty < x < \infty\}$ and $f(x) = \frac{1}{\pi(1+x^2)}$ for $-\infty < x < \infty$ (only).

H.W.: Given a c.d.f. $F(x) = \begin{cases} e^{x-3} & \text{for } x \leq 3 \\ [illegible] & \text{for } x > 3 \end{cases}$

① Find and sketch the p.d.f. $f(x)$.

② Is $F(x)$ conts. at $x=3$? Why?

Q10: Show that there does not exist any number (c) s.t. the following function f(x) would be a p.d.f.

$$f(x) = \begin{cases} \frac{c}{1+x} & \text{for } x > 0 \\ 0 & \text{o.w.} \end{cases}$$

Sol. If cond. ② is satisfied then f(x) is p.d.f.

I.e. T.P. $\int_{-\infty}^{\infty} f(x)\,dx \overset{?}{=} 1$ (if we can that) (لا يوجد عدد (c) يجعل هذه f(x) هي دالة احتمالية)

$$\int_0^\infty \frac{c}{1+x}\,dx = c\,Ln(1+x)\Big|_0^\infty = c[Ln(\infty) - Ln(1)]$$
$$= c[\infty - 0] = \infty$$

$\therefore$ c is not exist.

Then does not exist any no. (c) s.t. f(x) be a p.f.

H.W. Given a p.d.f. f(x):

$$f(x) = \begin{cases} ce^{-2x} & \text{for } x > 0 \\ 0 & \text{o.w.} \end{cases}$$

① Find the value c and sketch it's graph.

② Find the c.d.f. of X (F(x)) and sketch it's graph.

Q11: Show that the following function be a pr.f.

Sol. T.P. cond. ①

$$f(x) = \begin{cases} 2\sin x \cos x & \text{for } 0 < x < \frac{\pi}{2} \\ 0 & \text{o.w.} \end{cases}$$

Note: $2\sin x\cos x = \sin 2x$

Sol. T.P. cond. ① $f(x) \geq 0 \quad \forall x \in (0, \frac{\pi}{2})$

$$2\sin x\cos x \geq 0 \quad \forall x \in (0, \tfrac{\pi}{2})$$

T.P. cond. ②

$$\int_0^{\frac{\pi}{2}} 2\sin x\cos x\,dx = \int_0^{\frac{\pi}{2}} \sin 2x\,dx = \frac{-1}{2}\cos 2x\Big|_0^{\frac{\pi}{2}} = \frac{-1}{2}[\cos\pi - \cos 0]$$
$$= \frac{-1}{2}[-1-1] = 1$$

$\therefore$ By cond. ① and ② we get: f(x) is p.d.f.

Q_{12}: Show that if the following function be pr.f.:

$$f(x) = \begin{cases} \frac{2}{\pi}\sin^2(\frac{x}{2}) & \text{for } 0<x<\pi \\ 0 & \text{o.w.} \end{cases}$$

Sol. T.P. Cond. ①

i.e. $f(x) \geq 0 \quad \forall x \in (0,\pi)$

$\sin^2\frac{x}{2} \geq 0 \Rightarrow \frac{2}{\pi}\sin^2\frac{x}{2} \geq 0 \quad \forall x \in (0,\pi)$

T.P. Cond. ②

Note: $\sin^2 x = 1 - \cos x$

$$\frac{1}{\pi}\int_0^\pi 2\sin^2\frac{x}{2}\,dx = \frac{1}{\pi}\int_0^\pi (1-\cos x)\,dx$$

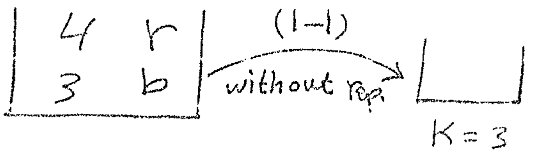
$$= \frac{1}{\pi}(x - \sin x)\Big|_0^\pi = \frac{1}{\pi}\left[(\pi-0)-(0-0)\right]$$

$$= \frac{\pi}{\pi} = 1$$

∴ $f(x)$ is a pr.d.f.

Q_{13}: A box has (4) red and (3) black balls. Choose a sample of (3) balls one by one without replacement. Let X be a number of black in a sample. Find the p.m.f. of X and sketch it's graph. Also find the Pr. that a sample has at least (2) red balls.

Sol.



$X \equiv$ no. of black balls in a sample.

∴ $X = 0,1,2,3 =$ d.r.v.

$f(x) = P(X=x) =$ p.m.f.

Event $(X=x)$: a sample have (x) black and (3-X) red balls.

→

Event $(X = x)$ has $P^{3}_{x} \, P^{4}_{3-x}$ and $n(S) = P^{7}_{3}$

$\underset{\text{no. of black balls}}{\uparrow}$ $\underset{\text{no. of red balls}}{\uparrow}$

$$f(x) = P(X = x) = \begin{cases} \dfrac{P^{3}_{x} \, P^{4}_{3-x}}{P^{7}_{3}} & \text{for } x = 0, 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$$

H.W. Find $F(x)$ of this problem.

Q14: Consider the following c.d.f.

$$F(x) = \begin{cases} 0 & \text{for } x < -1 \\ \frac{1}{2}(x^3 + 1) & \text{for } -1 \leq x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

Find ① $P(-1 < X < 1)$, $P(-1 < X \leq 1)$

② p.d.f. $f(x)$

③ $P(-2 < X < 2)$

Sol.

① $P(-1 \cancel{\leq} X < 1) = P(-1 < X \leq 1) = F(1) - F(-1)$ (by Th.)

$\underset{\text{X is c.r.v. \& F(x) is conts. at x = 1}}{\uparrow}$

$= 1 - \frac{1}{2}((-1)^3 + 1)$

$= 1$

② $$f(x) = F'(x) = \begin{cases} \frac{3}{2} x^2 & \text{for } -1 \leq x < 1 \\ 0 & \text{o.w.} \end{cases}$$

$\uparrow$ p.d.f. of X

Note: $F(x)$ is conts. at $x = -1$

③ $P(-2 < X < 2) = P(-2 < X \leq 2) = F(2) - F(-2)$ (by Th.)

$= 1 - 0$

$= 1$

Note $F(-2) = F(-2^{+}) - F(-2^{-})$

$= 0 - 0 = 0 \Rightarrow F$ is conts. at $x = -2$

Q15: Given a p.d.f.

$$f(x) = \begin{cases} b \sin x \cos x & \text{for } 0 < x < \frac{\pi}{2} \\ 0 & \text{o.w.} \end{cases}$$

Find the value of b.

Sol. $\because$ f(x) is a p.d.f. $\Rightarrow$ Cond. ② is satisfied.

$$\therefore b\int_0^{\frac{\pi}{2}} \sin x \cos x \, dx = 1$$

$$1 = b\left[\int_0^{\frac{\pi}{2}} \frac{1}{2} \sin 2x\right] dx \qquad (\text{since } \sin 2x = 2\sin x \cos x)$$

$$\frac{1}{2}\sin 2x = \sin x \cos x$$

$$1 = -\frac{b}{2} \cos 2x \Big|_0^{\frac{\pi}{2}}$$

$$1 = \frac{-b}{2}\left[\cos \pi - \cos 0\right]$$

$$1 = \frac{-b}{2}\left[-1 - (1)\right] \Rightarrow b = 1$$

$$\therefore f(x) = \begin{cases} \sin x \cos x & \text{for } 0 < x < \frac{\pi}{2} \\ 0 & \text{o.w.} \end{cases}$$

Q16: Given a d.f.

$$F(x) = \begin{cases} 0 & \text{for } x < -2 \\ \frac{x+3}{5} & \text{for } -2 \leq x < 2 \\ 1 & \text{for } x \geq 2 \end{cases}$$

(1) Find $P(X = -2)$, $P(-3 < X < -2)$

(2) Find f(x)

(3) Is F(x) conts. at $X = 2$? why?

Sol.

(1) $P(X = -2) = F(-2^{+}) - F(-2^{-})$ $\qquad$ by Th. $P(X = x) = F(x^{+}) - F(x^{-})$

$$= \left(\frac{-2+3}{5}\right) - 0 = \frac{1}{5}$$

$P(-3 < X < -2) + P(-2 < X \leq -2)$

since $\begin{pmatrix} F(x) \text{ is conts} \\ \text{at } x = -2 \end{pmatrix}$

ت →

(2) $f(x) = \frac{dF(x)}{dx} = F'(x) = \begin{cases} \frac{1}{5} & \text{for} \quad -2 < X < 2 \\ 0 & \text{o.w.} \end{cases}$

$P(-3 < x < -2) = \int_{-3}^{-2} f(x)\,dx = 0$

(3) $F(x)$ is conts. at $x = 2$ since $F(2^-) = F(2^+)$

$F(2^+) = 1$, $F(2^-) = \frac{2+3}{5} = \frac{5}{5} = 1$

Q17: Given the following p.m.f.

$$f(x) = \begin{cases} K\left(\frac{2}{3}\right)^{x-2} & \text{for} \quad x = 2, 3, 4, \ldots \\ 0 & \text{o.w.} \end{cases}$$

Find the value of K.

sol. ∵ $f(x)$ is p.m.f. then cond. ② is satisfied.

$$\therefore K \sum_{x=2}^{\infty} \left(\frac{2}{3}\right)^{x-2} = 1$$

Note $K \sum_{x=2}^{\infty} \left(\frac{2}{3}\right)^{x-2} = K \sum_{x=0}^{\infty} \left(\frac{2}{3}\right)^{x}$

$$\therefore 1 = K \sum_{x=0}^{\infty} \left(\frac{2}{3}\right)^{x} = K\left[\left(\frac{2}{3}\right)^0 + \left(\frac{2}{3}\right) + \cdots + \left(\frac{2}{3}\right)^n + \cdots\right]$$

$$1 = K\left[\underbrace{1 + \left(\frac{2}{3}\right) + \cdots + \left(\frac{2}{3}\right)^n + \cdots}_{\text{Geometric series}}\right]$$

الاساس $\rightarrow |r| = \left|\frac{2}{3}\right| < 1 \Leftrightarrow$ the series is convergent

$$1 = K\left[\frac{1}{1 - \left(\frac{2}{3}\right)}\right] = K \cdot \frac{1}{(1/3)} = 3K$$

$$\Rightarrow \quad 3K = 1 \Rightarrow \boxed{K = \frac{1}{3}}$$

$$\therefore f(x) = \begin{cases} \frac{1}{3}\left(\frac{2}{3}\right)^x & \text{for} \quad x = 0, 1, 2, \ldots \\ 0 & \text{o.w} \end{cases}$$

Q18: Given a set of 3 boys and 4 girls. Choose a sample of four students. If y be a random variable represents the number of boys in a sample:

1- Find the pr. distribution of y.

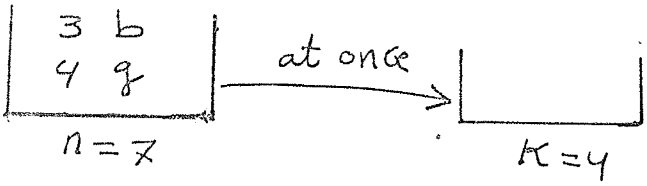
2- Find the c.d.f. of y. and sketch it's graph.

3- Find the pr. of the following by two methods:

(i) If a sample has at least (2) girls.

(ii) If a sample has at most (1) girl.

Sol.



$Y \equiv$ no. of boys in a sample.

$y = \{0, 1, 2, 3\} =$ d.r.v.

$n(s) = C^7_4$

$$(i)\ f(y) = P(Y=y) = \begin{cases} \dfrac{C^3_y \, C^4_{4-y}}{C^7_4} & \text{for } y = 0, 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$$

$$y = 0 \rightarrow f(0) = P(y=0) = \frac{C^3_0 \, C^4_4}{C^7_4} = \frac{1}{35}$$

$$y = 1 \rightarrow f(1) = P(y=1) = \frac{C^3_1 \, C^4_3}{C^7_4} = \frac{12}{35}$$

$$y = 2 \rightarrow f(2) = P(y=2) = \frac{C^3_2 \, C^4_2}{C^7_4} = \frac{18}{35}$$

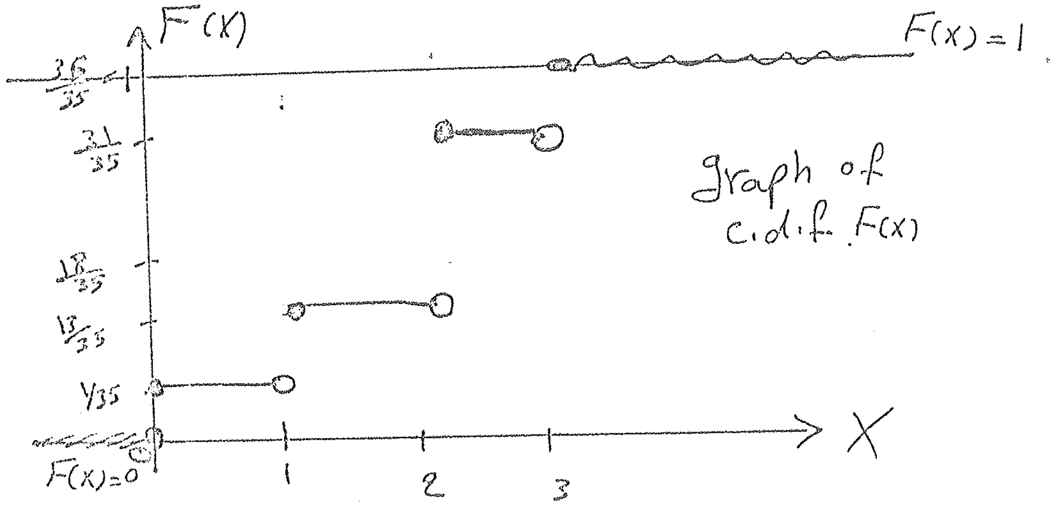
$$y = 3 \rightarrow f(3) = P(y=3) = \frac{C^3_3 \, C^4_1}{35} = \frac{4}{35}$$

pr. dist. of $X = \{(x_i, f(x_i))\ ;\ \forall x_i \in R_x\}$

$\therefore$ pr. dist. of $X = \{(0, \frac{1}{35}), (1, \frac{12}{35}), (2, \frac{18}{35}), (3, \frac{4}{35})\}$

(ii) To find c.d.f. $F(y)$

interval	integer	$f(y)$	$F(y)$
$y<0$	no integer	0	0
$0\leq y<1$	0	$1/35$	$1/35$
$1\leq y<2$	1	$12/35$	$13/35$
$2\leq y<3$	2	$18/35$	$31/35$
$y\geq 3$	3	$4/35$	1



(iii) $P(y\leq 2)=\sum_{y=0}^{2} f(y) = f(0)+f(1)+f(2)$

$= \frac{1}{35}+\frac{12}{35}+\frac{18}{35} = \frac{31}{35}$

or $P(y\leq 2) = F(2)$

$= \frac{31}{35}$

g	b
2	2
3	1
4	0

$\} \rightarrow y\leq 2$

$\underline{\underline{K=4}}$

$P(y=3) = f(3) = \frac{4}{35}$

or $P(y=3) = F(3^+) - F(3^-)$

$= 1 - \frac{31}{35} = \frac{4}{35}$

	g	b
لا يجوز	~~0~~	~~4~~
	1	3

$\Rightarrow y=3$

$\underline{\underline{K=4}}$

Q19: A box has (3) bad and (4) good tubes. Choose (3) tubes at once from this box.
Let X be a r.v. represents the number of bad tubes in a sample.

1- Find the pr. function of X and sketch it's graph.
2- Find the c.d.f. of X and sketch it's graph.
3- (i) Find $P(X<1 \,/\, 1<X<3)$.
(ii) Find the pr. that a sample has at least (1) good tube.

Sol:

$$\underset{n=7}{\boxed{\begin{matrix}3\ b\\ 4\ g\end{matrix}}} \xrightarrow{\text{at once}} \underset{K=3}{\boxed{\quad}} \quad , \; n(S) = C^7_3$$

$X \equiv$ no. of bad tube in a sample.

$\therefore R_X = \{0, 1, 2, 3\}$

$X = 0, 1, 2, 3 \equiv$ d.r.v.

Event $(X = x)$: a sample has x-bad and $(k-x)$ good tubes.

(1) $f(x) = P(X = x) =$ p.m.f.

$$\therefore f(x) = \begin{cases} \dfrac{C^3_x \; C^4_{3-x}}{C^7_3} & \text{for} \quad x = 0,1,2,3 \\ 0 & \text{o.w.} \end{cases}$$

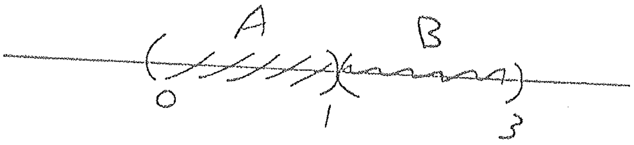
(2) c.d.f. $F(x) = \sum\limits_{x_j \leq x} f(x_j)$

interval	integer	$f(x)$ = p.m.f.	c.d.f. $F(x)$
$x<0$	no integer	0	0
$0 \leq x < 1$	0	4/35	4/35
$1 \leq x < 2$	1	18/35	22/35
$2 \leq x < 3$	2	12/35	34/35
$x \geq 3$	3	1/35	1

* Sketch it's graph.

$$(3)(i)\ P(\underbrace{X<1}_{A} \mid \underbrace{1<X<3}_{B}) = P(A|B) = \frac{P(AB)}{P(B)}$$

$$A = \{X : 0<X<1\}$$
$$B = \{X : 1<X<3\}$$



$$AB = \emptyset$$

$$\therefore P(X<1 \mid 1<X<3) = \frac{P(\emptyset)}{P(X=2)} = \frac{0}{f(2)} = \frac{0}{18/35} = 0$$

$$(ii)\ P(X\le 2) = F(2)$$
$$= \frac{34}{35}$$

$$\underline{\underline{\text{or}}}\quad P(X\le 2) = f(0)+f(1)+f(2)$$
$$= \frac{4}{35}+\frac{18}{35}+\frac{12}{35}$$
$$= \frac{34}{35}$$

g	b
1	2
2	1
3	0

$\} \to X \le 2$

K = 3

Probability
(Ch.3)

Group (1)

المرحلة الثالثة
اجب على جميع الاسئلة

Q1 Given a p.d.f. of X : الفصل الثالث

$$f(x) = \begin{cases} 2x & \text{for} \quad 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

Find the c.d.f. of X (F(x)).

Q2 If a r.v. $X \sim \text{unif}(0,10)$, Calculate احسب the pr. that:
① $P(X<3)$ & ② $P(X>6)$

Q3 Given a c.d.f. of X :

$$F(x) = \begin{cases} 0 & \text{for} \quad x<0 \\ \frac{x^2}{16} & \text{for} \quad 0\leq x<4 \\ 1 & \text{for} \quad x\geq 4 \end{cases}$$

Find the pr. of: ① $P(X=4)$, ② $P(X>0)$

Q4 Given a c.d.f. of X F(x), if F is continuous at $x=2$, IS $P(0<X<2) = F(2)-F(0)$? why ?.

Probability / Ch.3

Group (8)

اجب على جميع الاسئلة

Q1 Show that the following function is pr. function :

$$f(x) = \begin{cases} \frac{x+2}{25} & \text{for} \quad x=1,2,3,4,5 \\ 0 & \text{o.w.} \end{cases}$$

Q2 Given a c.d.f. of X :

$$F(x) = \begin{cases} 1-(1+x)e^{-x} & \text{for } x>0 \\ 0 & \text{for } x\leq 0 \end{cases}$$

Find: ① $P(X\leq 2)$, ② $P(1<X<3)$, ③ $P(X>4)$, ④ $P(X=0)$.

Q3 Answer اجب by (true) or (False) and given the reason السبب :
If a c.d.f. F(x) continuous at $x=2$, then $P(X=2) \neq 0$.

Q4 Given a set of integers $\{2, 4, 6, 8, 10\}$. Find the p.m.f. f(x) of X and sketch its graph.

Probability / Ch.3 Group P(2) اجب على جميع الاسئلة

Q1 Abox has (3) yellow and (2) red cards. Choose a sample of (3) cards. Let X be a number of yellow cards in a sample, then find:

① the pr.f. of X ($f(x)$).
② the c.d.f. of X ($F(x)$).
③ the pr. that a sample has (2) yellow cards.

Q2 Show that there does not exist any number (c) such that the following function f(x) would be a p.f. of X:

$$f(x) = \begin{cases} ce^{x} & \text{for } x>0 \\ 0 & \text{o.w.} \end{cases}$$

Q3 Given a p.d.f. of X:

$$f(x) = \begin{cases} \frac{1}{8}x & \text{for } 0 \leq x \leq 4 \\ 0 & \text{o.w.} \end{cases}$$

Find the value (t) such that $P(X \leq t) = \frac{1}{16}$.

المرحلة الثالثة

Probability /Ch.3 Group (7) اجب على جميع الاسئلة

Q1 Consider the c.d.f. of X:

$$F(x) = \begin{cases} 0 & \text{for } x<0 \\ x & \text{for } 0 \leq x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

① Find the p.f. of X ($f(x)$).
② IS F continuous at $x=1$? Why?.
③ Find $P(X=0)$.

Q2 Show that the following function is pr.f. of X:

$$f(x) = \begin{cases} \left(\frac{1}{2}\right)^{x} & \text{for } x = 1, 2, 3, \ldots \\ 0 & \text{o.w.} \end{cases}$$

Q3 رمية العملة

Toss a coin fourth times (4^{th} times). Find the c.d.f. of a r.v. X ($F(x)$), when X represents the numbers of tail (T) in a sample.

Probability / Ch.3 — Group (3) — اجب على جميع الاسئلة

Q₁ Given the d.f. $F(x)$ of X:

$$F(x) = \begin{cases} 0 & \text{for } x < -1 \\ \frac{x+1}{2} & \text{for } -1 \leq x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

Find: ① $P(X=1)$, ② $P(X<-1)$, ③ $P(-\frac{1}{2} < X < \frac{1}{2})$

Q₂ Find the value of (c) from the following Pr. function:

$$f(x) = \begin{cases} c\left(\frac{1}{4}\right)^x & \text{for } x = 1, 2, 3, \ldots \\ 0 & \text{o.w.} \end{cases}$$

Q₃ Is the following function is Pr. function? why?.

$$f(x) = \begin{cases} \frac{1}{5} & \text{for } x = 0, 1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$$

Q₄ Find the c.d.f. $F(x)$ from the following Pr.f. of X:

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2-x & 1 \leq x < 2 \\ 0 & \text{o.w.} \end{cases}$$

Probability / Ch.3 — Group (6) — اجب على جميع الاسئلة

Q₁ Is the following function is Pr. function? Why?

$$f(x) = \begin{cases} \frac{x-2}{5} & \text{for } x = 1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$$

Q₂ Given a c.d.f. F of X as follows:

$$F(x) = \begin{cases} 1-(1+x)e^{-x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

① Find $P(X \leq -2)$, $P(X > 4)$
② Is F continuous at $x=0$? why?

Q₃ State Cauchy distribution. (Note: only state) اكتب التوزيع فقط

Q₄ Given a c.d.f. of X:

$$F(x) = \begin{cases} 1-e^{-x} & \text{for } x \geq 0 \\ 0 & \text{o.w.} \end{cases}$$

Find the p.f. of X ($f(x)$).

Probability/Ch.3

Group (5)

المرحلة الثالثة
اجب على جميع الاسئلة

Q1 Given a p.d.f. of X:

$$f(x) = \begin{cases} ce^{x} & \text{for } x<0 \\ 0 & \text{o.w.} \end{cases}$$

① Find the value of c.
② Find F(x)

Q2 Show that the folloing function is p.f.:

$$f(x) = \begin{cases} 7(\frac{1}{8})^{x} & \text{for } x=1,2,3,\ldots \\ 0 & \text{o.w.} \end{cases}$$

Q3 If $X \sim$ uniform (0,1), find p.f. $f(x)$ of X.

Q4 Given a c.d.f. of X:

$$F(x) = \begin{cases} 0 & \text{for } x \leq -1 \\ \frac{1}{2}(x^{3}+1) & \text{for } -1<x<1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

Find the p.f. of X (f(x)).

Probability/Ch.3

Group (4)

المرحلة الثالثة
اجب على جميع الاسئلة

Q1 Find the Pr. distribution of X from the following Pr.f. of X:

$$f(x) = \begin{cases} \frac{1}{3} & \text{for } x=1,2,3 \\ 0 & \text{o.w.} \end{cases}$$

Q2 Show that there does not exist any number (c) such that the following function f(x) would be a p.f.:

$$f(x) = \begin{cases} ce^{-x} & \text{for } x<0 \\ 0 & \text{o.w.} \end{cases}$$

Q3 Given a set of 3-boys and 3-girls. A sample of (2) students are Choosing. If Z be a r.v. represents تمثل the number of boys in a sample, then find:

① the pr. distribution of Z.
② the c.d.f. of Z (F(Z)).
③ the Pr. that a sample has 3-boys.

$$f(x) = \begin{cases} 2x & 0<x<1 \\ 0 & \text{ow} \end{cases} \qquad \int_0^1 2x\,dx = x^2\Big]_0^1 = 1$$

$$y = 2x \qquad 0<y<2$$

$$x = \frac{y}{2} \longrightarrow \frac{dx}{dy} = J = \frac{1}{2}$$

$$g(y) = \begin{cases} \cancel{2}\left[\frac{y}{\cancel{2}}\right]\cdot\frac{1}{2} = y/2 & 0<y<2 \\ 0 & \text{ow} \end{cases}$$

$$\int \frac{y}{2}\,dy = \frac{y^2}{4}\Big]_0^2 = \frac{1}{4}(4-0) = 1$$

حلول أسئلة الامتحان الثاني (Ch.3) لمادة الاحتمالية / المرحلة (٣) / ٩/١/٢٠٠٦

Group (1)

1: Given a p.d.f. of X:

$$f(x) = \begin{cases} 2x & \text{for } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

Find the c.d.f. of X $(F(x))$.

sol.

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$= 2\int_{0}^{x} t\,dt = \frac{\cancel{2} t^2}{\cancel{2}}\Big|_0^x = x^2$$

$$\therefore F(x) = \begin{cases} x^2 & \text{for } 0 \le x < 1 \\ 0 & \text{o.w.} \end{cases}$$

الصفر دخل ضمن المساحة (وأول حد للتكامل)

Q2: If a r.v. $X \sim \text{unif}(0,10)$, Calculate the pr. that: احسب

① $P(X<3)$ ② $P(X>6)$

sol. $\because X \sim \text{uniform}(0,10)$

$$\therefore f(x) = \begin{cases} \frac{1}{10} & \text{for } 0 < x < 10 \\ 0 & \text{o.w.} \end{cases}$$

$$P(X<3) = \int_0^3 \frac{1}{10}\,dx = \frac{x}{10}\Big|_0^3 = \frac{1}{10}(3-0) = \frac{3}{10}$$

$$P(X>6) = \int_6^{10} \frac{1}{10}\,dx = \frac{x}{10}\Big|_6^{10} = \frac{1}{10}(10-6) = \frac{4}{10}$$

Q3: Given a c.d.f. of X:

$$F(x) = \begin{cases} 0 & \text{for } x<0 \\ \frac{x^2}{16} & \text{for } 0 \le x < 4 \\ 1 & \text{for } x \ge 4 \end{cases}$$

Find the pr. of: ① $P(X=4)$, ② $P(X>0)$

Sol. $P(X=4) = F(4^+) - F(4^-)$

$= 1 - \frac{4^2}{16} = 1 - \frac{16}{16} = 1-1 =$ Zero

$P(X>0) = 1 - P(X \leq 0)$

$= 1 - F(0)$ by def. of $F(x)$

$= 1 - F(0^+)$ by theorem $F(x) = F(x^+)$

$= 1 - \frac{0^2}{16} = 1 - 0 = 1$

Q4: Given a c.d.f. of X $(F(x))$, if F is continuous at $x=2$, Is $P(0<X<2) = F(2)-F(0)$? why?

Yes, since X is c.r.v. & $P(X=2)=0$

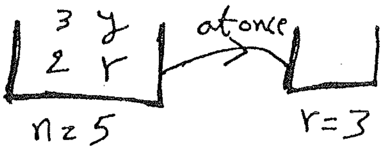
then $P(0<X<2) = P(0<X\leq 2) = F(2)-F(0)$ by theorem.

Group (2)

Q1: A box has (3) yellow and (2) red cards. Choose a sample of (3) cards. Let X be a number of yellow cards in a sample, then find:

① the pr. f. of X $(f(x))$.
② the c.d.f. of X $(F(x))$.
③ the pr. that a sample has (2) yellow cards.

Sol.



$X \equiv$ no. of yellow cards in a sample.

$x = 1, 2, 3$ (d.r.v.)

$$f(x) = \begin{cases} \frac{C^3_x \, C^2_{3-x}}{C^5_3} & \text{for } x = 1, 2, 3 \\ 0 & \text{o.w.} \end{cases}$$

)

interval	no. of integer	$f(x)$	$F(x)=\sum_{x_j\le x} f(x_j)$
$X<1$	no integer	0	0
$1\leq X<2$	1	$f(1)=\frac{C^{3}_{1}C^{2}_{2}}{C^{5}_{3}}$	$F(1)=f(1)+0$
$2\leq X<3$	2	$f(2)=\frac{C^{3}_{2}C^{2}_{1}}{C^{5}_{3}}$	$F(2)=0+f(1)+f(2)$
$X\geq 3$	3	$f(3)=\frac{C^{3}_{3}C^{2}_{0}}{C^{5}_{3}}$	$F(3)=0+f(1)+f(2)+f(3)$ $=1$

$$\therefore F(X)=\begin{cases} 0 & \text{for} \quad X<1 \\ F(1) & \text{for} \quad 1\leq X<2 \\ F(2) & \text{for} \quad 2\leq X<3 \\ F(3)=1 & \text{for} \quad X\geq 3 \end{cases}$$

③ $P(X=2)=P(\text{2-yellow cards in a sample})=f(2)=\frac{C^{3}_{2}C^{2}_{1}}{C^{5}_{3}}$

Q2 Show that there does not exist any number (c) such that the following function f(x) would be a p.f. of X :

$$f(x)=\begin{cases} c e^{x} & \text{for} \quad x>0 \\ 0 & \text{o.w.} \end{cases}$$

Sol. by condition (2) : (If f(x) p.f. then it satisfy this condition).

$$c\int_{0}^{\infty} e^{x}\,dx=1 \Longrightarrow c\int_{0}^{\infty} e^{x}\,dx = c\,e^{x}\Big|_{0}^{\infty}=c[e^{\infty}-e^{0}]$$
$$=c[\infty-1]=c(\infty)\neq 1$$

∴ there does not exist any number (c) such that the following function f(x) would be a p.f. of X.

Q3: Given a p.d.f. of X :

$$f(x)=\begin{cases} \frac{1}{8}x & \text{for} \quad 0\leq x\leq 4 \\ 0 & \text{o.w} \end{cases}$$

Find the Value (t) such that $P(X\leq t)=\frac{1}{16}$.

Sol.

$$P(X\leq t)=\frac{1}{8}\int_0^t x\,dx=\frac{1}{8}\cdot\frac{x^2}{2}\Big|_0^t=\frac{1}{16}(t^2-0^2)=\frac{t^2}{16}=\frac{1}{16}$$

$$\Rightarrow t^2=1 \Rightarrow t=\pm 1 \Rightarrow t=1 \text{ only since } t\in[0,4].$$

Group (3)

P1: Given the d.f. $F(x)$ of X:

$$F(x)=\begin{cases}0 & \text{for } x<-1\\ \frac{x+1}{2} & \text{for } -1\leq x<1\\ 1 & \text{for } x\geq 1\end{cases}$$

Find: ① $P(X=1)$, ② $P(X<-1)$, ③ $P(-\frac{1}{2}<X<\frac{1}{2})$

Sol.

) $P(X=1)=F(1^+)-F(1^-)$ (by theorem)

$$=1-\left(\frac{1+1}{2}\right)$$

$$=1-\frac{2}{2}=1-1=\text{Zero}$$

② $P(X<-1)=F(-1^-)$ (by theorem)

$$=0$$

③ $P(-\frac{1}{2}<X<\frac{1}{2})=P(-\frac{1}{2}<X\leq\frac{1}{2})$ since F is contn. at $x=\frac{1}{2}$

$=F(\frac{1}{2})-F(-\frac{1}{2})$ by theorem

$$=\frac{(\frac{1}{2}+1)}{2}-\frac{(-\frac{1}{2}+1)}{2}$$

$$=\frac{3}{4}-\frac{1}{4}=\frac{2}{4}=\frac{1}{2}$$

P2: Find the value of (c) from the following Pr. function:

$$f(x)=\begin{cases}c\left(\frac{1}{4}\right)^x & \text{for } x=1,2,3,\ldots\\ 0 & \text{o.w.}\end{cases}$$

Sol. by Cond. (2), we get: $\sum_{x=1}^{\infty} f(x)=1$

$$\therefore C \sum_{x=1}^{\infty} \left(\frac{1}{4}\right)^x = 1$$

$$C\left[\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\cdots\right] = \frac{1}{4} C\left[\underbrace{1+\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)^2+\cdots}_{\text{Geometric series}}\right]$$

where $r = \frac{1}{4} < 1 \Rightarrow$ this series is Convergent (by theorem)

$$\therefore \sum_{x=1}^{\infty} \left(\frac{1}{4}\right)^x = \frac{1}{1-\frac{1}{4}} = \frac{1}{3/4} = \frac{4}{3} \quad \text{(by theorem)}$$

$$\Rightarrow C \cancel{\frac{1}{4}}\left(\frac{\cancel{4}}{3}\right) = 1 \Rightarrow \frac{C}{3} = 1 \Rightarrow C = 3$$

$$\therefore f(x) = \begin{cases} 3\left(\frac{1}{4}\right)^x & \text{for } x = 1, 2, 3, \ldots \\ 0 & \text{o.w.} \end{cases}$$

Q₃: Is the following function is pr. function? why?

$$f(x) = \begin{cases} \frac{1}{5} & \text{for } x = 0, 1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$$

Sol. Cond. ② $\sum_{x=1}^{5} \frac{1}{5} = \frac{6}{5} > 1$

∴ Cond. ② doesnt satisfied.

∴ $f(x)$ is not pr. function.

Q₄: Find the c.d.f. $F(x)$ from the following pr.f of X:

$$f(x) = \begin{cases} X & \text{for } 0 < X < 1 \\ 2-X & \text{for } 1 \le X < 2 \\ 0 & \text{o.w.} \end{cases}$$

Sol. $$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_0^x t\,dt + \int_1^x (2-t)\,dt$$

$$= \frac{t^2}{2}\Big|_0^x + \left(2t - \frac{t^2}{2}\right)\Big|_1^x$$

$$= \frac{1}{2}x^2 + \left(2x - \frac{1}{2}x^2\right) - \left(2 - \frac{1}{2}\right)$$

$$= \underbrace{\left(2x - \frac{3}{2} - \frac{1}{2}x^2\right)}_{x \in [1,2]} + \underbrace{\left(\frac{1}{2}x^2\right)}_{x \in [0,1]}$$

$$\therefore F(x) = ?$$

$$F(X) = \begin{cases} 0 & \text{for} \quad X<0 \\ \frac{1}{2}X^2 & \text{for} \quad 0 \leq X<1 \\ 2X-\frac{1}{2}X^2-\frac{3}{2} & \text{for} \quad 1 \leq X<2 \\ 1 & \text{for} \quad X \geq 2 \end{cases}$$

Group(4)

Q1: Find the pr. distribution of X from the following pr.f of

$$f(x) = \begin{cases} \frac{1}{3} & \text{for} \quad x=1,2,3 \\ 0 & \text{o.w.} \end{cases}$$

sol.

$$\text{Pr. dist. of } X = \{ (x_i, f(x_i)) \; ; \; i=1,2,3 \}$$
$$= \{ (1,\tfrac{1}{3}), (2,\tfrac{1}{3}), (3,\tfrac{1}{3}) \}$$

Q2: Show that there does not exist any number (c) such that the following function f(x) would be a p.f.:

$$f(x) = \begin{cases} ce^{-X} & \text{for} \quad X<0 \\ 0 & \text{o.w.} \end{cases}$$

Sol.

If f(x) is p.f. then Cond.(2) is satisfied :

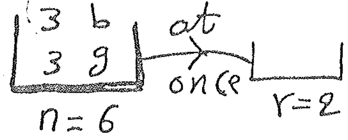
$$c\int_{-\infty}^{0} e^{-x}dx = -ce^{-x}\Big|_{-\infty}^{0} = -c[e^0 - e^{\infty}] = -c[1-\infty] = 1$$
$$= -c+\infty = 1$$
$$\Rightarrow \quad \therefore c \neq 1$$

∴ There does not exist any number (c) such that the following function f(x) would be a p.f.

Q3: Given a set of 3-boys and 3-girls. A sample of (2) students are Choosing. If Z be a r.v. represents the number of boys in a sample, then find:

① the Pr. distribution of Z.
② the c.d.f. of Z.
③ the pr. that a sample has 3-boys.

Sol.



Z ≡ no. of boys in a sample.
$Z = 0, 1, 2$

① $$f(z) = \begin{cases} \dfrac{C_z^3 \, C_{2-z}^3}{C_2^3} & \text{for } z = 0, 1, 2 \\ 0 & \text{o.w.} \end{cases}$$

pr. distribution of $Z = \{(z_i, f(z_i))\,;\, i = 0, 1, 2\}$

$$= \{(0, f(0)), (1, f(1)), (2, f(2))\}$$

$$= \left\{\left(0, \frac{C_0^3 C_2^3}{C_2^3}\right), \left(1, \frac{C_1^3 C_1^3}{C_2^3}\right), \left(2, \frac{C_2^3 C_0^3}{C_2^3}\right)\right\}$$

$$= \left\{\left(0, \frac{3(1)}{3}\right), \left(1, \frac{(3)3}{3}\right), \left(2, \frac{3(1)}{3}\right)\right\}$$

$$= \{(0,1), (1,3), (2,1)\}$$

②

$$F(z) = \begin{cases} 0 & \text{for } z < 0 \\ f(0) + 0 = F(0) & \text{for } 0 \leq z < 1 \\ f(1) + f(0) + 0 = F(1) & \text{for } 1 \leq z < 2 \\ f(2) + f(1) + f(0) + 0 = F(2) = 1 & \text{for } z \geq 2 \end{cases}$$

③ $P(\text{A sample has 3-boys}) = P(Z = 3) = f(3) = 0$

Group (5)

Q_1 : Given a p.d.f. of X :

$$f(x) = \begin{cases} ce^x & \text{for} \quad x<0 \\ 0 & \text{o.w.} \end{cases}$$

① Find the value of c.

② Find F(x)

Sol. ① by cond.(2), we get : $c\int_{-\infty}^{0} e^x dx = 1$

$$\Rightarrow \quad ce^x\Big|_{-\infty}^{0} = 1 \Rightarrow c[e^0 - e^{-\infty}] = 1 \Rightarrow c[1-0] = 1$$

$\therefore \; c = 1$

$$\therefore \; f(x) = \begin{cases} e^x & \text{for} \quad x<0 \\ 0 & \text{o.w.} \end{cases} \qquad \left(\begin{array}{c} x<0 \\ \Rightarrow -\infty < x < 0 \end{array} \right)$$

② $F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} e^t dt = e^t\Big|_{-\infty}^{x} = e^x - \underbrace{e^{-\infty}}_{=0}$

$= e^x$

$$\therefore \; F(x) = \begin{cases} e^x & \text{for} \quad x<0 \\ 1 & \text{for} \quad x \geq 0 \end{cases}$$

Q_2 : Show that the following function is P.f. :

$$f(x) = \begin{cases} 7(\frac{1}{8})^x & \text{for} \quad x = 1, 2, 3, \ldots \\ 0 & \text{o.w.} \end{cases}$$

Sol. Cond. ① : $7(\frac{1}{8})^1 = f(1) \geq 0$

$7(\frac{1}{8})^2 = f(2) \geq 0$

$\therefore \; f(x) \geq 0$ & Cond.(1) is satisfied.

Cond. ② $7\sum_{x=1}^{\infty} (\frac{1}{8})^x = 7[(\frac{1}{8})^1 + (\frac{1}{8})^2 + (\frac{1}{8})^3 + \ldots]$

$= (\frac{7}{8})[\underbrace{1 + (\frac{1}{8}) + (\frac{1}{8})^2 + \ldots}_{\text{Geometric series}}]$

by theorem & $r=\left|\frac{1}{8}\right|<1 \Rightarrow$ this series is convergent

by theorem, we get: $\sum_{x=0}^{\infty}\left(\frac{1}{8}\right)^{x} = \frac{1}{1-\frac{1}{8}} = \frac{1}{\frac{7}{8}} = \frac{8}{7}$

$\therefore\ 7\sum_{x=1}^{\infty}\left(\frac{1}{8}\right)^{x} = \cancel{\left(\frac{7}{8}\right)}\cdot\cancel{\left(\frac{8}{7}\right)} = 1$

$\therefore$ Cond.(2) is satisfied.

Q3: If $X \sim$ uniform$(0,1)$, find p.f. $f(x)$ of X.

Sol.

$$f(x)=\begin{cases} 1 & \text{for } 0<X<1 \\ 0 & \text{o.w.} \end{cases}$$

Q4: Given a c.d.f. of X

$$F(x)=\begin{cases} 0 & \text{for } X\leq -1 \\ \frac{1}{2}(X^{3}+1) & \text{for } -1<X<1 \\ 1 & \text{for } X\geq 1 \end{cases}$$

Find the p.f. of X ($f(x)$).

Sol.

$$f(x)=F'(X)=\begin{cases} 0 & \text{for } X\leq -1 \\ \frac{3}{2}X^{2} & \text{for } -1<X<1 \\ 0 & \text{for } X\geq 1 \end{cases}$$

$$f(x)=\begin{cases} \frac{3}{2}x^{2} & \text{for } -1\leq X\leq -1 \\ 0 & \text{o.w.} \end{cases}$$

← since F is continuous at $x=-1$

Group (6)

Q1: Is the following function is Pr. function? why?

$$f(x)=\begin{cases} \frac{x-2}{5} & \text{for } x=1,2,3,4,5 \\ 0 & \text{o.w.} \end{cases}$$

Sol. Cond.①: $f(1)=\frac{-1}{5}<0$, $f(2)=0$, $f(x)\geq 0$ $x=3,4,5$

$\therefore$ f is not pr. function.

Q2 : Given a c.d.f. F of X as follows :

$$F(x) = \begin{cases} 1-(1+x)e^{-x} & \text{for } x = 1,2,3,4,5 \\ 0 & \text{o.w.} \end{cases}$$

① Find $P(X \leq -2)$, $P(X>4)$

② Is F continuous at $x=0$? why?

Sol.

① $P(X \leq -2) = F(-2)$ by def. of F

$= 0$

$$P(X>4) = 1 - P(X \leq 4)$$
$$= 1 - F(4)$$
$$= 1 - [(1+4)e^{-4}]$$

② Yes , since $F(0^+) = F(0^-) = \text{Zero}$

Q3 : State Cauchy distribution .

$$f(x) = \frac{1}{\pi(1+x^2)} \quad \text{for} \quad -\infty < x < \infty$$

Q4 : Given a c.d.f. of X :

$$F(x) = \begin{cases} 1-e^{-x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

Find the pr. f. of X $(f(x))$.

Sol.

$$f(x) = F'(x) = \begin{cases} e^{-x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases} \longrightarrow$$

since $F(0^-) = F(0^+)$ & F is conts. at $x=0$

Group (7)

Consider the c. d.f. of X :

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \leq x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

① Find the P.f. of X ($f(x)$).

② Is F conts. at $x=1$? Why?

③ Find $P(X=0)$.

sol.

① $f(x) = F'(x) = \begin{cases} 0 & \text{for } x<0 \\ 1 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{for } x \geq 1 \end{cases} = \begin{cases} 1 & \text{for } 0<x<1 \\ 0 & \text{for o.w.} \end{cases}$

② Yes, since $F(1^+) = F(1^-) = 1$.

③ $P(X=0) = F(0^+) - F(0^-)$

$= 0 - 0 =$ Zero

Q2: Show that the following function is Pr. f. of X :

$$f(x) = \begin{cases} (\frac{1}{2})^x & \text{for } x=1,2,3,\dots \\ 0 & \text{o.w.} \end{cases}$$

sol.

Cond. ① $(\frac{1}{2})^1 > 0,\ (\frac{1}{2})^2 > 0, \dots$

$\therefore f(x) \geq 0 \quad \forall x = 1,2,3,\dots$

Cond. ② $\sum_{x=1}^{\infty} (\frac{1}{2})^x = [(\frac{1}{2})^1 + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \dots]$

$= (\frac{1}{2}) [\underbrace{1 + (\frac{1}{2}) + (\frac{1}{2})^2 + \dots}_{\text{Geometric Series}}] ; r = |\frac{1}{2}| < 1$

$= \frac{1}{2} (\frac{1}{1-\frac{1}{2}}) = \frac{1}{\cancel{2}} \cdot \cancel{2} = 1$

Q3: Toss a coin fourth times (4^{th} times). Find the c.d.f. of a r.v. X, when X represents the numbers of tail ($\boxed{T}$) in a sample.

Sol.

$$f(x) = \begin{cases} \frac{C_x^4}{16} & \text{for } x = 0,1,2,3,4 \\ 0 & \text{o.w.} \end{cases} \quad \Big| \quad n(s) = 2^4 = 16$$

$$F(x) = \sum_{x_j \leq x} f(x_j)$$

$$F(x) = \begin{cases} 0 & \text{for} \quad X < 0 \\ F(0) = 0 + f(0) & 0 \leq X < 1 \\ F(1) = 0 + f(0) + f(1) & 1 \leq X < 2 \\ F(2) = 0 + f(0) + f(1) + f(2) & 2 \leq X < 3 \\ F(3) = 0 + f(0) + f(1) + f(2) + f(3) & 3 \leq X < 4 \\ F(4) = 1 = 0 + f(0) + f(1) + f(2) + f(3) + f(4) & X \geq 4 \end{cases}$$

Group (8)

Q₁: Show that the following function is pr. function:

$$f(x) = \begin{cases} \frac{x+2}{25} & \text{for} \quad x = 1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$$

sol. Cond. ②: $\frac{1}{25}\sum_{x=1}^{5}(X+2) = \frac{1}{25}[3+4+5+6+7] = \frac{25}{25} = 1$

Cond. ① $f(1) = \frac{3}{25} > 0, f(2) = \frac{4}{25} > 0, \ldots, f(5) = \frac{7}{25} > 0$

$\therefore f(X) \geq 0 \quad \forall X = 1, 2, 3, 4, 5$

$\therefore f(x)$ is pr. f.

Q₂: Given a c.d.f. of X:

$$F(x) = \begin{cases} 1-(1+x)e^{-x} & \text{for } X > 0 \\ 0 & \text{for } X \leq 0 \end{cases}$$

Find: ① $P(X \leq 2) = F(2)$ by def. of F

$= 1-(1+2)e^{-2}$

② $P(1 < X < 3) = P(1 \leq X \leq 3)$ since F is conts. at x=3

$= F(3) - F(1)$

$= (1-(1+3)e^{-3}) - (1-(1+1)e^{-1})$

③ $P(X > 4) = 1 - P(X \leq 4) = 1 - F(4) = 1-(1-(1+4)e^{-4})$

④ $P(X=0) = F(0^+) - F(0^-) = 1-(1+0)e^{0} - 0$

$= 0$

by theorem & $r = \left|\frac{1}{8}\right| < 1 \Rightarrow$ this series is convergent

by theorem, we get: $\sum_{x=0}^{\infty}\left(\frac{1}{8}\right)^x = \frac{1}{1-\frac{1}{8}} = \frac{1}{\frac{7}{8}} = \frac{8}{7}$

$$\therefore\ 7\sum_{x=1}^{\infty}\left(\frac{1}{8}\right)^x = \cancel{\left(\frac{7}{8}\right)}\cdot\cancel{\left(\frac{8}{7}\right)} = 1$$

$\therefore$ Cond.(2) is satisfied.

Q_3: If $X \sim$ uniform$(0,1)$, find p.f. $f(x)$ of X.

Sol.

$$f(x) = \begin{cases} 1 & \text{for } 0 < X < 1 \\ 0 & \text{o.w.} \end{cases}$$

Q_4: Given a c.d.f. of X

$$F(x) = \begin{cases} 0 & \text{for } X \leq -1 \\ \frac{1}{2}(X^3+1) & \text{for } -1 < X < 1 \\ 1 & \text{for } X \geq 1 \end{cases}$$

Find the p.f. of X ($f(x)$).

Sol.

$$f(x) = F'(x) = \begin{cases} 0 & \text{for } X \leq -1 \\ \frac{3}{2}X^2 & \text{for } -1 < X < 1 \\ 0 & \text{for } X \geq 1 \end{cases}$$

$$f(x) = \begin{cases} \frac{3}{2}x^2 & \text{for } -1 \leq X \leq -1 \\ 0 & \text{o.w.} \end{cases} \longleftarrow \text{since } F \text{ is continuous at } x=-1$$

Group (6)

Q_1: Is the following function is pr. function? why?

$$f(x) = \begin{cases} \frac{x-2}{5} & \text{for } x = 1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$$

Sol. Cond.① : $f(1) = \frac{-1}{5} < 0$, $f(2) = 0$, $f(x) \geq 0$ $x = 3, 4, 5$

$\therefore$ f is not pr. function.

Q2 : Given a c.d.f. F of X as follows :

$$F(x) = \begin{cases} 1-(1+x)e^{-x} & \text{for } x=1,2,3,4,5 \\ 0 & \text{o.w.} \end{cases}$$

① Find $P(X \leq -2)$, $P(X > 4)$

② Is F continuous at $x=0$? why?

Sol.

① $P(X \leqslant -2) = F(-2)$ by def. of F

$= 0$

$P(X > 4) = 1 - P(X \leqslant 4)$

$= 1 - F(4)$

$= 1 - [(1+4)e^{-4}]$

② Yes , since $F(0^+) = F(0^-) =$ Zero.

Q3 : State Cauchy distribution .

$$f(x) = \frac{1}{\pi(1+x^2)} \quad \text{for} \quad -\infty < x < \infty$$

Q4 : Given a c.d.f. of X :

$$F(x) = \begin{cases} 1-e^{-x} & \text{for } x \geqslant 0 \\ 0 & \text{for } x < 0 \end{cases}$$

Find the pr. f. of X $(f(x))$.

Sol.

$$f(x) = F'(x) = \begin{cases} e^{-x} & \text{for } x \geqslant 0 \\ 0 & \text{for } x < 0 \end{cases} \longrightarrow \text{since}$$

$F(0^-) = F(0^+)$
& F is conts. at $x=0$

Group (7)

Consider the c.d.f. of X :

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \leq x < 1 \\ 1 & \text{for } x \geqslant 1 \end{cases}$$

① Find the P.f. of X ($f(x)$).

② Is F conts. at $x=1$? why ?

③ Find $P(X=0)$.

Sol.

① $f(x) = F'(x) = \begin{cases} 0 & \text{for } x<0 \\ 1 & \text{for } 0\leq x\leq 1 \\ 0 & \text{for } x\geq 1 \end{cases} = \begin{cases} 1 & \text{for } 0<x<1 \\ 0 & \text{for o.w.} \end{cases}$

② Yes, since $F(1^+) = F(1^-) = 1$.

③ $P(X=0) = F(0^+) - F(0^-)$
$= 0 - 0 = \text{Zero}$

Q2: Show that the following function is Pr. f. of x :

$$f(x) = \begin{cases} (\frac{1}{2})^x & \text{for } x=1,2,3,\ldots \\ 0 & \text{o.w.} \end{cases}$$

Sol.

Cond. ① $(\frac{1}{2})^1 > 0$, $(\frac{1}{2})^2 > 0$, ...

$\therefore f(x) \geq 0 \quad \forall x = 1,2,3,\ldots$

Cond. ② $$\sum_{x=1}^{\infty} (\tfrac{1}{2})^x = [(\tfrac{1}{2})^1 + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^3 + \cdots]$$
$$= (\tfrac{1}{2})\,[\underbrace{1+(\tfrac{1}{2})+(\tfrac{1}{2})^2+\cdots}_{\text{Geometric series}}] \;;\; r=|\tfrac{1}{2}|<1$$
$$= \tfrac{1}{2}\left(\frac{1}{1-\frac{1}{2}}\right) = \tfrac{1}{\cancel{2}}\cdot\cancel{2} = 1$$

Q3: Toss a coin fourth times (4^{th} times). Find the c.d.f. of a r.v. X, when X represents the numbers of tail (T) in a sample.

Sol.

$$f(x) = \begin{cases} \dfrac{C^4_x}{16} & \text{for } x=0,1,2,3,4 \\ 0 & \text{o.w.} \end{cases}$$

$n(S) = 2^4 = 16$

$$F(x) = \sum_{x_j \leq x} f(x_j)$$

$$F(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ F(0) = 0 + f(0) & 0 \leq x < 1 \\ F(1) = 0 + f(0) + f(1) & 1 \leq x < 2 \\ F(2) = 0 + f(0) + f(1) + f(2) & 2 \leq x < 3 \\ F(3) = 0 + f(0) + f(1) + f(2) + f(3) & 3 \leq x < 4 \\ F(4) = 1 = 0 + f(0) + f(1) + f(2) + f(3) + f(4) & x \geq 4 \end{cases}$$

Group (8)

Q₁: Show that the following function is pr. function:

$$f(x) = \begin{cases} \frac{x+2}{25} & \text{for} \quad x = 1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$$

sol. Cond. ② : $\frac{1}{25} \sum_{x=1}^{5} (x+2) = \frac{1}{25} [3+4+5+6+7] = \frac{25}{25} = 1$

Cond. ① $f(1) = \frac{3}{25} > 0, \; f(2) = \frac{4}{25} > 0, \; \dots, \; f(5) = \frac{7}{25} > 0$

$\therefore f(x) \geq 0 \quad \forall x = 1, 2, 3, 4, 5$

$\therefore f(x)$ is pr.f.

Q₂: Given a c.d.f. of X:

$$F(x) = \begin{cases} 1-(1+x)e^{-x} & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x \leq 0 \end{cases}$$

Find: ① $P(X \leq 2) = F(2)$ by def. of F

$= 1-(1+2)e^{-2}$

② $P(1 < X < 3) = P(1 < X \leq 3)$ since F is conts. at $x=3$

$= F(3) - F(1)$

$= (1-(1+3)e^{-3}) - (1-(1+1)e^{-1})$

③ $P(X > 4) = 1 - P(X \leq 4) = 1 - F(4) = 1 - (1-(1+4)e^{-4})$

④ $P(X=0) = F(0^+) - F(0^-) = 1-(1+0)e^{0} - 0$

$= 0$

Q_3: Answer by (true) or (False) and given the reason:
If a c.d.f. $F(x)$ Continuous at $x=2$, then $P(X=2)\neq 0$.

Sol. No, since if F conts. at $x=2$, then $F(2^+)=F(2^-)$

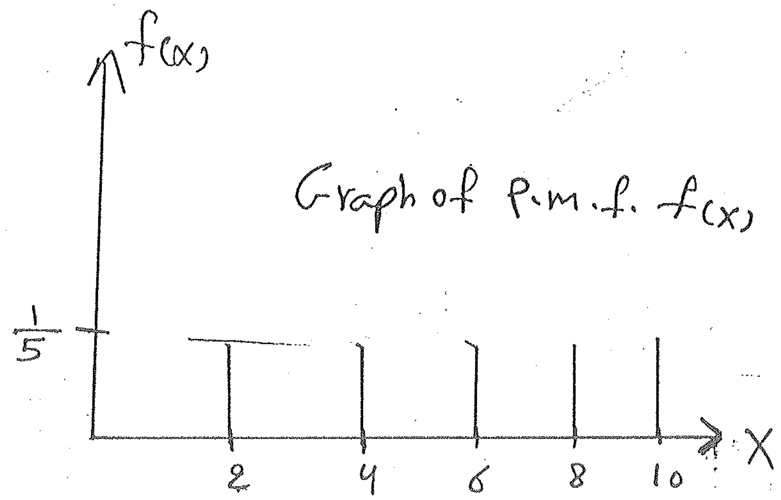
and $P(X=2) = F(2^+) - F(2^-)$
$= \text{zero}$

Q_4: Given a set of integers $\{2,4,6,8,10\}$. Find the p.m.f. $f(x)$ of X and sketch its graph.

Sol.

$X \sim$ unif. (5)

$$f(x) = \begin{cases} \frac{1}{5} & \text{for } x = 2,4,6,8,10 \\ 0 & \text{o.w.} \end{cases}$$



Graph of p.m.f. $f(x)$

الأحتمالية

الفصل الرابع

أ. عباس نجم سلمان

$e^{0} = 1$

$e^{\infty} = \infty$

$e^{-\infty} = 0 \longrightarrow \ln 0 = -\infty$

$\ln 1 = 0$

$\ln \infty = \infty$

CHAPTER FOUR

Expectation and Variance:

Def: Let x be a. r. v either d.r.v. or c.r.v."E(x)" is called the "Expectation of x" or "expected Value of x" or "mean of x" And denoted by μ .

Defined as followes:

1- $E(x) = \sum_{\forall x} x\, f(x)$, when x is a d.r.v.

2- $E(x) = \int_{-\infty}^{\infty} xf(x)\,.dx$ *when x is ac.r.v.*

Note: 1- the value of E (x) is constant.

2- E (x) exists if $\sum_{\forall x} |x|\, f(x) < \infty$, when x is d.r.v.

3- E (x) exists if $\int_{-\infty}^{\infty} |x| f(x) dx < \infty$, when x is c.r.v.

Ex "1": Given p.m.f $f(x) = \begin{cases} \dfrac{\binom{3}{x}}{8} & \text{for } x = 0.2.3\,,1\,. \\ 0 & \text{o.w} \end{cases}$

Find E (X)?

Sol: $E(x) = \sum_{x=0}^{3} x\, f(x)$

X	$f(x)=\dfrac{\binom{3}{x}}{8}$	x.f(x)
0	$\binom{3}{0}\Big/8 = \frac{1}{8}$	0
1	$\binom{3}{1}\Big/8 = \frac{3}{8}$	$\frac{3}{8}$

2	$\binom{3}{2}/8=\frac{3}{8}$	$\frac{6}{8}$
3	$\binom{3}{3}/8=\frac{1}{8}$	$\frac{3}{8}$
	$\sum_{x=0}^{3} f(x)=1$	$\sum_{x=0}^{3} x f(x)=\frac{12}{8}$

$$\therefore E(x)=\sum_{x=0}^{3} x f(x)=\frac{3}{2}$$

H.W: given ap. M.f $f(x)=\begin{cases}\frac{x}{15} & \text{for } x=1,2,3,4,5\\ 0 & 0.w.\end{cases}$

Find the expected Value of x.

Ex "2": Given a p.d.f $f(x)=\begin{cases}\frac{x}{8} & \text{for } 0<x<4\\ 0 & 0.w.\end{cases}$ Find E(x) ?

Sol: $E(x)=\int_{-\infty}^{\infty} x f(x)\,dx=\int_{0}^{4} x\frac{x}{8}\,dx=\int_{0}^{4}\frac{x^2}{8}\,dx$

$$=\frac{1}{8}\frac{x^3}{3}\Big|_0^4=\frac{1}{24}[64-0]=\frac{64}{24}=\frac{8}{3}$$

H.W: Given ap. d.f $f(x)=\begin{cases}3x^2 & \text{for } 0<x<1\\ 0 & 0.w.\end{cases}$

Find E(x)?

تمارين ←

$+1 < X < e$

Ex: Given ap. d.f $f(x)=\begin{cases} \frac{1}{x} & \text{for } \cancel{0 \le x \le 1} \\ 0 & o.w \end{cases}$

$-1 < X < 2.72 \simeq e$

Dose E(x) exist?

Sol: $E(x) = \int_{1}^{e} x \frac{1}{x} dx = \int_{1}^{e} dx = 1 - e$

$\therefore E(x)$ is exist.

Ex: Given ap .d.f $f(x)= \begin{cases} \frac{1}{x2} & \cancel{0 \le x \le 1} \ \ x > 1 \\ 0 & \text{o.w.} \end{cases}$

Dose E(x) exist?

Sol: $E(x)= \int_{1}^{\infty} x \frac{1}{x^2} dx = \int_{1}^{\infty} \frac{1}{x} dx = \ln x \Big|_{1}^{\infty} = [\infty - 0] = +\infty \not< \infty$

$\therefore E(x)$ isnot exist.

Expectation of afunction of x:

Def: Let x be ar.v. and let g(x) be afunction of x then

$E[g(x)] = \sum_{\forall x} g(x) f(x)$, if x is d.r.v.

$= \int_{-\infty}^{\infty} g(x)\, f(x)\, dx$, if x is c.r.v.

Ex "1": Given ap.m.f $f(x)=\begin{cases} \frac{x}{10} & \text{for } x = 1,2,3,4 \\ 0 & \text{o.w.} \end{cases}$

find $E(x^2)$?

sol: $g(x)=x^2$

$E[g(X)] = \sum_{\forall x} g(X) f(x)$

$$E[x^2] = \sum_{x=1}^{4} (x)^2 \frac{x}{10} = \sum_{x=1}^{4} \frac{x^3}{10} = \left[\frac{1}{10} + \frac{8}{10} + \frac{27}{10} + \frac{64}{10}\right]$$

$$=\frac{100}{10}=10$$

Ex"2": Given ap.d.f $f(x)=\begin{cases}\frac{x}{8} & \text{for } 0<x<4\\ 0 & \text{o.w.}\end{cases}$

find $E(\sqrt{x})$?

sol: $g(x)=\sqrt{x}=x^{\frac{1}{2}}$

$$E[g(x)]=E(x^{\frac{1}{2}})=\int_0^4 x^{\frac{1}{2}}\frac{x}{8}dx=\frac{1}{8}\int_0^4 x^{\frac{3}{2}}dx=\frac{1}{8}.\frac{2}{5}x^{\frac{5}{2}}\Big|_0^4$$

$$\frac{1}{20}\left[(\sqrt{4})^5-0\right]=\frac{1}{20}[32]=\frac{32}{20}$$

Note: if f(x) be a p. d.f of a r.v. x, then E(b)=b, where b is constant.

Proof: case "1": If x is a d.r.v. with p.m.f. f(x)

$$E(b)=\sum_{x} b\,f(x)=b\underset{1}{\sum_{x} f(x)}=b\ .$$

Case "2": If x is a c.r.v. with p.d.f. f(x).

$$E(b)=\int_{-\infty}^{\infty} b f(x)\,dx=b\underset{1}{\int_{-\infty}^{\infty} f(x)\,dx}=b\ .$$

Properties of Expectation :

Theorem "1" If x is ar.v. have ap.f. f(x), and E(x) exists.

Let y=ax+b, a b∈R, then E(y)=aE(x)+b.

Sol:case :"1" If x is a c. r.v. With P.d.f f(x)

Y=g(x)=ax+b

$$E(y)=E[g(x)]=\int_{-\infty}^{\infty} g(x)\,f(x)\,dx$$

$$E\left[ax+b\right]=\int_{-\infty}^{\infty}(ax+b)\ f(x)dx=\int_{-\infty}^{\infty} ax\,f(x)\ dx+\int_{-\infty}^{\infty}b\,f(x)\ dx$$

$$=a\int_{-\infty}^{\infty} x\,f(x)\ dx+b\int_{-\infty}^{\infty}f(x)\ dx=a\,E(x)+b$$

case"2": If x is ad.r.v. with p.m.f f(x).

y=g(x)=ax+b

$$E(y)=E\left[g(x)\right]=\sum_{\forall x}g(x)\ f(x)$$

$$E\left[ax+b\right]=\sum_{\forall x}(ax+b)\ f(x)=\sum\ ax\ f(x)+\sum b\,f(x)$$

$$=a\sum x\,f(x)+b\sum f(x)=aE(x)+b$$

theorem "2" : let x be a r.v. if u(x) and v(x) are two functions of x. then:

$$E\left[u(x)\mp v(x)\right]=E\left[u(x)\right]\mp E\left[v(x)\right]$$

proof: " case "1": If x is ad.r.v. with p.m.f f(x) let g(x)=u(x)∓v(x)

$$E\left[u(x)\ \mp v(x)\right]=E\left[g(x)\right]=\sum_{\forall x}g(x)\ f(x)$$

$$=\sum_{\forall x}\left[u(x)\mp v(x)\right]f(x)=\sum_{\forall x}u(x)f(x)\mp\sum_{\forall x}v(x)\,f(x)$$

$$=E\left[u(x)\right]\mp E\left[v(x)\right]$$

case "2": If x is ac.r.v. with p.d.f f(x).

let g(x)=u(x) ∓v(x)

$$E\left[u(x)\mp v(x)\ \right]=E\left[g(x)\right]=\int_{-\infty}^{\infty}g(x)\ f(x)dx$$

$$=\int_{-\infty}^{\infty}\left[u(x)\mp v(x)\right]f(x)dx=\int_{-\infty}^{\infty}u(x)f(x)dx\mp\int_{-\infty}^{\infty}v(x)\ f(x)\ dx$$

$$=E\left[u(x)\right]\mp E\left[v(x)\right]$$

– Ex "1" Given ap.d.f $f(x)=\begin{cases}\frac{x+2}{18} & \text{for } -2<x<4\\ 0 & 0.w\end{cases}$

find $E\left[2x^{3}-1\right]$ & $E\left[(x+2)^{2}\right]$

Sol: $E\left[2x^{3}-1\right] \Rightarrow 1.g(x)=2x^{3}-1 \Rightarrow E[g(x)]=\int_{-2}^{4} g(x)\, f(x)\,dx$

Or 2. $E\left[2x^{3}-1\right]=2E(x^{3})-1$

$$E(\overset{3}{x})=\int_{-2}^{4} x^{3}\left(\frac{x+2}{18}\right)dx=\frac{1}{18}\int_{-2}^{4}(x^{4}+2x^{3})dx$$

$$=\frac{1}{18}\left[\frac{x^{5}}{5}+\frac{x^{4}}{2}\right]_{-2}^{4}=\frac{1}{18}\left[\left(\frac{1024}{5}+\frac{256}{2}\right)-\left(\frac{-32}{5}+\frac{16}{2}\right)\right]$$

$$=\frac{1}{18}\left(\frac{1056}{5}+\frac{240}{2}\right)=-\frac{1}{18}\left(\frac{1056}{5}+120\right)$$

$E\left[(x+2)^{2}\right] \Rightarrow 1.g(x)=(x+2)^{2} \Rightarrow E[g(x)]=\int_{-2}^{4} g(x)\, f(x)\,dx$.

Or 2. $E\left[x^{2}+4x+4\right]=E(x^{2})+4E(x)-4$

$$=\int_{-2}^{4} x^{2}\, f(x)\,dx+4\int_{-2}^{4} x\, f(x)\,dx+4$$

$=\ \dots$

H.W: 1 Given ap.d.f., $f(x)=\begin{cases}\frac{x}{2} & 0<x<4\\ 0 & o.w.\end{cases}$

Find E [(x+2)^{2}] ?

2. Given ap. m.f, $f(x)=\begin{bmatrix}\frac{x}{10} & \text{for } x=1,2,3,4\\ 0 & \text{o.w.}\end{bmatrix}$ find $E\left[2x^{3}-1\right], E\left[(x-1)^{3}\right]$?

$(x-1)^{3}=(x^{3}-3x^{2}+3x-1)$

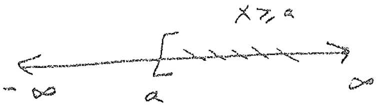
✓ theorem "3": let x be ar.v.

a. If $\exists(a)$ such that $p(x \geq a) = 1$, then $E(x) \geq a$.

b. If $\exists(b)$ such that $p(x \leq b) = 1$, then $E(x) \leq b$.

c. If $p(a \leq x \leq b) = 1$, then $a \leq E(x) \leq b$.

Proof: a. case "1": If x is ac.r.v. with p.d.f f(x)



$$P(x \geq a) = 1 \Rightarrow \int_a^\infty f(x)\,dx = 1$$

$$\because f(x) > 0 \quad \text{for } a \leq x < \infty$$
$$=0 \qquad \text{o.w.}$$

$$Rx = \{x : a \leq x < \infty\}$$

$$E(x) = \int_a^\infty x\, f(x)\,dx \geq \int_a^\infty a\, f(x)\,dx = a \underbrace{\int_a^\infty f(x)\,dx}_{} = 1$$
$$= a$$

$$\therefore E(x) \geq a$$

Case (2) if x is d.r.v. with p.m.f f(x)

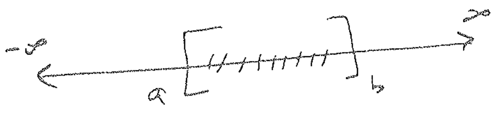
$$E(x) = \sum_{x=a}^{\infty} x\, f(x) \geq \sum a\, f(x) = a \sum f(x) = 1$$
$$= a$$

$\therefore E(x) \geq a$.

b. Similary of (a).

c. Case "1": If x is ad.r.v. have ap.m.f f(x)

$$P(a \leq x \leq b) = 1 \Rightarrow \sum_{x=a}^{b} f(x) = 1$$



$$\therefore f(x) > 0 \quad \text{for } a \leq x \leq b$$
$$=0 \qquad \text{o.w}$$

$$E(x) = \sum_{x=a}^{b} x f(x) \geq \sum a f(x) = a \sum f(x) = 1$$

$$\therefore E(x) \geq a \ldots\ldots\ldots (1)$$

$$E(x) = \sum_{x=a}^{b} x f(x) \leq \sum b f(x) = b \sum f(x) = 1$$

$$\therefore E(x) \leq b \ldots\ldots\ldots (2)$$

$\therefore by "1" \& "2" we get $a \leq E(x) \leq b$

theorem 4": If $p(x \geq a) = 1$ and $E(x) = a$ then $p(x = a) = 1$ and $p(x > a) = 0$.

proof: case “1”: If x is ac.r.v. from ap.d.f f(x)

$$\therefore p(x \geq a) = 1 \Rightarrow \int_a^\infty f(x)\,dx = 1$$

$\therefore f(x) > 0$ for $a \leq x < \infty$

$=0$ ow

$$E(x) = \int_a^\infty x\ f(x)\ dx = a = a.1 = a\int_a^\infty f(x)dx = \int_a^\infty af(x)dx$$

$$\int_a^\infty x\,f(x)\,dx = a\int_a^\infty f(x)\,dx \Rightarrow \int_a^\infty x\,f(x)\,dx = \int_a^\infty a\,f(x)\,dx$$

..this inquality hold only when x=a

i.e. $(x>a)=\phi \Rightarrow P(X > a) = 0$

$$p(x \geq a) = p[(x = a) \cup (x > a)] = p(x = a) + p(x > a) \Rightarrow 1 = p(x = a) + 0 \Rightarrow p(x = a) = 1$$

case “2”: If x is ad.r.v from ap.m.f f(x)

$$\therefore p(x \geq a) = 1 \Rightarrow \sum_{x=a}^{\infty} f(x) = 1$$

$\therefore f(x) > 0$ for $a \leq x < \infty$

$=0$ ow

$$E(x) = \sum_{x=a}^{\infty} xf(x) = a = a.1 = a.\sum_{x=a}^{\infty} f(x) = \sum_{x=a}^{\infty} af(x)$$

$$\sum_{x=a}^{\infty} xf(x) = \sum_{x=a}^{\infty} af(x)$$

This inequality hold only when x =a

i.e. $(x>a)=\phi \Rightarrow p(x > a) = 0$

$$p(x \geq a) = [(x = a) \cup (x > a)] = p(x = a) + p(x > a)$$

$$\Rightarrow 1 = p(x = a) + 0 \Rightarrow p(x = a) = 1$$

variance of random variable:

Def: let x be ar.v. the variance of x denoted by v(x) or δ^2 is defined as

: $v(x)=E\{[x - E(x)]^2\}$

$\because E(x)=\mu$, then $v(x) = E[(x-\mu)^2]$

note: since $(x-\mu)^2 \geq 0$ then $E[(x-\mu)^2] \geq 0$

$\therefore v(x) \geq 0$ alwayes.

properties of variance:

theorem “5” let x be ar.v.,then $v(x)=E(x^2)-[E(x)]^2$

Proof: $v(x) = E\{[x - E(X)]^2\}$

$\because V(x)=E\{x^2 - 2E(x)x + [E(x)]^2\}, E(x) \text{ constant.}$

$v(x)=E(x^2)-2E(x)E(x)+[E(x)]^2$

$v(x)=E(x^2)-2[E(x)]^2+[E(x)]^2 \Rightarrow v(x) = E(x^2) - [E(x)]^2$

note: 1. $v(x)=0 \Leftrightarrow E(x^2)=E[(x)]^2$

2. $\because v(x) \geq 0 \Rightarrow \therefore E(x^2) \geq [E(x)]^2$

3. $v(b)=0$, b is constant.

Theorem "6" let x be ar.v. and v(x) exist , If y=ax+b ; $a,b \in R$ then $v(y) = a^2 v(x)$.

Proof: $y=ax+b \Rightarrow E(y) = aE(x) + b$

$V(y)=E\{[y - E(y)]^2\} = E\{[(ax+b) - (aE(x)+b)]^2\}$

$=E\{[ax - aE(x)]^2\} = E\{a^2[x - E(x)]^2\}$

$= a^2E\{[X - E(x)]^2\} = a^2V(x)$

Ex: Given ap.d.f $f(x) = \begin{cases} 3x^2 & for\ 0 < x < 1 \\ 0 & o.w. \end{cases}$

a. find E(x) & V(x). b. If y=1-2x, then find E(y), v(y).

SOL .

$a. E(x) = \int_0^1 X(3X^2)dx = 3\frac{X^4}{4}\Big|_0^1 = \frac{3}{4}$

$v(x)=E(x^2) - [E(x)]^2$

$E=(x^2)= \int_0^1 x^2(3x^2)dx = \frac{3}{5}x^5 \Big|_0^1 = \frac{3}{5}$

$\therefore v(x) = \frac{3}{5} - \frac{9}{16} = \frac{48-45}{80} = + \frac{3}{80}$

b. $\because y = (-2)x + 1 \Rightarrow \therefore E(y) = (-2)E(x) + 1$

$\therefore E(y) = (-2)(\frac{3}{4}) + 1 = \frac{-3}{2} + 1 = -\frac{1}{2}$

$v(y)=(-2)^2\ v(x)=4.(\frac{3}{80}) = \frac{12}{80}$

H.w.: Given ap.d.f. $f(x) = f(x) = \begin{cases} 6x(1-x) & for\ 0 < x < 1 \\ 0 & 0.w. \end{cases}$

If y=2-3x, then find E(y) & v(y)

Theorem "7" v(x)=0 iff $\exists K$ where k is constant such that $p(x = k) = 1$

$\Leftarrow$ suppose that $p(x=k)=1$,t.p $V(X)=0$

Poof: $f(x)>0 \quad for\ x=k$

$\quad =0 \qquad 0.w.$

$v(x)=v(x=k)=0$ by note.

$\Rightarrow$ suppose $V(X)=0$

,t.p $p(x=k)=1$

$$\because v(x)=0 \Rightarrow E\{[x-E(x)]^2\}=0$$

$$Y=[x-E(x)]^2 \Rightarrow E(Y)=0$$

$$by\ th.\ "4" \Rightarrow [E(x)=a \rightarrow p(x=a)=1]$$

$$\therefore p(y=0)=1 \Rightarrow p\{[x-E(x)]^2=0\}=1$$

$$\therefore p\{(x-E(x))=0\}=1 \Rightarrow \therefore p[x=E(x)]=1$$

$$\therefore x=k \Rightarrow E(x)=E(k)=k \Rightarrow \therefore p[x=k]=1$$

* *Existence of Mean and Variance:*

$E(x)$ exists iff $E(|x|)<\infty$

$\because V(x)=E(x^2)-[E(x)]^2 \Rightarrow \therefore V(x)$ exists iff $E(x)$ & $E(x^2)$ exist

Ex. "1": Given acauchy p.d.f $f(x)=\dfrac{1}{\pi(1+x^2)}$ for $-\infty<x<\infty$

Show that E(x) dose not exist?

Sol.: $E(|x|)=\int\limits_{-\infty}^{\infty}|x|\dfrac{1}{\pi(1+x^2)}dx=2\int\limits_{0}^{\infty}x\dfrac{1}{\pi(1+x^2)}dx$

$=\dfrac{1}{\pi}\int\limits_{0}^{\infty}\dfrac{2x}{1+x^2}dx=\dfrac{1}{\pi}\ln(1+x^2)|_0^{\infty}=\dfrac{1}{\pi}[\ln\infty-\ln$ [illegible] $]\not<\infty$

$E(|x|)\not<\infty \Rightarrow E(x)$ dose not exist

Exercises: 1. Let x be a c.r.v. have a p.d.f f(x) where

$f(x)>0$ for $0<x<b<\infty$, Show that $E(x)=\int\limits_{0}^{b}[1-F(x)]dx$

$\quad =0$ o.w.

Hint.: $f(x)=\dfrac{dF(x)}{dx} \leftrightarrow f(x)dx=dF(x)$

2. If x is d.r.v. have p.M.f $f(x)>0$. for x = -1, 0, 1

$\qquad =0$ o.w.

a. If $f(0)=\frac{1}{2}$, Find $E(x^2)$. $\sum f(x)=1 \Rightarrow f(-1)+f(0)+f(1)=1$

b. If $f(0)=\frac{1}{2}$, and $E(x)=\frac{1}{6}$, Find f(-1), f(1)

3. Given a p.d.f $f(x)=\begin{cases}1-|x| & \text{for } |x|<1\\ 0 & \text{o.w.}\end{cases}$

a. Find E(x) & V(x), b. If y= 2-3 x, find E(y) & V(y).

Hint

$$f(x)=\begin{cases}1-|x| & -1<x<1\\ 0 & \text{o.w.}\end{cases} = \begin{cases}1+x & -1<x<0\\ 1-x & 0\le x<1\\ 0 & \text{o.w.}\end{cases}$$

Moments of Random Variables:

Def.: Let x be a r.v. either d.r.v. or c.r.v.. let $K \in I^+$ then $E(x^k)$ is called "the k^{th} moment of x" or "the moment of order k of x".

when $k = 1 \Rightarrow E(x^1) = 1^{st}$ moment of $x = \mu$

$E(x^2) = 2^{nd}$ moment of x

Note: $E(x^k)$ exists iff $E(|x|^k) < \infty$

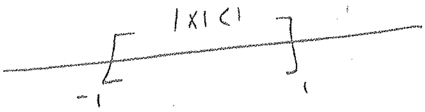
بدون برهان Theorem "8": If $E(x^k)$ exists then $E(x^j)$ exists, $j < k$ and $j, k \in I^-$

Proof: case "1" If x is c.r.v. with p.d.f f(x)

$\because E(x^k)$ exists $\Rightarrow \therefore E(|x|^k) < \infty$

T.p $E(x^j)$ exists, T.p $E(|x|^j) < \infty$

$$E(|x|^j) = \int_{-\infty}^{\infty} |x|^j f(x)dx$$



$$E(|x|^j) = \int_{|x|\le 1}^{-\infty} |x|^j f(x)dx + \int_{|x|>1} |x|^j f(x)dx$$

2^{nd} central moment of R.V.X is equal to V(x).

Ex.: Let x be a r.v.s.t $E(x) = 1$, $E(x^2) = 2$ and $E(x^3) = 5$. Find the 3^{rd} central moment of x.

Sol.: $E[(x-\mu)^3] = E\{x^3 - 3\mu x^2 + 3\mu^2 x - \mu^3\}$

$$= E(x^3) - 3\mu E(x^2) + 3\mu^2 E(x) - \mu^3$$

$$= 5 - 3 \,.\, 1 \,.\, 2 + 3 \,.\, 1 \,.\, 1 - 1 = 1$$

Exercises: 1. If x ~ uniform (a, b); a, b $\in$ R. Find the valued of 1^{st} central moment of x and also find the 2^{nd} central moment of x.

2. Let $\mu = E(x)$ and $\delta^2 = V(x)$ showthat $E[(x-\mu)^4] \geq \delta^4$

i.e. 4^{th} central moment of x is greater than or equal the square of variance of x.

Moment Generating Function (M.g.f).

Def.: A moment generating function (M.g.f) of a r.v.x is a function that determines all moments of x, denoted by $M_X(t)$.suppose that $t \in (-h, h)$, $h > 0$.

If $E[e^{tX}]$ exists $\forall t \in (-h, h)$ then $M_X(t) = E[e^{tX}]$, $-h < t < h$

There are two cases of $M_X(t)$.

Case "1": If x is a d.r.v. from a p.M.f f(x)

$$M_x(t) = E[e^{tx}] = \sum_{\forall x} e^{tx} f(x)$$

Case "2": If x is a c.r.v. have a p.d.f f(x)

$$M_x(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

Ex.: Given a p.d.f $f(x) = \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{o.w} \end{cases}$

Find $M_X(t)$ and sketch it's graph.

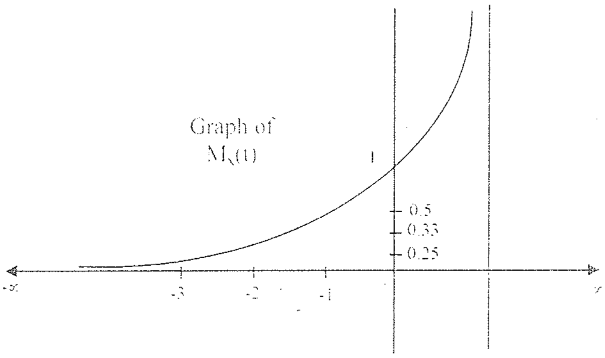
Sol.: $M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} \cdot e^{-x} dx = \int_0^{\infty} e^{-(1-t)x} dx$

This integration exist only when (1-t) > 0, i.e. t < 1

$$M_x(t) = \frac{-1}{1-t} e^{-(1-t)x} \Big|_0^\infty = \frac{-1}{1-t}\left[\underset{=0}{e^{-\infty}} - \underset{=1}{e^{0}}\right] = \frac{1}{1-t}$$

$$M_x(t) = \frac{1}{1-t} \qquad for \quad t < 1$$

t	$M_x(t)$
1	∞
0	1
-1	0.5
-2	0.33
-3	0.25
.	
-∞	0



Ex.: Given a p.M.f.

$$f(x) = \begin{cases} \frac{x}{15} & \text{for } x = 1, 2, 3, 4, 5 \\ 0 & \text{o.w} \end{cases}$$

Find $M_x(t)$.

Sol.: $$M_x(t) = E(e^{tx}) = \sum_{x=1}^{5} e^{tx} f(x) = \sum_{x=1}^{5} \frac{x}{15} e^{tx}$$

$$= \frac{1}{15}\sum_{x=1}^{5} xe^{tx} = \frac{1}{15}[1\cdot e^{t} - 2\cdot e^{2t} + 3\cdot e^{3t} + 4\cdot e^{4t} + 5e^{5t}] \quad for \ -\infty < t < \infty$$

Theorem "9": Let x be a r.v. have a M.g.f. $M_x(t)$, then

$$M_x(0) = 1,\ M'_x(0) = E(x),\ M''_x(0) = E(x^2),\ M'''_x(0) = E(x^3), \ldots,$$
$$M_x^{(k)}(0) = E(x^k)$$

Proof: $* M_x(t) = E(e^{tx})$ by Macclaurin series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^k}{k!} + \ldots$$

$$\therefore e^{tx} = 1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \ldots + \frac{t^kx^k}{k!} + \ldots$$

$$M_x(t) = E[1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \ldots + \frac{t^kx^k}{k!} + \ldots]$$

$$M_x(t) = 1 + tE(x) + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + \ldots + \frac{t^k}{k!}E(x^k) + \ldots$$

$M_x(t=0)=M_x(0)=1$

$$*\ M'_x(t)=\frac{dM_x(t)}{dt}=E(x)+\frac{2t}{2!}E(x^2)+\frac{3t^2}{3!}E(x^3)+\ldots+\frac{kt^{k-1}}{k!}E(x^k)+\ldots$$

$M'_x(0)=E(x)$

$$*\ M''_x(t)=E(x^2)+\frac{6t}{3!}E(x^3)+\ldots+\frac{k(k-1)t^{k-2}}{k!}E(x^k)+\ldots$$

$M''_x(0)=E(x^2)$

Simillary we can find $E(x^3)$, $E(x^4)$, ... $E(x^k)$, . . .

Note: $M_x(t)=M_x(0)+tM'_x(0)+\frac{t^2}{2!}M''_x(0)+\ldots+\frac{t^k}{k!}M_x^{(k)}(0)$

This series is called the M. g.f. by Macclaurin series.

Ex.: Given $M_x(t)=\frac{1}{1-2t}$, $t<\frac{1}{2}$ Find $E(x)$ and $V(x)$

Sol.: $M_x(t)=(1-2t)^{-1}$, $t<\frac{1}{2}$

$M'_x(t)=-(1-2t)^{-2}\cdot(-2)=2(1-2t)^{-2}$

$\therefore E(x)=M'_x(0)=2(1-0)^{-2}=2$

$V(x)=E(x^2)-[E(x)]^2$

$M''_x(t)=8(1-2t)^{-3}$

$\therefore E(x^2)=M''_x(0)=8(1-0)^{-3}=8$

$\therefore V(x)=8-2^2=4\geq 0$

Theorem "10": Let x be a r.v. have a M.g.f $M_x(t)$. If $Y=ax+b$. a, b, $\in R$ then $M_y(t)=e^{bt}\cdot M_x(at)$

Proof: $Y=ax+b$

$M_y(t)=E(e^{ty})\,(by\ def.)=E[e^{t(ax+b)}]=E[e^{(at)x+bt}]$

$M_y(t)=[e^{(at)x}\cdot e^{bt}]=e^{bt}E[e^{(at)x}]=e^{bt}\cdot M_x(at)$

Since $M_x(t)=E(e^{tx})\Rightarrow M_x(at)=E[e^{(at)x}]$

Ex: Given $M_x(t)=\frac{1}{1-3t}$ for $t<\frac{1}{3}$, If y= 1-2x , find $M_y(t)$.

Sol: y= (-2)x+1 , a=-2 , b=1

By th. (10) $\Rightarrow M_y(t) = e^{bt} M_x(at) = e^t M_x(-2t)$

$\because M_x(t) = \frac{1}{1-3t}$ for $t < \frac{1}{3}$

$M_x(-2t) = \frac{1}{1-3(-2t)} = \frac{1}{1+6t}$ for $t > \frac{-1}{6}$

$M_y(t) = e^t \cdot \frac{1}{1+6t}$ for $t > \frac{-1}{6}$

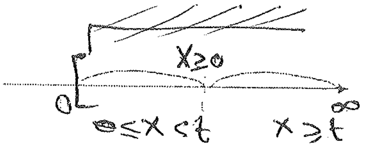
The b$\overset{u}{o}$nded of probability:

Theorem "11": ((Markov inquality))

If x is ar.v. and if $P(x \geq 0) = 1$ then $P(x \geq t) \leq \frac{E(x)}{t}$, for $t > 0$

Proof: case "1": if x is a c.r.v with a p.d.f f(x)

$$\because p(x \geq 0) = 1 \Rightarrow \int_0^\infty f(x)dx = 1$$



$, (t>0)$

$$f(x) > 0 \quad \text{for } x \geq 0$$
$$= 0 \quad \text{O.W}$$

$$E(x) = \int_0^\infty x f(x)dx = \int_0^t x f(x)dx + \int_t^\infty x f(x)dx \geq \int_t^\infty x f(x)dx \geq \int_t^\infty t f(x)dx. \ [x \geq t]$$

$$\therefore E(x) \geq \int_t^\infty t f(x)dx \Rightarrow E(x) \geq t \int_t^\infty f(x)dx$$

$$E(x) \geq t. \quad p(x \geq t)$$

$$\therefore p(x \geq t) \leq \frac{E(x)}{t}$$

Case "2": If x is a d.r.v. with a p.m.f. f(x) by the same method.

Theorem "12": "Chebyshev inqualities"

Let x be a r.v. where V(x) exists,then:

1. $p(|x-\mu| \geq t) \leq \frac{V(x)}{t^2}$, $t>0$, 2. $p(|x-\mu| < t) \geq 1 - \frac{V(x)}{t^2}$, $t>0$,

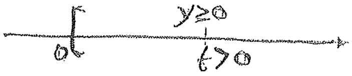
Note: 1. $\frac{V(x)}{t^2}$ *is called the upper buond of* $p(|x-\mu| \geq t)$

2. $1 - \frac{V(x)}{t^2}$ *is called the lower buond of* $p(|x-\mu| < t)$

Proof: 1. $V(x) = E[(x-\mu)^2] \geq 0$

$$Let\ y=(x-\mu)^2 \geq 0 \Rightarrow E(y)=E[(x-\mu)^2 = V(x) \geq 0$$

∴ by Markov inquality, we get



$$p(v \geq t^2) \leq \frac{E(y)}{t^2} \Rightarrow p[(x-\mu)^2 \geq t^2] \leq \frac{V(x)}{t^2}$$

$$p[|x-\mu| \geq t] \leq \frac{V(x)}{t^2}$$

2. $p(A^c) = 1 - p(A)$

$$p[|x-\mu| \geq t^2] = 1 - p[|x-\mu| < t^2] \leq \frac{V(x)}{t^2}$$

$$p[|x-\mu| < t^2] \geq 1 - \frac{v(x)}{t^2}$$

Ex. "1": If x ~ unif. $(-\sqrt{3}, \sqrt{3})$ then:

a. Find the upper bound of $p(|x-\mu| \geq \frac{3}{2})$

b. Find the value of $p(|x-\mu_x| \geq \frac{3}{2})$

Sol.:

a. $f(x) = \begin{cases} \frac{1}{\sqrt{3}+(\sqrt{3})} = \frac{1}{2\sqrt{3}} & \text{for } -\sqrt{3} < x < \sqrt{3} \\ 0 & \text{o.w.} \end{cases}$

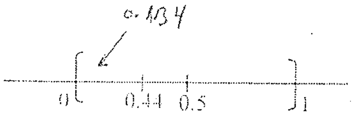
$$u.b = \frac{V(x)}{t^2}, \qquad t = \frac{3}{2}$$

$$E(x) = \int_{\sqrt{3}}^{\sqrt{3}} X \frac{1}{2\sqrt{3}} dx = \frac{1}{4\sqrt{3}} X^2 \Big|_{-\sqrt{3}}^{\sqrt{3}} = 0$$

$$E(x^2) = \int_{-\sqrt{3}}^{\sqrt{3}} x^2 \frac{1}{2\sqrt{3}} dx = \frac{1}{6\sqrt{3}} x^3 \Big|_{-\sqrt{3}}^{\sqrt{3}} = \frac{1}{6\sqrt{3}}[(\sqrt{3})^3 - (-\sqrt{3})^3] = \frac{1}{6\sqrt{3}}[3\sqrt{3} + 3\sqrt{3}] = 1$$

$$\therefore V(x) = E(x^2) - [E(x)]^2 = 1 - 0 = 1$$

$$\therefore u.b = \frac{V(x)}{t^2} = \frac{1}{(\frac{3}{2})^2} = \frac{4}{9} = 0.44$$



b. $p(|x - \mu| \geq \frac{3}{2}) = p(|x - 0| \geq \frac{3}{2}) = p(|x| \geq \frac{3}{2})$

$$= 1 - p(|x| < \frac{3}{2}) = 1 - p(-\frac{3}{2} < x < \frac{3}{2}) = 1 - \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{1}{2\sqrt{3}} dx$$

$$= 1 - \frac{1}{2\sqrt{3}} x \Big|_{-\frac{3}{2}}^{\frac{3}{2}} = 1 - \frac{1}{2\sqrt{3}}(\frac{3}{2} + \frac{3}{2}) = 1 - \frac{3}{2\sqrt{3}} = 1 - \frac{\sqrt{3}}{2}$$

$$= 1 - 0.866 = 0.134$$

Ex. "2": Given a p.d.f $f(x) = \begin{cases} \frac{2x}{9} & \text{for } 0 < x < 3 \\ 0 & \text{o.w.} \end{cases}$

a. Find the lower bound of $p(\frac{5}{4} < x < \frac{11}{4})$

b. Find the value of $p(\frac{5}{4} < x < \frac{11}{4})$

Sol.: a. $L.b = 1 - \frac{V(x)}{t^2}$

$$E(x) = \int_0^3 x \cdot \frac{2x}{9} dx = \frac{2}{27} x^3 \Big|_0^3 = \frac{2}{27}(27 - 0) = 2$$

$$E(x^2) = \int_0^3 x^2 \cdot (\frac{2x}{9}) dx = \frac{2}{36} x^4 \Big|_0^3 = \frac{1}{18}(81) = \frac{9}{2} = 4.5$$

$$\therefore V(x) = E(x^2) - [E(x)]^2$$

$$= 4.5 - 4 = 0.5 = \frac{1}{2}$$

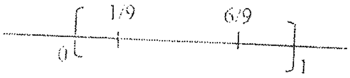
$$p(\frac{5}{4} < x < \frac{11}{4}) = p(\frac{5}{4} - 2 < x - 2 < \frac{11}{4} - 2) = p(-\frac{3}{4} < x - 2 < \frac{3}{4})$$

$$= p(|x - 2| < \frac{3}{4})$$

$$\therefore t = \frac{3}{4}$$

$$\therefore L.b = 1 - \frac{V(x)}{t^2} = 1 - \frac{\frac{1}{2}}{\frac{9}{16}} = 1 - \frac{1}{2} \times \frac{16}{9} = 1 - \frac{8}{9} = \frac{1}{9}$$

b. $p(\frac{5}{4} < x < \frac{11}{4}) = \int_{\frac{5}{4}}^{\frac{11}{4}} \frac{2x}{9} dx$



$= \frac{1}{9} x^2 |_{\frac{5}{4}}^{\frac{11}{4}} = \frac{1}{9}[\frac{121}{16} - \frac{25}{16}] = \frac{1}{9}[\frac{96}{16}] = \frac{6}{9}$

Median of Distribution of r.v.:

Def.: The median (m) is a value of x such that satisfing the two following inqualities:

$$p(x < m) < \frac{1}{2} \quad \& \quad p(x \le m) \ge \frac{1}{2}$$

By properties of c.d.f F(x)

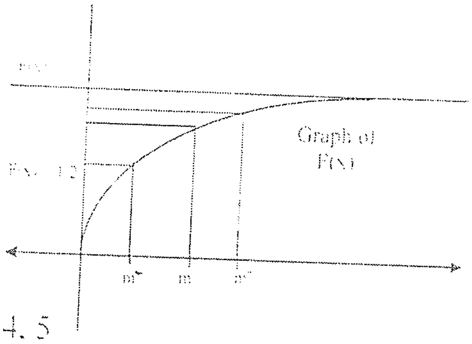
$p(x < m) = F(\overline{m}), \quad p(x \le m) = F(m) = F(m^-)$

$i.e.: p(x < m) = F(\overline{m}) < \frac{1}{2}, \quad p(x \le m) = F(m^-) \ge \frac{1}{2}$

Note ① *If* $\overline{m} = m = m^-$ *then*

$$p(x \le m) = F(m) = \frac{1}{2}$$

② *Thevalueof medianisuniqu*



Ex.: Given a p.m.f $\quad f(x) = \begin{cases} \frac{x}{15} & \text{for } x = 1, 2, 3, 4, 5 \\ 0 & \text{o.w.} \end{cases}$

Find the median of x.

Sol.: $p(x < m) < \frac{1}{2} \quad \& \quad p(x \le m) \ge \frac{1}{2}$

Suppose that m =1

$p(x < 1) = 0 < \frac{1}{2}, \quad p(x \le 1) = f(1) = \frac{1}{15} \not\ge \frac{1}{2}$

$\therefore m \ne 1$

Suppose that m = 2

$p(x<2) = f(1) = \frac{1}{15} < \frac{1}{2}$

$p(x \le 2) = f(1) + f(2) = \frac{1}{15} + \frac{2}{15} = \frac{1}{15} \ngeq \frac{1}{2}$

$\therefore m \neq 2$

Suppose that m=3

$p(x<3) = f(1) + f(2) = \frac{1}{15} + \frac{2}{15} = \frac{1}{5} < \frac{1}{2}$

$p(x \le 3) = f(1) + f(2) + f(3) = \frac{1}{15} + \frac{2}{15} - \frac{3}{15} = \frac{6}{15} \ngeq \frac{1}{2}$

$\therefore m \neq 3$

Suppose that m = 4

$\frac{12}{30} < \frac{15}{30}$

$p(x<4) = f(1) + f(2) + f(3) = \frac{6}{15} < \frac{1}{2}$

$p(x \le 4) = f(1) + f(2) + f(3) + f(4) = \frac{10}{15} \ge \frac{1}{2}$

$\therefore m = 4$ *is median*

Ex.: Given a p.d.f $f(x) = \begin{cases} \frac{1}{x^2} & \text{for } x > 1 \\ 0 & \text{o.w.} \end{cases}$

Find the median of x?

Sol.: $p(x<m) < \frac{1}{2}$ & $p(x \le m) \ge \frac{1}{2}$

$$p(x<m) = \int_1^m \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^m = -[\frac{1}{m} - 1] < \frac{1}{2}$$

$$-\frac{1}{m} + 1 < \frac{1}{2} \Rightarrow -\frac{1}{m} < -\frac{1}{2} \Rightarrow m < 2 \quad .....(1)$$

$$p(x \le m) = \int_1^m \frac{1}{x^2}\,dx \ge \frac{1}{2}$$

$$-[\frac{1}{m} - 1] \ge \frac{1}{2} \Rightarrow m \ge 2 \quad ....(2)$$

From (1) & (2) ⇒ median = 2

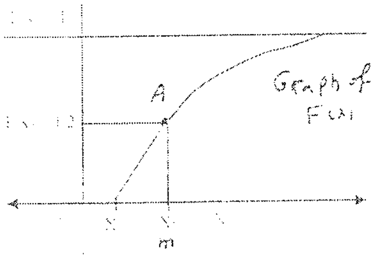
Note: We can also find the value of (Median) from the graph of F(x) such that. At the point $F(x)=\frac{1}{2}$, draw a line to cut the curve of F(x) at A, draw a line to cut the X-axis at m (median) from ex. "2" above

$$f(x)=\begin{cases}\frac{1}{x^2} & \text{for } x>1\\ 0 & \text{o.w.}\end{cases}$$

$$F(x)=p(X\le x)=\int_1^x \frac{1}{t^2}\,dt=1-\frac{1}{x}\Rightarrow F(x)=\begin{cases}1-\frac{1}{x} & \text{for } x\ge 1\\ 0 & \text{o.w.}\end{cases}$$

or : $F(x)=\frac{1}{2}$

$$1-\frac{1}{x}=\frac{1}{2}\Rightarrow -\frac{1}{x}=-\frac{1}{2}\Rightarrow x=2$$



To Find the Mode of Dist. Of R.V.X

Def.: Mode is a value of a r.v.x that maximize f(x)

i.e. If x_1 = mode. ⟶ then $f(x_1)$ is a Max.

Note: $F(x_1)$ is Max. ⇔ $f''(x_1)<0$

Ex.: Given a p.d.f. $f(x)=\begin{cases}12x^2(1-x) & 0<x<1\\ 0 & \text{o.w.}\end{cases}$

Find the mode of x?

Sol.: 0 < mode < 1

$f(x)=12x^2-12x^3\Rightarrow \quad f'(x)=24x-36x^2$

$\Rightarrow \quad 12x(2-3x)=0$ either $x=0$ or $x=\frac{2}{3}$

$f''(x)=24-72x\Rightarrow f''(0)=24>0 \quad \Rightarrow f(0)$ is min.

$f''(\frac{2}{3})=24-72(\frac{2}{3})=-24<0\Rightarrow f(\frac{2}{3})$ is max

$x_1=\frac{2}{3}=$ mode

CHAPTER FOUR

Def.: Percentile

It is a value of x say (x_0) such that $p(x \le x_o) = \frac{t}{100}$ $\quad 0 < t < \infty$

denoted by P_t, $\quad 0 < t < 100$

i.e.: $\quad P_t = x_o$

Ex.: Given a p.d.f $f(x) = \begin{cases} \frac{x}{2} & , 0 < x < 2 \\ 0 & , \quad \text{o.w.} \end{cases}$

Find P_{40}, $\quad P_{65}$

Sol.: let $p_{40} = x_o \quad , t = 40$

$p(x \le x_o) = \frac{t}{100}$, $\quad p(x \le x_o) = \frac{40}{100} = 0.4$

$\int_0^{x_0} \frac{x}{2}\,dx = \frac{x^2}{4}\Big|_0^{x_0} = 0.4$

$[x^2 - 0^2] = 4(0.4) \Rightarrow x^2 - 1.6 \Rightarrow x_o = \sqrt{1.6} = 1.26$

Exercises

① Given a p.f.

$f(-1)=\frac{1}{8}$, $f(0)=\frac{6}{8}$, $f(1)=\frac{1}{8}$

Find the u.b. of $p(|X| \geq 2\sigma)$.

sol.

$$E(X)=\sum_{x=-1}^{1} x f(x) = (-1)f(-1)+(0)f(0)+(1)f(1)$$
$$= -\frac{1}{8}+0+\frac{1}{8} = 0$$

$$\boxed{E(X)=\mu=0}$$

$$E(X^2)=\sum_{x=-1}^{1} x^2 f(x) = (-1)^2 f(-1)+(0)^2 f(0)+(1)^2 f(1)$$
$$= \frac{1}{8}+0+\frac{1}{8}=\frac{1}{4}$$

$$\delta^2 = V(X) = E(X^2)-[E(X)]^2 = \frac{1}{4}-0=\frac{1}{4}$$

$$\boxed{\sigma=\frac{1}{2}}$$

$$p(|X| \geq 2\sigma) = p(|X-0| \geq 2\cdot\frac{1}{2}) = p(|X-\mu| \geq 1)$$
$$= p(|X-\mu| \geq t) \leq \frac{V(X)}{t^2} \Rightarrow t=1$$

$$u.b = \frac{V(X)}{t^2} = \frac{(\frac{1}{4})}{1} = \frac{1}{4}$$

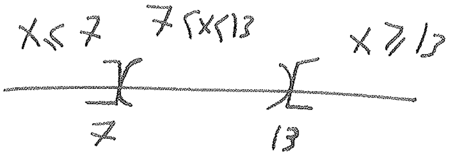
Note If we have to find the exact value of above pr. :

$$p(|X| \geq 2\cdot\sigma) = p(|X| \geq 2\cdot\frac{1}{2}) = p(|X| \geq 1) = 1-p(|X|<1)$$
$$= 1-p(-1<X<1) = 1-p(X=0) = 1-f(0)$$
$$= 1-\frac{6}{8}=\frac{2}{8}$$

② If X is a r.v. with $E(X)=10$, $p(X \geq 7)=0.1$, $p(X \geq 13)=0.3$, then show that $V(X) \geq \frac{9}{2}$.

sol. $p(|X-\mu|<t) \geq 1-\frac{V(X)}{t^2}$

$p(|X-\mu| \geq t) \leq \frac{V(X)}{t^2}$



$$p[(X \leq t) \cup (7<X<13) \cup (X \geq 13)] = p(S)$$
$$p(X \leq 7)+p(7<X<13)+p(X \geq 13)=1$$
$$0.2 + p(7<X<13) + 0.3 = 1$$
$$p(7<X<13) = 0.5 = \frac{1}{2}$$
$$\therefore E(X)=\mu=10$$

$$P(7<X<13) = P(7-10< X-10<13-10)$$

$1/2$

$$= P(-3< X-10< 3)$$

$$= P(|X-10|<3)$$

$$= 1-P(|X-10|\geq 3)$$

$$\therefore \boxed{P(|X-10|\geq 3)=\frac{1}{2}}$$

$$\frac{V(X)}{t^2} \geq \underset{\substack{\nearrow \\ t=3}}{P(|X-10|\geq 3)} = \frac{1}{2} \Rightarrow \frac{V(X)}{3^2} = \frac{1}{2}$$

$$\therefore \boxed{V(X) = \frac{9}{2}}$$

---

③ How many times must we toss a dice such that the pr. to get $d=1$ or 6 at least once more than (0.8)?

$P(x\geq 1) > 0.8$

مش $\boxed{6}$

[illegible]

sol. $A = d=1$ or 6

$$P(A) = \frac{2}{6} = \frac{1}{3} = P \Rightarrow \boxed{1-P=\frac{2}{3}}$$

$X \equiv$ number of tosser

$x = 0,1,2,3,\dots,n$

$X \sim b(n,\frac{1}{3})$

$$f(x,n,\tfrac{1}{3}) = \begin{cases} \binom{n}{x}\left(\frac{1}{3}\right)^x\left(\frac{7}{3}\right)^{n-x} & x=0,1,2,\dots,n \\ 0 & \text{o.w.} \end{cases}$$

If $P(X\geq 1)>0.8$, Find n?

$P(X\geq 1)>0.8$

$1-P(X<1)>0.8$

$1-f(x=0)>0.8$

$-f(0)>-1+0.8=-.2$

$f(0)<0.2$

$\binom{n}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^n<0.2$

$\left(\frac{2}{3}\right)^n<0.2$

if $n=1$, $\frac{2}{3}=.67 \not< .2$

if $n=2$, $\left(\frac{2}{3}\right)^2=\frac{4}{9}=.44 \not< .2$

if $n=3$, $\left(\frac{2}{3}\right)^3=\frac{8}{27}=.29 \not< .2$

if $n=4$, $\left(\frac{2}{3}\right)^4=\frac{16}{18} \approx .19 < .2$

$n=4$ is sifficient

if $n=5$, $\left(\frac{2}{3}\right)^5=\frac{32}{243} \approx .12 < .2$

then $n\geq 4$ s.t. $\left(\frac{2}{3}\right)^n<.2$

$\therefore$ we must toss a die at least (4) time.

④ Let $X \sim b(2,p)$ & $Y \sim b(4,p)$. If $p(X \geq 1) = \frac{5}{9}$, Find $p(Y \geq 1)$.

sol. ∵ $X \sim b(2,p)$

$$\therefore f(x,2,p) = \begin{cases} \binom{2}{x} p^x (1-p)^{2-x} & \text{for } x = 0,1,2 \\ 0 & \text{o.w.} \end{cases}$$

$$P(X \geq 1) = \frac{5}{9} \Rightarrow 1 - P(X < 1) = \frac{5}{9} \Rightarrow 1 - P(X = 0) = \frac{5}{9}$$

$\therefore P(X=0) = \frac{4}{9} = f(0)$

$\binom{2}{0} p^0 (1-p)^2 = \frac{4}{9}$

$1 \cdot 1 \cdot (1-p)^2 = \frac{4}{9}$

$(1-p)^2 = \frac{4}{9}$

$(1-p) = \mp \frac{2}{3}$

If $1-p = \frac{2}{3}$

then $\boxed{p = \frac{1}{3}}$

If $1-p = -\frac{2}{3}$

since $p = 1 + \frac{2}{3} = \frac{5}{3} > 1$

$0 < p < 1$

$\therefore \boxed{p = \frac{1}{3}}$

$\therefore y \sim b(4,b)$

$$\therefore f(y,4,b) = \begin{cases} \binom{4}{y} p^y (1-p)^{4-y} & ; y = 0, \dots, 4 \\ 0 & \text{o.w.} \end{cases}$$

$\because p = \frac{1}{3} \Rightarrow 1-p = \frac{2}{3}$

$$\begin{aligned} P(Y \geq 1) = 1 - P(Y<1) &= 1 - P(y=0) \\ &= 1 - f(0) \\ &= 1 - \binom{4}{0}\left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^4 \end{aligned}$$

$$\therefore P(Y \geq 1) = 1 - \left(\frac{2}{3}\right)^4 = 1 - \frac{16}{27} = \frac{11}{27} < 1$$

⑤ If $X \sim G(\frac{1}{2})$, find Median of X

sol. $X \sim G(p)$; $p = \frac{1}{2}$, $q = 1 - \frac{1}{2} = \frac{1}{2}$

$$f(x;p) = \begin{cases} \frac{1}{2}\left(\frac{1}{2}\right)^{x-1} & \text{for } x = 1,2,3,\dots \\ 0 & \text{o.w.} \end{cases}$$

$P(X<m) \leq \frac{1}{2}$ & $P(X \leq m) \geq \frac{1}{2}$, $m \equiv$ median

If $m=1$, $P(X<1) = P(X=0) = 0 < \frac{1}{2}$

$P(X \leq 1) = P(X=1) = \frac{1}{2}$

then $\boxed{m=1}$

Can $m = 2$?

$P(X<2) = P(X=1) = \frac{1}{2}$

$P(X \le 2) = P(X=1,2) = P(X=1) + P(X=2) = \frac{1}{2} + \frac{1}{2}(\frac{1}{2})$

$= \frac{3}{4} > \frac{1}{2}.$

$\therefore \boxed{m=2}$

There are two Values of Median 1,2.

$⑥$ Given $M_X(t) = (\frac{2}{3} + \frac{1}{3}e^t)^9$, Show that:

$$P(\mu - 2\sigma < X < \mu + 2\sigma) = \sum_{x=1}^{5} \binom{9}{x}(\frac{1}{3})^x(\frac{2}{3})^{9-x} = P(0 < X < 6)$$

Sol.

$M_X(t) = (\frac{2}{3} + \frac{1}{3}e^t)^9 \Longrightarrow P = \frac{1}{3}, n = 9, 1-b = \frac{2}{3}$

$\therefore \boxed{X \sim b(9, \frac{1}{3})}$

$M_X(t) = (1-P+Pe^t)^n$

$$f(x, 9, \frac{1}{3}) = \begin{cases} \binom{9}{x}(\frac{1}{3})^x(\frac{2}{3})^{9-x} & \text{for } X = 0, 1, 2, \dots, 9 \\ 0 & \text{o.w.} \end{cases}$$

$E(X) = \mu = np, \sigma^2 ; V(X) = np(1-p)$

$E(x) = 9(\frac{1}{3})$; $V(x) = 3(\frac{2}{3})$

$= 3$; $= 2$

$\mu = 3$; $\sigma^2 = 2$

$\sigma = \sqrt{2}$

$\mu - 2\sigma = 3 - 2\sqrt{2} = 3 - 2(1.4) = 3 - 2.8 = 0.2$ $\boxed{\sqrt{2} = 1.4}$

$\mu + 2\sigma = 3 + 2.8 = 5.8$

$P(\mu - 2\sigma < X < \mu + 2\sigma) = P(.2 < X < 5.8) = P(1 \le X \le 5)$

$= \sum_{x=1}^{5} f(x)$

$= \sum_{x=1}^{5} \binom{9}{x}(\frac{1}{5})^x(\frac{2}{3})^{9-x}$

$= P(0 < X < 6)$

⑦ If $X \sim b(n,P)$, and $Y = \frac{X}{n}$, Prove that :

$\forall \varepsilon > 0, \lim_{n \to \infty} P(|Y-P| \geq \varepsilon) = 0$

X

Sol.

$P(|X-\mu| \geq t) \leq \frac{V(X)}{t^2} = u.b.$

$P(|X-\mu| < t) \geq 1 - \frac{V(X)}{t^2} = L.b.$

$P(|\mu - P| \geq \varepsilon) = P(|\frac{X}{n} - P| \geq \varepsilon) = P(|\frac{X-nP}{n}| \geq \varepsilon)$

$= P(|X-nP| \geq n.\varepsilon)$

$\boxed{\mu = nP}$

$P(|X-nP| \geq n.\varepsilon) \geq \frac{V(X)}{t^2}$, $\boxed{V(X) = nP(1-P)}$

$= \frac{nP(1-P)}{n^2\varepsilon^2}$

$= \frac{P(1-P)}{n\varepsilon}$

σ^2 $n \to \infty$; $P(|X-nP| \geq n.\varepsilon) \to 0$

$\therefore \lim_{n\to\infty} P(|X-\mu| \geq t) = \lim_{n\to\infty} P(|X-nP| \geq n.\varepsilon)$

$= 0$ By $(0 \leq P(A) \leq 1)$

$P(|Y-P| < \varepsilon) = P(|\frac{X}{n} - P| \geq \varepsilon) = P(|\frac{X-nP}{P}| < \varepsilon)$

$= P(|X-nP| < n.\varepsilon)$; $\mu = nP$, $t = n.\varepsilon$

$< 1 - \frac{V(X)}{t^2}$; $V(X) = nP(1-P)$

$= 1 - \frac{nP(1-P)}{n^2\varepsilon^2} = 1 - \frac{P(1-P)}{n\varepsilon^2}$

$\therefore \lim_{n\to\infty} (|Y-P| < \varepsilon) = \lim_{n\to\infty} (1 - \frac{P(1-P)}{n\varepsilon^2}) = 1 - 0 = 1$

⑧ Suppose that $X_1, X_2, \dots, X_n$ are n- independent random variables, and for $i = 1, 2, \dots, n$ let ψ_i be the M.g.f. of X_i. Let $Y = \sum_{i=1}^n x_i$ and let ψ be the M.g.f. of Y. Then for any t s.t. $\psi_i(t)$ exist,

$\psi(t) = \prod_{i=1}^n \psi_i(t)$.

6

Sol. X_i $(i = 1, 2, \dots, n)$ is indep. r.v.s

ψ_i $(i = 1, 2, \dots, n)$ be M.g.f. of X_i

If $y = \sum_{i=1}^{n} X_i$, then if ψ be the M.g.f. of y, then forevery t s.t. ($\psi_i(t)$ exists), then

$$\psi(t) = \prod_{i=1}^{n} \psi_i(t) \quad \text{s.t.} \quad y = \sum_{i=1}^{n} X_i$$

$$\psi_y(t) = E(e^{ty}) = E[e^{(\Sigma x_i)t}] = E[e^{tx_1+tx_2+\cdots+tx_n}]$$

$$= E(e^{tx_1}) \cdot E(e^{tx_2}) \cdots E(e^{tx_n})$$

$$= (\psi_1(t))\,(\psi_2(t)) \cdots (\psi_n(t))$$

$$= \prod_{i=1}^{n} \psi_i(t)$$

⑨ Let X be a r.v. with M.g.f. ψ_1, Let $Y = aX+b$; where $a, b \in \mathbb{R}$ and let ψ_2 denote the M.g.f. of Y, then for any value of t such that $\psi_1(at)$ exists, $\psi_2(t) = \exp(bt) \cdot \psi_1(at)$.

sol.

$$\psi_2(t) = E(e^{ty}) = E[e^{(ax+b)t}] = E[e^{axt} \cdot e^{bt}] = e^{bt} E(e^{axt})$$

$$= e^{bt} \cdot \psi_1(at) = \exp(bt)\,\psi_1(at)$$

since $\psi_1(t) = E(e^{tx})$

$\therefore \quad \psi_1(at) = E(e^{atx})$

نظرية
4 درجة

⑩ If X is a r.v. with $M_X(t)$ exists for $t \in (-h, 0)$, then

$$P(X \leq a) \leq \exp(-at) \cdot M_X(t)$$

sol. $M_X(t) = E(e^{tx})$

$$y - e^{tx} > 0$$

$$e^{at} > 0$$

$\therefore \; P(X \geq t) \leq \dfrac{E(X)}{t}$ by Markave

$$P(Y \geq e^{at}) \leq \frac{E(Y)}{e^{at}}$$

$$P(e^{tx} \geq e^{at}) \leq \exp(at) \cdot M_X(t)$$

$$\therefore P(X \leq a) \leq \exp.(at) \cdot M_X(t)$$

X
تحذير

⑪ Given a m.g.f. $M_X(t) = e^{3t^2+2t}$ for $-\infty<t<\infty$

Find the L.b. of $P(-\frac{1}{2}<X<\frac{9}{2})$.

Sol.

$$M'_X(t) = \left(e^{3t^2+2t}\right)(6t+2)$$

$$E(X) = M'_X(0) = e^0(0+2)$$

$$M''_X(t) = \left(e^{3t+2t}\right)(6) + (6t+2)^2\, e^{3t^2+2t}$$

$$E(X^2) = (e^0)(6) + (2)^2(e^0) = 6+4=10$$

$$V(X) = E(X^2) - [E(X)]^2 = 10-(2)^2 = 6$$

$$\therefore P\left(-\frac{1}{2}<X<\frac{9}{2}\right) = P\left(-\frac{1}{2}-2<X-2<\frac{9}{2}-2\right)$$

$$= P(-2.5<X-2<2.5)$$

$$= P(|X-2|<2.5) \Rightarrow t=\frac{5}{2}=2.5$$

$$L.b. = 1-\frac{V(X)}{t^2} = 1-\frac{6}{(\frac{5}{2})^2} = 1-6.\frac{4}{25} = 1-\frac{24}{25}$$

$$= \frac{1}{25} \simeq 0.04$$

⑫ Given a p.m.f. $f(x) = \begin{cases} \frac{x}{10} & \text{for } X=0,1,2,3,4 \\ 0 & \text{o.w.} \end{cases}$

Find the pr. Percentite at the point $X=3.5$.

Sol.

$$P(X\leq x_0) = \frac{t}{100}$$

$$P(X\leq 3.5) = \frac{6}{100} \Rightarrow \sum_{X=0}^{3} f(x) = \frac{t}{100}$$

$$f(0)+f(1)+f(2)+f(3) = \frac{t}{100}$$

$$\frac{0}{10}+\frac{1}{10}+\frac{2}{10}+\frac{3}{10} = \frac{t}{100} \Rightarrow \frac{6}{10} = \frac{t}{100} \Rightarrow 10t=600$$

$$\boxed{t=60\,\%}$$

⑬ Find the mean, median and mode of Cauchy dist. if exists.

Sol. $f(x) = \dfrac{1}{\pi(1+X^2)}$ $\quad -\infty<X<\infty \quad$ (Cauchy dist.)

Mean: $E(X)$ exist only when $E(|X|)<\infty$

$$E(|X|) = \int_{-\infty}^{\infty}|X|\frac{1}{\pi(1+X^2)}dX = \int_0^{\infty}\frac{2X}{\pi(1+X^2)}dx = \frac{1}{\pi}Ln(1+X^2)\Big|_0^{\infty}$$

$$= \frac{1}{\pi}[Ln(\infty)-Ln(0)] = \infty$$

$E(X)$ not exist.

Median $P(X\le m)\ge\frac{1}{2}$ & $P(X<m)\le\frac{1}{2}$

$P(X<m)\le\frac{1}{2}$

$\int_{-\infty}^{m}\frac{1}{\pi(1+x^2)}dx\le\frac{1}{2}$

$\frac{1}{\pi}\int_{-\infty}^{m}\frac{1}{1+x^2}dx\le\frac{1}{2}$

$\frac{1}{\pi}\tan^{-1}x\Big|_{-\infty}^{m}\le\frac{1}{2}$

$\frac{1}{\pi}[\tan^{-1}(m)-\underset{-\pi/2}{\tan^{-1}(\infty)}]\le\frac{1}{2}$

$\tan^{-1}(m)+\frac{\pi}{2}\le\frac{\pi}{2}$

$\tan^{-1}(m)\le 0$

$m\le\tan(0)$

$\therefore m\le 0$

$P(X\le m)\ge\frac{1}{2}$

$\int_{-\infty}^{m}\frac{1}{\pi(1+x^2)}dx\ge\frac{1}{2}$

$\frac{1}{\pi}\int_{-\infty}^{m}\frac{1}{(1+x^2)}dx\ge\frac{1}{2}$

$\frac{1}{\pi}\tan^{-1}x\Big|_{-\infty}^{m}\ge\frac{1}{2}$

$\frac{1}{\pi}[\tan^{-1}(m)-\tan^{-1}(\infty)]\geq\frac{1}{2}$

$\tan^{-1}(m)\ge 0$

$m\ge\tan(0)$

$\therefore m\ge 0$

$\therefore \boxed{m_e=0}$ median.

Mode $f(x)=\frac{1}{\pi}(1+x^2)^{-1}\Rightarrow f'(x)=-\frac{1}{\pi}(1+x^2)^{-2}(2x)=0$

$\frac{-2x}{\pi(1+x^2)^2=0}\Longrightarrow x=0$

$f''(x)=\frac{2}{\pi}(1+x^2)^{-3}(2x)(2x)-\frac{1}{\pi}(1+x^2)^{-2}(2)$

$f''(0)=0-\frac{2}{\pi}<0\Rightarrow mode=0$

$\therefore \boxed{M_o=0}$ mode.

⑭ Given a p.d.f. $f(x)=\begin{cases}e^x & \text{for } x<0\\ 0 & \text{o.w.}\end{cases}$

Find $M_X(t)$ and sketch it's graph. Also, find $E(x)$ by two methods.

Sol.

$M_X(t)=E(e^{tx})=\int_{-\infty}^{0}e^{tx}e^{x}dx=\int_{-\infty}^{0}e^{x(1+t)}dx$

This integration exists only when $(1+t)>0\Longrightarrow\boxed{t>-1}$

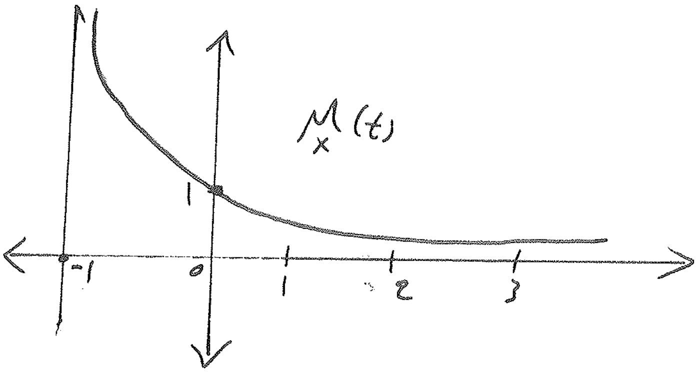
$$M_x(t) = \frac{1}{1+t} e^{x(1+t)} \Big|_{-\infty}^{0} = \frac{1}{1+t}\left[e^{0} - e^{-\infty}\right] = \frac{[1-0]}{1+t} = \frac{1}{1+t}$$

$$M_x(t) = \begin{cases} \frac{1}{1+t} & \text{for } t > -1 \\ 0 & \text{o.w.} \end{cases}$$

must $1+t > 0$

$t > -1$

t	$M_x(t)$
-1	∞
0	1
1	$1/2$
2	$1/3$
3	$1/4$
$\vdots$	
∞	0



① $E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{0} x \cdot e^{x}\,dx$

$u = x \quad , dv = e^{x}$

$du = 1 \quad , v = e^{x}$

$$= uv\Big|_{-\infty}^{0} - \int_{-\infty}^{0} v\,du$$

$$= xe^{x}\Big|_{-\infty}^{0} - \int_{-\infty}^{0} e^{x}\,dx$$

$$= \left[0 \cdot e^{0} - (-\infty)e^{-\infty}\right] - e^{x}\Big]_{-\infty}^{0} = [0-0] - [e^{0} - e^{-\infty}]$$

$$= -1$$

$$M_x(t) = \frac{1}{1+t} = (1+t)^{-1} \quad ; t > -1$$

$$M_x'(t) = -1(1+t)^{-1}(1)$$

$$E(X) = M_x'(0) = (-1)[1+0]^{-1} = -1$$

⑮ If $X \sim unif(0,2)$, Find the u.b. of $p(|X-\mu| \geq 2)$.

Sol: $u.b. = \frac{V(x)}{t^2} \quad , \quad t = 2$

ليش ?

$$f(x) = \begin{cases} \frac{1}{2-0} = \frac{1}{2} & 0 < x < 2 \\ 0 & \text{o.w.} \end{cases}$$

$$\mu = E(X) = \int_{0}^{2} x\, f(x)\,dx = \int_{0}^{2} x \cdot \frac{1}{2}\,dx = \frac{1}{2} \cdot \frac{x^2}{2}\Big|_{0}^{2} = \frac{1}{4}[4-0] = \frac{4}{4} = 1$$

$$\boxed{\mu = E(X) = 1}$$

$$E(X^2) = \int_{0}^{2} x^2 \cdot \frac{1}{2}\,dx = \frac{1}{2}\frac{x^3}{3}\Big|_{0}^{2} = \frac{1}{6}[8-0] = \frac{8}{6} = \frac{4}{3}$$

$$V(X) = E(X^2) - [E(X)]^2$$
$$= \frac{4}{3} - (1)^2$$
$$= \frac{4}{3} - \frac{3}{3} = \frac{1}{3}$$
$$\text{U.b.} = \frac{V(X)}{t^2} = \frac{(\frac{1}{3})}{(2)^2} = \frac{1/3}{4} = \frac{1}{12}$$

(16) Find the pr. percentite at the point $x=1$, if Given a p.d.f.

$$f(x) = \begin{cases} \frac{3}{8}x^2 & \text{for } 0<x<2 \\ 0 & \text{o.w.} \end{cases}$$

Sol.

$$P(X \leq 1) = \frac{t}{100} \Rightarrow \int_0^1 f(x)\,dx = \frac{t}{100} \Rightarrow \frac{3}{8}\int_0^1 x^2\,dx = \frac{t}{100}$$

$$\frac{3}{8} \cdot \frac{x^3}{3}\Big|_0^1 = \frac{1}{8} = \frac{t}{100} \Rightarrow t = \frac{100}{8} = 12.5\ \% = 0.125$$

(17) A Coin is tossed 4-times, $X \equiv$ number of heads. Find the mean, median and mode of X (if exists).

sol.

$$f(x) = \begin{cases} \frac{\binom{4}{x}}{16} & x = 0,1,2,3,4 \\ 0 & \text{o.w.} \end{cases}$$

X	$f(x) = P(X=x)$
0	$f(0) = \frac{1}{16}$
1	$f(1) = \frac{4}{16}$
2	$f(2) = \frac{6}{16}$
3	$f(3) = \frac{4}{16}$
4	$f(4) = \frac{1}{16}$

$$\sum_{x=0}^{4} f(x) = \frac{16}{16} = 1$$

$f(2)$ is a Maximam

$\therefore$ mode $= 2$

Median

~~If $m = 0$~~

$P(X<0) \leq \frac{1}{2}$; $P(X \leq 0) \geq \frac{1}{2}$

$0 \leq \frac{1}{2}$ $\qquad f(0) = \frac{1}{16} \ngeq \frac{1}{2}$

$\therefore \boxed{m \neq 0}$

If $m=1$

$P(X<1) \leq \frac{1}{2}$; $P(X \leq 1) \geq \frac{1}{2}$

$f(0) = \frac{1}{16} < \frac{1}{2}$; $f(0) + f(1) \geq \frac{1}{2}$

$\frac{1}{16} + \frac{4}{16} \geq \frac{1}{2}$

$\frac{15}{16} \ngeq \frac{1}{2}$

$\boxed{\therefore m \neq 1}$

if $m = 2$

$P(X < 2) \leq \frac{1}{2}$; $P(X \geqslant 2) \geqslant \frac{1}{2}$

$f(0) + f(1) \leq \frac{1}{2}$; $f(0) + f(1) + f(2) \geqslant \frac{1}{2}$

$\frac{1}{16} + \frac{4}{16} \leq \frac{1}{2}$; $\frac{1}{16} + \frac{4}{16} + \frac{6}{16} \geqslant \frac{1}{2}$

$\frac{5}{16} \leq \frac{1}{2}$; $\frac{11}{16} \geqslant \frac{1}{2}$

$\therefore$ median $= \boxed{m = 2}$

Mean

$$E(X) = \sum_{x=0}^{4} x\, f(x) = 0 \cdot f(0) + 1 \cdot f(1) + 2 \cdot f(2) + 3 \cdot f(3) + 4 \cdot f(4)$$

$$= 1 \cdot \frac{4}{16} + 2 \cdot \frac{6}{16} + 3 \cdot \frac{4}{16} + 4 \cdot \frac{1}{16}$$

$$= \frac{4}{16} + \frac{12}{16} + \frac{12}{16} + \frac{4}{16} = \frac{32}{16} = 2 = \mu$$

$\therefore \boxed{\mu = 2}$

---

(18) If $X \sim b(n, p)$, then find the value of (n) s.t. $P\left(\left|\frac{X}{n} - p\right| < 0.1\right) \geqslant 0.95$

Sol. By theorem (chebyshive theorem)

$$P\left(\left|\frac{X}{n} - p\right| < 0.1\right) \geqslant 0.95 \quad \overset{\text{مقارنة}}{\Longleftrightarrow} \quad P\left(\left|\frac{X}{n} - \mu\right| < t\right) \geqslant 1 - \frac{V(X)}{t^2}$$

$$= P(|X - np| < (0.1)n) = P\left(|X - \mu| < \frac{n}{10}\right) \Rightarrow \boxed{t = \frac{n}{10}} , \mu = np$$

$$L.b. = 1 - \frac{V(X)}{t^2}$$

$$V(X) = np(1-p)$$

$$L.b. = 1 - \frac{np(1-p)}{\left(\frac{n}{10}\right)^2} = 1 - \left[np(1-p)\frac{100}{n^2}\right]$$

$$= 1 - \frac{100\, p(1-p)}{n} \geqslant 0.95$$

$$\frac{100\, p(1-p)}{n} \leq 0.05 = \frac{5}{100} = \frac{1}{20}$$

$$2000\, p(1-p) \leq n$$

$\boxed{n \geqslant 2000\, p(1-p)}$ $\forall p$, $0 < p < 1$

6

⑲ If X has a M.g.f. as follows:

$M_X(t) = \frac{2e^t}{5(1-\frac{3}{5}e^t)}$ for $t < Ln(\frac{5}{3})$, then find $P(X>7|X>3)$.

Sol.

$$M_X(t) = \frac{(\frac{2}{5})e^t}{(1-\frac{3}{5}e^t)} = \frac{pe^t}{1-qe^t}$$

memoryless

$\therefore X \sim G(p=\frac{2}{5})$

$\therefore P(X>7/X>3) = P(X>4) = (q)^4 = (\frac{3}{5})^4.$

⑳ If X has a p.M.f.

$$f(x) = \begin{cases} (\frac{2}{3})(\frac{1}{3})^{x-1} & \text{for } x = 1, 2, 3, \dots \\ 0 & \end{cases}$$

6

Find $P(X>5|X>2)$

① which distribution that X have

② write ~~the~~ m.g.f. of X, E(x) & Var(x)

③ Compute P(X>5|X>4)

Sol. $X \sim G(\frac{2}{3})$, $p = \frac{2}{3}$, $q = \frac{1}{3}$

$\therefore P(X>5/X>2) = P(X>3) = (q)^3 = (\frac{1}{3})^3 = \frac{1}{27}$

㉑ If $X \sim P(\lambda)$, then show that:

$E(x) = V(x) = \lambda$

desire the Expected value of X and the variance of X

$\therefore M_X(t) = e^{\lambda(e^t-1)}$

$M'_X(t) = e^{\lambda(e^t-1)}(\lambda e^t)$

6

$E(x) = M'_X(0) = e^0(\lambda)e^0 = \lambda$

$M''_X(t) = e^{\lambda(e^t-1)}(\lambda e^t) + (\lambda e^t)e^{\lambda(e^t-1)}(\lambda e^t)$

$E(x^2) = M''_X(t=0) = 1.(\lambda) + (\lambda).(1).(\lambda)$

$E(x^2) = \lambda + \lambda^2$

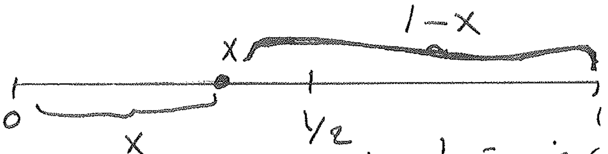
$V(x) = E(x^2) - [E(x)]^2$

$= \lambda + \lambda^2 - \lambda^2 = \lambda$

$$\boxed{E(X) = V(X) = \lambda}$$

(22) A point X is chosen at random from a stick of length one unite, and then the stick is broken at the chosen point into two unequal parts, find the expected value of longer part.

sol.



مسطرة (stick)

تنقسم المسطرة الى قسمين غير متساويين ، فاحد الاقسام صغير X و الآخر كبير (1−X)

∴ X is shorter part, $(1-X)$ is longer part.

∴ $X \in (0, 1/2)$

∴ $X \sim \text{unif.}(0, \frac{1}{2})$

$\boxed{4}$

$$f(x) = \begin{cases} \frac{1}{1/2 - 0} = 2 & \text{for } 0 < x < \frac{1}{2} \\ 0 & \text{o.w.} \end{cases}$$

Find $E[\text{longer part}]$

i.e Find $E(1-x)$?

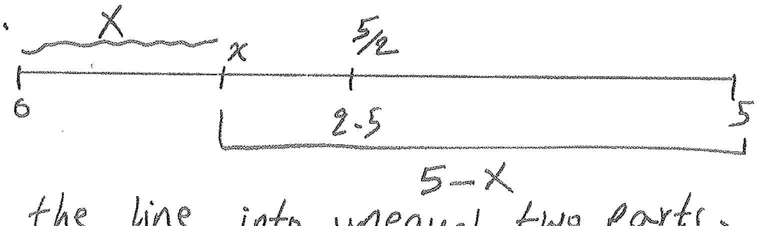
$$E(1-x) = 1 - E(X)$$

$$= 1 - \int_0^{1/2} x \cdot (2)\, dx = 1 - \frac{\cancel{2}}{\cancel{2}} x^2 \Big|_0^{1/2} = 1 - [(\tfrac{1}{2})^2 - 0]$$

$$= 1 - 1/4 = 3/4$$

$\therefore E(1-x) = \frac{3}{4}$ (القيمة المتوقعة للجزء الأكبر)

(23) A point X is chosen on aline of long 5 cm. This chosen point divides the line into two unequal parts. Find the expectation of shorter part.



sol.

$\boxed{4}$

Let X divide the line into unequal two parts.

∴ X is the shorter part, $(5-X)$ is the longer part.

∴ $X \in (0, \frac{5}{2})$

∴ $X \sim \text{unif}(0, \frac{5}{2})$

$$f(x) = \begin{cases} \frac{1}{\frac{5}{2} - 0} = \frac{2}{5} & \text{for } 0 < x < \frac{5}{2} \\ 0 & \text{o.w.} \end{cases}$$

Find $E(\text{shorter part}) = E(x)$?

$$E(x) = \int x f(x)\,dx = \int_0^{3/2} x \left(\tfrac{2}{5}\right) dx = \frac{\cancel{2}}{5} \cdot \frac{x^2}{\cancel{2}} \Big|_0^{5/2}$$
$$= \frac{1}{5}\left[\left(\tfrac{5}{2}\right)^2 - 0\right] = \frac{1}{\cancel{5}} \cdot \frac{\cancel{25}}{4} = \frac{5}{4} = 1.25$$

(24) Given ~~a~~ p.d.f.

$$f(x) = \begin{cases} x^2 & \text{for } 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$$

Find mode of X.

لسؤال امتحاني

$\int_0^1 12x^2 = \frac{12}{3}x^3 \Big|_0^1$
$= 4[1-0$

$\int_0^1 9x^2 = 1$
$\frac{9}{3}x^3 \Big|_0^1 = 1$
$\frac{9}{3} = 1$
$9 = 3$

Sol. $f(x) = 12x^2 - 2x^3$
$f'(x) = 24x - 36x^2 = 0$
$12x(2-3x) = 0$
$(12x = 0)$ or $(2 - 3x = 0)$
$x_1 = 0$ or $x_2 = \frac{2}{3}$

$f''(x) = 24 - 72x$
af $(x=0) \Rightarrow f''(0) = 24 \Rightarrow f(0)$ is min. then mode $\neq 0$
af $(x = \frac{2}{3}) \Rightarrow f''(\frac{2}{3}) = 24 - 72(\frac{2}{3}) = -24 < 0$ then $f(\frac{2}{3})$ is Max.
$\therefore$ mode $= \frac{2}{3}$

(25) If X has a poisson dist. with Parameter m, then
$E(x) = V(x) = m$.

Theorem

Derive

Sol. $M_x(t) = e^{m(e^t - 1)}$ (by theorem)
$M'_x(t) = e^{m}(e^{t-1}) \cdot m e^t$
$\therefore E(x) = M'_x(0) = m\, e^{m(e^0 - 1)} = m$
$M''_x(t) = e^{m(e^t-1)} \cdot me^t + me^t \cdot e^{m(e^t-1)} \cdot (me^t)$
$E(x^2) = e^0 m e^0 + me^0 \cdot e^0 (me^0)$
$= m + m^2$
$V(x) = E(x^2) - [E(x)]^2$
$V(x) = (m + \cancel{m^2}) - \cancel{m^2}$
$V(x) = m = E(x)$

㉖ Given a m.g.f. of X, $M_X(t) = \frac{1}{1-2t}$ for $t < \frac{1}{2}$
Find the mean and variance of X.

Sol. $$M_X(t) = \frac{1}{1-2t} = (1-2t)^{-1}$$

$$M_X'(t) = -(1-2t)^{-2}(-2) = 2(1-2t)^{2}$$

$$M_X''(t) = -4(1-2t)^{-3}(-2) = 8(1-2t)^{-3}$$

$$M_X'(0) = 2 \quad , \quad M_X''(0) = 8$$

$\therefore E(X) = 2$, $E(X^2) = 8$ (by theorem)

$$V(X) = E(X^2) - [E(X)]^2$$
$$= 8 - (2)^2 = 4$$

4

㉗ If X has a uniform dist. on $(-\sqrt{3}, \sqrt{3})$, then find the upper bound of $P(|X-M| \geq \frac{3}{2})$.

Sol. $$U.b. = \frac{V(X)}{t^2}$$

$$f(x) = \frac{1}{\sqrt{3}-(-\sqrt{3})} = \begin{cases} \frac{1}{2\sqrt{3}} & \text{for } -\sqrt{3} < x < \sqrt{3} \\ 0 & \text{o.w.} \end{cases}$$

$$E(X) = \int_{-\sqrt{3}}^{\sqrt{3}} x\left(\frac{1}{2\sqrt{3}}\right)dx = 0$$

$$E(X^2) = \int_{-\sqrt{3}}^{\sqrt{3}} x^2\left(\frac{1}{2\sqrt{3}}\right)dx = 1$$

$$\therefore V(X) = E(X^2) - (E(X))^2$$
$$= 1 - 0^2 = 1 \Rightarrow V(X) = 1$$

$$U.b. = \frac{V(X)}{t^2} = \frac{1}{(\frac{3}{2})^2} = \frac{4}{9} = 0.44$$

㉘ If X has a geometric dist. with $p = \frac{1}{4}$, then find $P(X>8 | X>3)$.

Sol. $p = \frac{1}{4} \Rightarrow q = \frac{3}{4}$

$$P(X>8|X>3) = P(X>5) = [P(A^c)]^5 = q^5 = \left(\frac{3}{4}\right)^5$$

6

㉙ If $X \sim b(n,p)$, then the M.g.f. of X is

$$M_X(t) = (1-p+pe^t)^n$$

Sol.

$\because X \sim b(n,p)$

$$\therefore f(x,n,p) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{for } x = 0,1,2,\dots,n \\ 0 & \text{o.w.} \end{cases}$$

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{n} e^{tx} f(x)$$

$$= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x}$$

since $(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} b^x a^{n-x}$

Let $b = pe^t$ and $a = 1-p$

then $(1-p+pe^t)^n = \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x} = M_X(t)$

$$\therefore \boxed{M_X(t) = (1-p+pe^t)^n}$$

㉚ Given a p.d.f. $f(x) = \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{o.w.} \end{cases}$

Find and sketch graph of $M_X(t)$.

① which distribution that x have
② derive $M_X(t)$
③ $E(x)$ $V(x)$

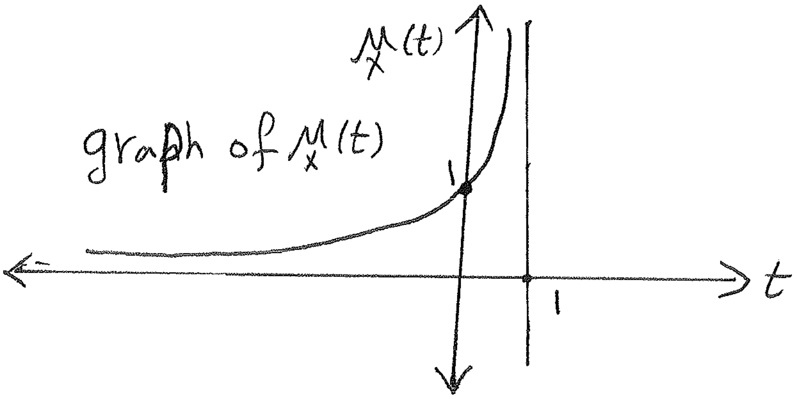
Sol.

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx}(e^{-x})\,dx = \int_0^\infty e^{-(1-t)x}\,dx$$

$e^{-(1-t)x}$ exists when $t < 1$

$$M_X(t) = \frac{-1}{1-t}\int_0^\infty e^{-(1-t)x}(-(1-t))\,dx = \frac{1}{1-t} \text{ for } t < 1$$

t	$M_X(t) = \frac{1}{1-t}$
1	∞
0	1
−1	.5
⋮	
$-\infty$	.0



(31) Given $M_x(t) = \frac{1}{1-3t}$ for $t < \frac{1}{3}$, If $y = 1-2X$, then find $M_y(t)$.

Sol. $M_y(t) = e^{bt} M_x(at)$, $y = ax+b$

$M_x(t) = \frac{1}{1-3t}$ for $t < \frac{1}{3}$

$M_y(t) = e^{t} M_x(-2t)$, $y = 1-2X$

where $M_x(-2t) = \frac{1}{1+6t}$ for $t > \frac{-1}{6}$

(32) If a r.v. X with $M_x(t)$ for $-h<t<h$

show that $P(X \geq a) \leq e^{-at} M_x(t)$ for $0<t<h$

and $P(X \leq a) \leq e^{-at} M_x(t)$ for $-h<t<0$

Proof $M_x(t) = E(e^{tx})$; $y = e^{tx}$

$e^{at} > 0$; $a>0$ & $t>0$

$P(X \geq t) \leq \frac{E(X)}{t}$ (by theorem)

$P(Y \geq e^{at}) \leq \frac{E(Y)}{e^{at}}$

$P(e^{tx} \geq e^{at}) \leq \frac{E(e^{tx})}{e^{at}}$

$P(\ln e^{tx} \geq \ln e^{at}) \leq \frac{M_x(t)}{e^{at}} = e^{-at} M_x(t)$

$P(tx \geq at) \leq e^{-at} M_x(t)$ for $0<t<h$

$P(X \geq a) \leq e^{-at} M_x(t)$ —— ①

$P(tx \geq at) \leq e^{-at} M_x(t)$

$P(X \leq a) \leq e^{-at} M_x(t)$ for $-h<t<0$ —— ②

(33) If X has a geometric dist. with parameter P, then the M.g.f. of X is $M_x(t) = \frac{Pe^t}{1-qe^t}$ for $t < \ln(\frac{1}{q})$.

Sol. $X \sim G(P,q) \Rightarrow f(x) = \begin{cases} P\,q^{x-1} & , x = 1,2,3,\dots \\ 0 & \text{o.w.} \end{cases}$

derive $\boxed{6}$

$$M_X(t) = E(e^{tx})$$
$$= \sum_{x=1}^{\infty} e^{tx} f(x)$$
$$= \sum_{x=1}^{\infty} e^{tx}(pq^{x-1}) = \frac{p}{q}\sum_{x=1}^{\infty}(q e^{t})^{x}$$
$$= \frac{p}{q}[qe^{t} + (qe^{t})^{2} + \cdots + (qe^{t})^{n} + \cdots]$$
$$= \frac{p}{q} \cdot qe^{t}[1 + qe^{t} + q^{2}e^{2t} + \cdots + q^{n-1}e^{(n-1)t} + \cdots]$$

$$\therefore r = \frac{qe^{t}}{1} = qe^{t}$$

$$|r| < 1 \Longrightarrow qe^{t} < 1 \Rightarrow e^{t} < \frac{1}{q} \Rightarrow \ln e^{t} < \ln\left(\frac{1}{q}\right)$$
$$\Rightarrow t < \ln\left(\frac{1}{q}\right)$$

$$S = \underbrace{1 - qe^{t} + (qe^{t})^{2} + \cdots}_{\text{Geometriq series}} , \quad S = \frac{1}{1-r} = \frac{1}{1-qe^{t}}$$

$$M_X(t) = \frac{pe^{t}}{1-qe^{t}} \quad \text{for } t < \ln\left(\frac{1}{q}\right)$$

(34) $X \sim N(\mu, \sigma^{2})$, $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$; find $E(Z), V(Z)$.

Sol.
$$E(Z) = E\left(\frac{X-\mu}{\sigma}\right)$$
$$= E\left(\frac{X}{\sigma} - \frac{\mu}{\sigma}\right)$$
$$= E\left(\frac{X}{\sigma}\right) - \frac{\mu}{\sigma}$$
$$= \frac{1}{\sigma}E(x) - \frac{\mu}{\sigma}$$
$$= \frac{\mu}{\sigma} - \frac{\mu}{\sigma} = 0$$

$$\operatorname{var}(Z) = V\left(\frac{X}{\sigma} - \frac{\mu}{\sigma}\right)$$
$$= \frac{V(X)}{\sigma^{2}} - V\left(\frac{\mu}{\sigma}\right)$$
$$= \frac{\sigma^{2}}{\sigma^{2}} - 0$$
$$= 1$$

$$M_Z(t) = e^{(tZ)} = E\left[e^{\left(\frac{x}{\sigma} - \frac{\mu}{\sigma}\right)t}\right]$$
$$= e^{-\frac{\mu}{\sigma}t} E\left[e^{\frac{x}{\sigma}t}\right]$$
$$= e^{-\frac{\mu}{\sigma}t} M_{\frac{x}{\sigma}}(t)$$
$$= \cancel{e^{-\frac{\mu}{\sigma}t}}\ \cancel{e^{\frac{\mu}{\sigma}t}} + \frac{\sigma^{2}t}{2\sigma}$$
$$= e^{\frac{\sigma^{2}t^{2}}{2\sigma^{2}}} = e^{t^{2}/2} = e^{0+\frac{t^{2}}{2}}$$

$$\therefore Z \sim N(0,1)$$

6

㉟ If a r.v. X has Gamma dist. with parameters (a) & (b), then show that $V(x) = b \cdot E(x)$

Sol. $\because X \sim G(a,b)$

$\therefore E(x) = a.b$ & $V(x) = a.b^2$

6

$\therefore V(X) = (a.b).b$

$= E(x).b$

$= bE(x)$

OR $\because X \sim G(a,b)$

$\therefore M_X(t) = (1-bt)^{-a}$

$M'_X(t) = (-a)(1-bt)^{-a-1}.(-b) = (ab)(1-bt)^{-(a+1)}$

$M'_X(0) = (a.b)(1-0)^{-(a+1)} = a.b = E(x)$; (Since $E(x) = M'_X(0)$)

$E(x^2) = ?$

$E(x^2) = M''_X(0)$

$M''_X(t) = (a.b)(a+1)(1-bt)^{-(a+2)}.(-b) = (ab^2)(a+1)(1-bt)^{-(a+2)}$

$E(x^2) = M''_X(0) = ab^2(a+1)(1-0) = a^2b^2 + ab^2$

$\therefore V(X) = E(x^2) - [E(x)]^2$

$= (a^2b^2) + (ab^2) - (ab)^2 = ab^2$

$\therefore V(x) = (a.b).b = E(x).b \Rightarrow V(x) = bE(x)$

㊱ If a m.g.f. of X is as follows:

$M_X(t) = (1-2t)^{-7}$, then find the p.d.f. of x, E(x) & V(x).

Sol. $\beta = 2, \alpha = 7 \rightarrow r = 14$

$\therefore X \sim X^2(14)$ chi-squar with (14) d.f.

$$\therefore f(x) = \begin{cases} \dfrac{x^6 e^{-\frac{x}{2}}}{\Gamma(7) 2^7} & \text{for } 0 < x < \infty \\ 0 & \text{o.w.} \end{cases}$$

6

$E(X) = r = 14$, $\sigma^2 = V(X) = 2r = 2(14) = 28$.

Find $P(X \leq a) = 0.95$

the value of a

(37) Given a p.d.f.

$$f(x) = \begin{cases} \frac{2x}{9} & \text{for } 0<x<3 \\ 0 & \text{o.w.} \end{cases}$$

Find the lower bounded of $P(\frac{5}{4} < X < \frac{11}{4})$.

Sol: $l.b = 1 - \frac{V(X)}{t^2}$

we must find $E(X)$, $E(X^2)$:

$$E(X) = \int_0^3 x f(x)dx = \int_0^3 x(\frac{2}{9}x)dx = \frac{2}{9}[\int_0^3 x^2 dx]$$

$$= \frac{2}{9}[\frac{x^3}{3}]_0^3 = \frac{2}{27}[27-0] = 2$$

$$E(X^2) = \int_0^3 x^2 f(x)dx = \int_0^3 x^2(\frac{2}{9}x)dx = \frac{2}{9}[\int_0^3 x^3 dx]$$

$$= \frac{2}{36}x^4\Big]_0^3 = \frac{1}{\underset{2}{\cancel{18}}}[\overset{9}{\cancel{81}}] = \frac{9}{2} = 4.5$$

$$V(X) = E(X^2) - [E(X)]^2$$
$$= 4.5 - (2)^2$$
$$= 4.5 - 4 = 0.5 = \frac{1}{2}$$

$$P(\frac{5}{4} < X < \frac{11}{4}) = P(\frac{5}{4} - 2 < X-2 < \frac{11}{4} - 2)$$
$$= P(-\frac{3}{4} < X-2 < \frac{3}{4})$$
$$= P(|X-2| < \frac{3}{4})$$

$\therefore t = \frac{3}{4}$

$$l.b = 1 - \frac{V(X)}{t^2} = 1 - \frac{\frac{1}{2}}{(\frac{3}{4})^2} = 1 - \frac{1/2}{9/16} = 1 - \frac{8}{9}$$
$$= \frac{1}{9}$$

$$\therefore \boxed{l.b = \frac{1}{9}}$$

(38) Given a p.d.f. of X

$$f(x) = \begin{cases} \frac{1}{4}e^{-\frac{x}{4}} & \text{for } X>0 \\ 0 & \text{o.w.} \end{cases}$$

① Find $M_x(t)$ ② If $y = 2-X$, then find $M_y(t)$

Sol.

Sol.

① $M_X(t) = E(e^{tX}) = \frac{1}{4}\int_0^\infty e^{tx} \cdot e^{-\frac{1}{4}x}\,dx$

$= \frac{1}{4}\int_0^\infty e^{-(\frac{1}{4}-t)\cdot x}\,dx$

This integration exist only when $(\frac{1}{4}-t) > 0$

i.e $\frac{1}{4} > t \rightarrow t < \frac{1}{4}$

$\therefore M_X(t) = \frac{1}{4} \cdot \frac{-1}{(\frac{1}{4}-t)} e^{-(\frac{1}{4}-t)x}\Big|_0^\infty = \frac{-1}{4(\frac{1}{4}-t)}[e^{-\infty}-e^{0}]$

$= \frac{-1}{4(\frac{1}{4}-t)}[0-1]$

$= \frac{1}{4(\frac{1}{4}-t)}$ for $t < \frac{1}{4}$

② $Y = 2-X$, $b=2$, $a=-1$

$M_Y(t) = e^{bt} \cdot M_X(at)$

$= e^{2t} \cdot M_X(t)$

$M_X(-t) = \frac{1}{4(\frac{1}{4}+t)}$ for $-t < \frac{1}{4}$

$M_X(-t) = \frac{1}{4(\frac{1}{4}+t)}$ for $t > \frac{-1}{4}$

$\therefore M_Y(t) = e^{2t} \cdot \frac{1}{4(\frac{1}{4}+t)}$ for $t > \frac{-1}{4}$

(f(x,y) ≡ J.P.$\frac{d}{m}$.f.) (M) خلاصة للتدريس

(٢٠١٠/٢٨٠)

Chapter (5)

note: (Pr. dist. = Pr. fun. = f)

Probability Distribution of Two Random Variables

(I) Joint Probability Density Function (J.P.d.f.)

Def. Let X and Y be two c.r.v.'s. A function f defined over XY-plane is a joint p.d.f. of X and Y if for a subset $R \in XY$-plane, then:

$$\boxed{P[(x,y)\in R] = \iint_R f(x,y)\,dx\,dy}$$

if $R = \{(x,y);\ a<x<b\ \&\ c<y<d\}$

Then

$$P[(x,y)\in R] = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

$$= \int_a^b\int_c^d f(x,y)\,dy\,dx$$

Review

ch.5 (Pr. dist. of X & Y)

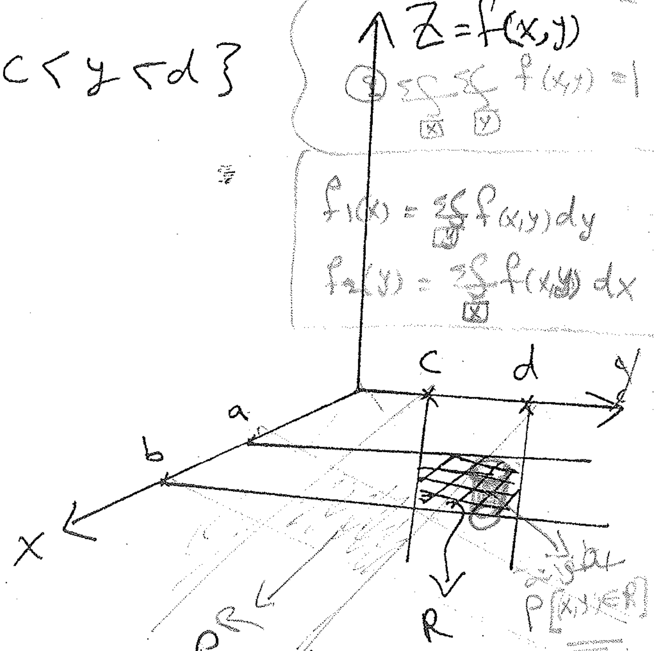
f(x,y) is J.P.m.f.

① $f(x,y) \geq 0\ \forall(x,y)\in R$

② $\sum_{\boxed{x}}\sum_{\boxed{y}} f(x,y) = 1$

$f_1(x) = \sum_{\boxed{y}} f(x,y)\,dy$

$f_2(y) = \sum_{\boxed{x}} f(x,y)\,dx$



Note: (مقارنة بين J.Pr.f و Pr.f)

① f(x,y) is a $\overset{\text{مجسمة}}{\text{Solid}}$ over XY-plane. ⟶ f(x) is a curve

② R can be □, ▭, △, ○, ⬭, … ⟶ (a,b) $\overline{a\quad b}$

R ≡ Domian of func. f(x,y)

③ $P[(x,y)\in R]$ is the $\overset{\text{حجم}}{\text{Volum}}$ $\overset{\text{داخل}}{\text{inside}}$ $\overset{\text{المجسم}}{\text{Solid}}$ of f(x,y). ⟶ مساحة تحت المنحنى Area under the curve

(X متغير)

Properties of Joint p.d.f.

① A joint p.d.f. f(x,y) satisfies two conditions:

ⓐ $f(x,y) \geq 0 \qquad \forall (x,y) \in R$

ⓑ $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

أنواع Regions

Ex. حل ١٢٥ الرسم

① $R = \{(x,y);\ 0<x<1,\ 0<y<1\}$

② $R = \{(x,y);\ 0<x<y<1\}$

③ $R = \{(x,y);\ x^2+y^2 \leq 1\}$

② Also:

∴ $f(x,y)$ is a function of x and y, from $f(x,y)$, we can find a function of X alone $f_1(x)$ and a function of Y alone $f_2(y)$.
s.t.

جزئي

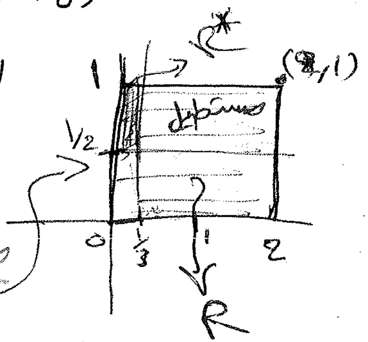
$f_1(x)$ is called Marginal p.d.f. of X
where $f_1(x) = \int_{\square}^{\square} f(x,y)\,dy$; limits of integral follows limits of y

$f_2(y)$ is called Marginal p.d.f. of Y
where $f_2(y) = \int_{\square}^{\square} f(x,y)\,dx$; limits of integral follows limits of x

---

Example ①: Given a joint p.d.f. (J.p.d.f.) $f(x,y)$:

$$f(x,y) = \begin{cases} Kx^2y & \text{for } 0<x<2\ ,\ 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

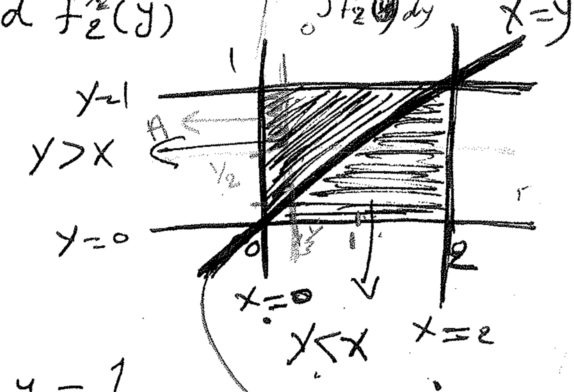


ملاحظة: في هذا المثال $\iint dxdy = \iint dydx$ لأن X لا تعتمد على Y (حدود مفصولة ثابتة) معزولة isolation X, Y

ⓐ Find the value of K,
ⓑ Find $P(0<x<\frac{1}{3}, \frac{1}{2}<y<2)$, ⓕ $P(X>\frac{1}{2})$, $P(y<\frac{1}{3})$

R^* $\quad \int_{1/2} f_1(x)\,dx \quad \int_0^{1/3} f_2(y)\,dy$

ⓒ Find $P(X+Y \geq 1)$, ⓓ Find $f_1(x)$ and $f_2(y)$



Sol. By cond. (2)

ⓐ $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

$R = \{(x,y) : 0<x<2\ ,\ 0<y<1\}$

$\int_0^1\int_0^2 K\,x^2y\,dx\,dy = 1 \Rightarrow \int_0^1 Ky\left(\frac{x^3}{3}\Big|_0^2\right)dy = 1$

$\frac{1}{3}\int_0^1 Ky\,(8-0)\,dy = 1 \Rightarrow \frac{8K}{3}\int_0^1 y\,dy = 1$

الخط (المساواة) لا يعطي المنطقة R لهذا يجب تحديد أكبر x أو أصغر

$\frac{8K}{3}\,\frac{y^2}{2}\Big|_0^1 = 1 \Rightarrow \frac{8K}{6}[1^2-0] = 1 \Rightarrow \frac{4}{3}K = 1 \Rightarrow \boxed{K=\frac{3}{4}}$

ⓑ $f(x,y) = \begin{cases} \frac{3}{4}x^2y & 0<x<2,\ 0<y<1 \\ 0 & \text{o.w} \end{cases}$

$$P(0<x<\tfrac{1}{3},\ \tfrac{1}{2}<y<2) = \int_{1/2}^{1}\int_{0}^{1/3} \frac{3}{4}x^2y\,dx\,dy = \cdots = \frac{1}{32(9)}$$

$$= \frac{1}{288}$$

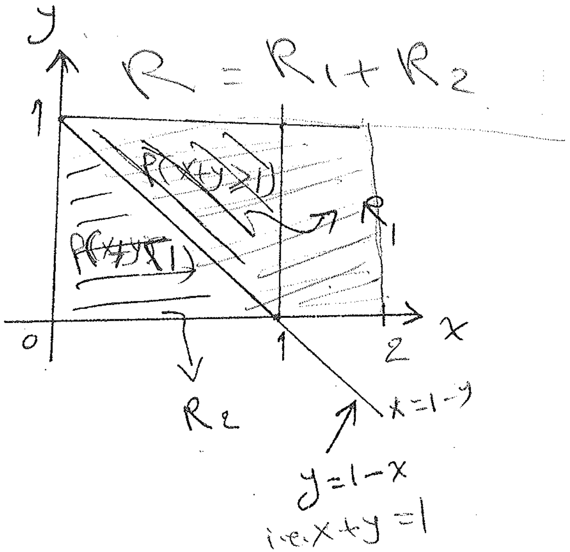
الاحتمال المطلوب

ⓒ $P(x+y\geq 1) = ?$ ⓓ $P(X+y<1) = ?$

Consider $(x+y)=1 \Rightarrow y=1-x$

X	$y=1-x$
0	1
1	0

$\xrightarrow{\text{الرسم}}$



$P(x+y\geq 1)$ = Volum of $f(x,y)$ over R_1

$P(x+y<1)$ = Volum of $f(x,y)$ over R_2

$P(x+y\geq 1) = 1 - P(x+y<1)$

المساواة لا تؤثر [illegible]

$$P(x+y<1) = \int_0^1\int_0^{1-x} f(x,y)\,dy\,dx$$

$$= \int_0^1\left[\int_0^{1-x}\frac{3}{4}x^2y\,dy\right]dx = \int_0^1 \frac{3}{4}x^2\,\frac{y^2}{2}\Big|_0^{1-x}dx$$

$$= \int_0^1 \frac{3}{8}x^2[(1-x)^2-0]\,dx$$

$$= \frac{3}{8}\int_0^1 x^2[1-2x+x^2]\,dx$$

$$= \frac{3}{8}\left[\frac{x^3}{3} - 2\frac{x^4}{4} + \frac{x^5}{5}\right]\Big|_0^1$$

$$= \frac{3}{8}\left[\left(\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right)-(0-0+0)\right] = \frac{3}{8}\left[\frac{10-15+6}{30}\right]$$

$$= \frac{3}{8}\left(\frac{1}{30}\right) = \frac{1}{80}$$

$$P(x+y<1) = \frac{1}{80} \Rightarrow P(x+y\geq 1) = 1-\frac{1}{80} = \frac{79}{80} \in [0,1]$$

ⓓ To find $f_1(x)$ & $f_2(y)$

$$f_1(x) = \int_0^1 f(x,y)\,dy$$

$$f_1(x) = \int_0^1 \frac{3}{4}x^2 y\,dy = \frac{3}{4}x^2\frac{y^2}{2}\Big|_0^1 = \frac{3}{8}x^2[1-0] = \frac{3}{8}x^2$$

$$f_1(x) = \begin{cases} \frac{3}{8}x^2 & \text{for } 0<x<2 \\ 0 & \text{o.w.} \end{cases}$$

$$f_2(y) = \int_0^2 f(x,y)\,dx$$

$$= \int_0^2 \frac{3}{4}x^2 y\,dx = \frac{3}{4}y\frac{x^3}{3}\Big|_0^2 = \frac{1}{4}y[8-0] = \frac{8}{4}y$$

$$= 2y$$

$$f_2(y) = \begin{cases} 2y & \text{for } 0<y<1 \\ 0 & \text{o.w.} \end{cases}, \quad P(y<\tfrac{1}{3}) = \int_0^{1/3} 2y\,dy = \cancel{2}y^2\Big|_0^{1/3} = \frac{1}{9}$$

(لم يشرح)

Example ⑩ Given a J. p.d.f.

* (يسمى احادي وثنائي حسب عدد المتغيرات)

univariat	احادي one
Bivariat	ثنائي two
multivariat	متعدد n

$$f(x,y) = \begin{cases} 2 & \text{for } 0<x<y<1 \\ 0 & \text{o.w.} \end{cases}$$

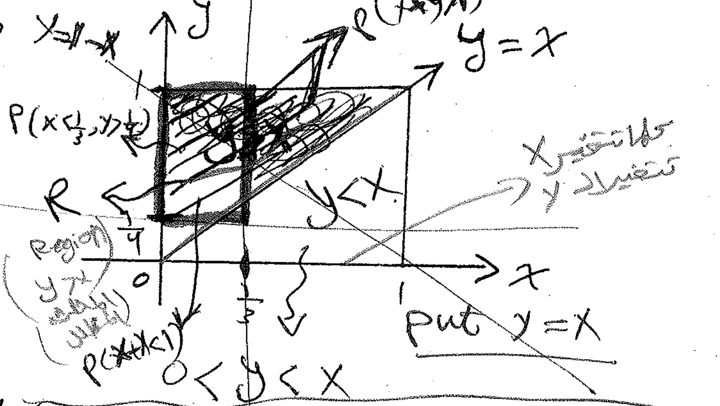
Find $f_1(x)$, $f_2(y)$, $P(X>\frac{1}{2})$, $P(y<\frac{1}{2})$, $P(X<\frac{1}{3})$,

Find
- ① $P(X>\frac{1}{2})$
- ② $P(Y<\frac{1}{2})$
- ③ $P(X>\frac{1}{2}, Y<\frac{1}{2})$

Sol. $P(X>1/2, Y<1/2) = \int\int_{1/2}^{y_2} f(x,y)\,dy\,dx = $ ——

$R = \{(x,y);\ 0<x<y<1\}$, Consider $y = x$, then:

$$R = \begin{cases} 0<x<y & ;\ x<y<1 \\ 0<x<1 & ;\ 0<y<1 \end{cases}$$



put y=X

0<y<x

$$f_1(x) = \int_x^1 2\,dy \quad \text{for } 0<x<1$$

$$= 2y\Big|_x^1 = 2[1-x]$$

$$f_1(x) = \begin{cases} 2(1-x) & \text{for } 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

*(H.W. Prove that f_1 & f_2 are pr. fun.)

$$f_2(y) = \int_0^y 2\,dx \quad \text{for } 0<y<1$$

$$= 2x\Big|_0^y = 2y$$

$$f_2(y) = \begin{cases} 2y & \text{for } 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

2010/2/10

Find $P(X+Y<1) = ?$ → $X+y=1$, $y=1-x$

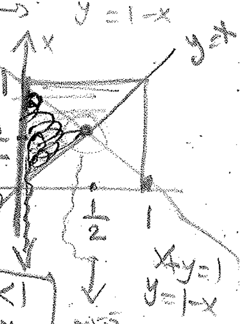
تقاطع المستقيمان :

$y=x$, $y=1-x$

$x=1-x$ → $\boxed{X=\frac{1}{2}}$

$(x,y) = (\frac{1}{2},\frac{1}{2})$

$$\therefore P(X+Y<1) = \int_0^{\frac{1}{2}}\int_0^{1-x} f(x,y)\,dy\,dx$$



$x+y<1$
$y<1-x$

$$P(X>\tfrac{1}{3})=\int_{\frac{1}{3}}^{1} f_1(x)\,dx$$

$$=\int_{\frac{1}{3}}^{1} 2(1-x)\,dx=\frac{8}{18}$$

$$P(Y<\tfrac{1}{2})=\int_0^{\frac{1}{2}} 2y\,dy=\cdots=\frac{1}{4}$$

$$P(X<\tfrac{1}{3},Y>\tfrac{1}{4})=\underbrace{\int_{\frac{1}{4}}^{1}\underbrace{\int_0^{\frac{1}{3}} 2\,dx}_{x}\,dy}_{y}=\cdots=\frac{1}{2}$$

* لو لم يطلب f_1 و f_2 ويطلب $P(X<\frac{1}{4})$ $P(Y>\frac{1}{4})$ نحسب ال f ضمنياً implicit

Find $P(X+Y\geq 1)=?$

(II) Joint Probability Mass Function (J.P.m.f.)

<u>Def.</u> Let X and Y be two d.r.v.. A function $f(x,y)$ is a joint p.m.f. of X and Y iff

$$\boxed{f(x,y)=P(X=x,\,Y=y)\quad\forall (x,y)\in R}$$

X is d.r.v.
$f(x)$ is p.m.f.
$f(x)=P(X=x)$
$f(2)=P(X=2)$

and satisfies two conditions:

(a) $f(x,y)\geq 0\quad\forall (x,y)\in R$

(b) $\sum\limits_{\forall y}\sum\limits_{\forall x} f(x,y)=1$ $\forall R$

Also,

$f_1(x)=\sum\limits_{\forall y} f(x,y)$, limit of sumation followes limit of y.

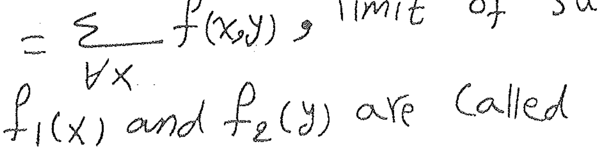
$f_2(x)=\sum\limits_{\forall x} f(x,y)$, limit of Sumation followes limit of X.

where $f_1(x)$ and $f_2(y)$ are called Marginal P.m.f. of X and Y respectively.

بالتعاقب

Example③: A box has (3) balls (1), (2), (3). Choose two balls one by one without replac.; find $f(x,y)$.

<u>Sol.</u>



$$S=\left\{\begin{matrix} \underset{(x_1,y_1)}{(1,2)} & \underset{(x_2,y_1)}{(2,1)} & \underset{(x_3,y_1)}{(3,1)} \\ \underset{(x_1,y_3)}{(1,3)} & \underset{(x_2,y_3)}{(2,3)} & \underset{(x_3,y_2)}{(3,2)} \end{matrix}\right\}\ ;\ \text{S has "6" elts.}$$

OR A box has (3) balls which are numbered 1,2,3.
مرقمة

كم حالة لسحب 2 من 3 بدون ارجاع

$P^3_2 = \frac{3!}{1!} = 6 = n(S)$ (ارطع)

سؤال مقترح 2

نفس السؤال مع تغيير طريقة السحب (1-1) with rep

$n(S) = (3)^2 = 9$

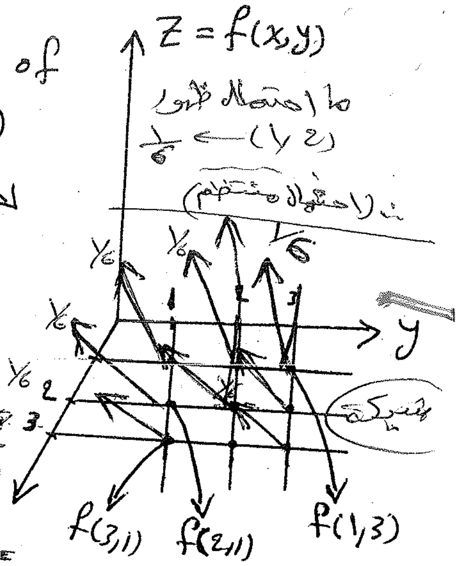
$X = 1^{st}$ chosen ball ; $x = 1, 2, 3$

تمثل الاختيار الاول

$Y = 2^{nd}$ chosen ball ; $y = 1, 2, 3$ $\} \Rightarrow X \neq Y$

تمثل الاختيار الثاني

Graph of $f(x,y)$



$P(X=2, Y=1) = \frac{1}{6} = f(2,1)$

$P(X=3, Y=2) = \frac{1}{6} = f(3,2)$

لا يوجد هذه الحالة ex $f(1,1) = 0$

$$\boxed{P(X \geq 2, Y \leq 2) = \sum_{x=2,3} \sum_{y=1,2} f(x,y)}$$

حل الاكزام = 1/6

$f(x,y) = P(X = x; Y = y) = \frac{1}{6}$

قيمة الدالة ثابتة ← uniform(K)

$$f(x,y) = \begin{cases} \frac{1}{6} & \text{for } x = 1,2,3 \text{ ; } y = 1,2,3 \text{ \& } x \neq y \\ 0 & \text{o.w.} \end{cases}$$

$f(X=2) = f_1(2)$

الرسم الهندسي بالبعد (3) (3-dim.)

$\boxed{P(X=1) = f_1(1)}$

Find: $f_1(x) = \sum_{\forall y} f(x,y) = \begin{cases} \frac{3}{6} & , x = 1,2,3 \end{cases}$ & $f_2(y) = \sum_{\forall x} f(x,y) = \sum_{x=1}^{3} \frac{1}{6} = \frac{3}{6} \begin{cases} \frac{3}{6} & , y = 1,2,3 \\ 0 & \text{o.w.} \end{cases}$

Example ④ Given a J.p.m.f. $f(x,y)$ (لم يحل)

$$f(x,y) = \begin{cases} \frac{1}{21}(x+y) & \text{for } x = 1,2,3 \\ & \quad\; y = 1,2 \\ 0 & \text{o.w.} \end{cases}$$

$P(X \geq 2) = \sum_{x=2}^{3} f_1(x)$
$= f_1(2) + f_1(3)$
$= \frac{3}{6} + \frac{3}{6} = 1$

$P(Y \leq 2) = ?$

Find $P(X \geq 2, Y \leq 2)$, $f_1(x)$, $f_2(y)$

sol.

$P(X \geq 2, Y \leq 2) = \sum_{y=1}^{2} \sum_{x=2}^{3} \frac{1}{21}(x+y)$

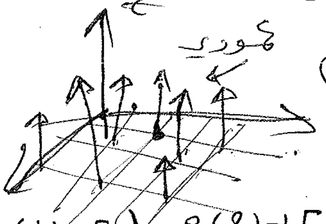
$= \frac{1}{21} \sum_{y=1}^{2} [(2+y) + (3+y)]$

$= \frac{1}{21} \sum_{y=1}^{2} [2y+5] = \frac{1}{21} [(2(1)+5) + 2(2)+5]$

$= \frac{1}{21} [7+9] = \frac{16}{21}$

نفس المثال اعلاه ولكن مع الارجاع

$n(S) = 3^2 = 9$



$f(x,y) = \frac{1}{9}$

$P(X \geq 2, Y \leq 2)$

$= \sum_{x=2}^{3} \sum_{y=1}^{2} \frac{1}{6}$

$= \sum \left(\frac{1}{6} + \frac{1}{6} \right)$

$= \left(\frac{2}{6} \right) \left[\frac{1}{6} + \frac{1}{6} \right]$

$= \frac{4}{6}$

$P(X=3, Y=1) = f(3,1) = \frac{1}{21}(3+1) = \frac{4}{21}$

$f_1(x) = \sum_y f(x,y)$

$f_1(x) = \sum_{y=1}^{2} \frac{1}{21}(x+y) = \frac{1}{21}[(x+1) + (x+2)]$

$= \frac{1}{21}[3+2x]$

$$f_1(x) = \begin{cases} \frac{1}{21}[3+2x] & \text{for } x = 1,2,3 \\ 0 & \text{o.w.} \end{cases}$$

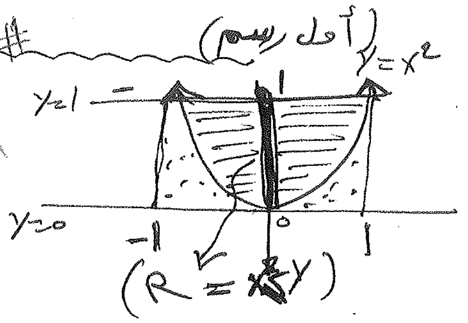
$$f_2(y) = \sum_{x} f(x,y)$$

$$= \sum_{x=1}^{3} \frac{1}{21}(x+y)$$

$$= \frac{1}{21}\left[(1+y)+(2+y)+(3+y)\right]$$

$$= \frac{1}{21}(6+3\cancel{y}) = \frac{3}{21}(y+2) = \frac{1}{7}(y+2)$$

$$f_2(y) = \begin{cases} \frac{1}{7}(y+2) & \text{for } y=1,2 \\ 0 & \text{o.w.} \end{cases}$$



Example ⑤ Given a J.p.d.f.

$$f(x,y) = \begin{cases} cx^2y & \text{for } \boxed{x^2\leq y\leq 1} \\ 0 & \text{o.w.} \end{cases}$$ Region

ⓐ Find the value of c ⓑ Find $P(X\leq Y)$

Sol.

$$R = \{(x,y) ; 0\leq x^2 \leq y\leq 1\} = \left\{\begin{matrix} 0\leq x^2\leq y, & x^2\leq y\leq 1 \\ 0\leq x^2\leq 1, & 0\leq y\leq 1 \end{matrix}\right\}$$

تثبيت R من البداية لحساب الحدود والتكامل

suppose $y = x^2 \Rightarrow x = \mp\sqrt{y}$

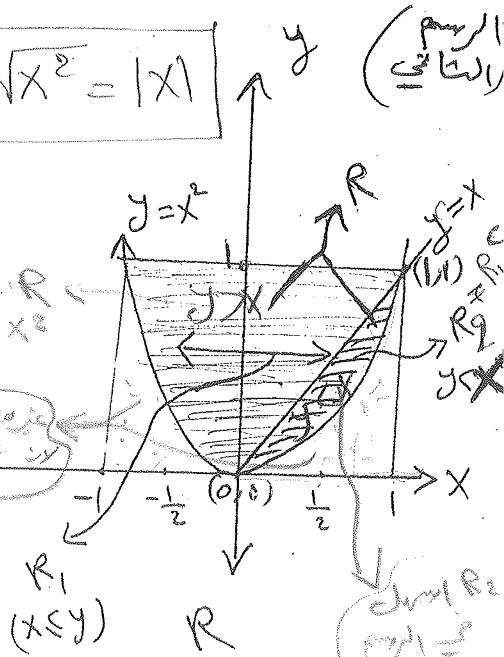
$0\leq x^2\leq y$	$0\leq x^2\leq y<1$
$\lvert x\rvert \leq \sqrt{y}$	$0\leq y\leq 1$
$-\sqrt{y}\leq x\leq\sqrt{y}$	
$x^2\leq 1$	$y=x^2$
$\lvert x\rvert\leq 1$	$x=\mp\sqrt{y}$
$-1\leq x\leq 1$	

$$\boxed{\text{by } \sqrt{x^2}=\lvert x\rvert}$$

R: لان $y > x^2$ (ومن الصعب التكامل بها)

$$R = \left\{\begin{matrix} -\sqrt{y}\leq x\leq\sqrt{y} ; & x^2\leq y\leq 1 \\ -1\leq x\leq 1 ; & 0\leq y\leq 1 \end{matrix}\right\}$$

غير مطلوبة لان $y<x^2$



x	$y=x^2$
0	0
$\mp\frac{1}{2}$	$\frac{1}{4}$
∓ 1	1

Consider $y = x^2$

$y=x^2$, $y=x$: لايجاد نقاط التقاطع

$x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$x=0, x=1$

$\Longrightarrow (0,0), (1,1)$

R

$0\leq x^2\leq y\leq 1$

R2 اسهل في الرسم والتكامل

لذلك

$P(Y>X) = 1 - P(X\geq Y)$

ⓐ By Cond.(2)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$$

$$\int_{-1}^{1}\int_{x^2}^{1} c\,x^2 y\,dy\,dx = 1$$

$$1 = \frac{c}{2}\int_{-1}^{1} x^2 y^2 \Big|_{x^2}^{1} dx = \frac{c}{2}\int_{-1}^{1} x^2(1-x^4)\,dx$$

$$1 = \frac{c}{2}\int_{-1}^{1}[x^2 - x^6]\,dx = \frac{c}{2}\left[\frac{x^3}{3} - \frac{x^7}{7}\right]_{-1}^{1}$$

$$1 = \frac{c}{2}\left[(\frac{1}{3} - \frac{1}{7}) - (\frac{-1}{3} - \frac{-1}{7})\right]$$

$$1 = \frac{c}{2}\left[(\frac{1}{3} - \frac{1}{7}) + (\frac{1}{3} - \frac{1}{7})\right]$$

$$1 = \frac{c}{2}(\frac{2}{3} - \frac{2}{7}) = c(\frac{1}{3} - \frac{1}{7}) = c(\frac{7-3}{21}) = \frac{4}{21}c$$

$$\Rightarrow \boxed{c = \frac{21}{4}}$$

$$\therefore f(x,y) = \begin{cases} \frac{21}{4}x^2y & \text{for } x^2 \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

(الرسم الثاني)

ⓑ To find $P(X \leq Y)$:

$$R = \begin{Bmatrix} -\sqrt{y} \leq x \leq \sqrt{y} \,, & x^2 \leq y \leq 1 \\ -1 \leq x \leq 1 \,, & 0 \leq y \leq 1 \end{Bmatrix}$$

$P(X \leq Y) \doteq$ volum of $f(x,y)$ over R_1

<u>but</u> $P(Y \geq X) = 1 - P(Y < X)$

$P(Y < X) =$ Volum of $f(x,y)$ over R_1^c

$$P(Y<X) = \int_0^1\int_{x^2}^{x} \frac{21}{4}x^2y\,dy\,dx$$

(x=0 q x=1) لي

$$= \frac{21}{8}\int_0^1 x^2y^2\Big|_{x^2}^{x} dx$$

$$= \frac{21}{8}\int_0^1 x^2[x^2 - x^4]\,dx = \frac{21}{8}\int_0^1[x^4 - x^6]\,dx$$

$$= \frac{21}{8}\left[\frac{x^5}{5} - \frac{x^7}{7}\right]\Big|_0^1$$

$$= \frac{21}{8}\left[\left(\frac{1}{5} - \frac{1}{7}\right) - (0-0)\right]$$

$$= \frac{21}{8}\left[\frac{7-5}{35}\right] = \frac{\cancel{21}^{3}}{\cancel{8}_{4}} \cdot \frac{\cancel{2}}{\cancel{35}_{5}} = \boxed{\frac{3}{20}}$$

$$\therefore P(X \leq Y) = 1 - P(X > Y) \equiv \int_0^1 (x^2 - x)\,dx \quad (\text{المساحة بين منحنيين})$$

$$= 1 - P(Y < X)$$

$$= 1 - \frac{3}{20} = \boxed{\frac{17}{20}}$$

ⓒ To find $f_1(x)$ and $f_2(y)$

$$R = \left\{ \begin{array}{ll} -\sqrt{y} \leq x \leq \sqrt{y} & , \; x^2 \leq y \leq 1 \\ -1 \leq x \leq 1 & , \; 0 \leq y \leq 1 \end{array} \right\}$$

$$f_1(x) = \int_{x^2}^{1} \frac{21}{4} x^2 y \, dy \quad \text{for } -1 \leq x \leq 1$$

$$= \frac{21}{4} x^2 \frac{y^2}{2}\Big|_{x^2}^{1} = \frac{21}{8} x^2 (1 - x^4)$$

$$f_1(x) = \begin{cases} \frac{21}{8} x^2 (1 - x^4) & \text{for } -1 \leq x \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f_2(y) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{21}{4} x^2 y \, dx \quad \text{for } 0 \leq y \leq 1$$

$$= \frac{21}{4} y \frac{x^3}{3}\Big|_{-\sqrt{y}}^{\sqrt{y}} = \frac{7}{4} y \left[y^{\frac{3}{2}} + y^{\frac{3}{2}}\right] = \frac{7}{2} y^{\frac{5}{2}}$$

$$f_2(y) = \begin{cases} \frac{7}{2} y^{\frac{5}{2}} & 0 \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

$\boxed{P(X+Y<1) = ?}$ (H.w.)

Stochastic Independence

الاستقلال التصادفي

Def: X and Y are stochastically independence iff :

$$\boxed{f_1(x) f_2(y) = f(x,y) \quad \forall (x,y)}$$

مقارنة مع

A & B are indep. events:
P(A) P(B) = P(AB)

شرح بـ($\frac{1}{2}$ ساعة)

الوقت للتدريس

Denoted by (s. indep.)

Example : Given a J.p.m.f

$x \backslash y$	1	2	3	4	$f_1(x)$ بالالوان
1	$f(1,1) = .1$	$f(1,2) = 0$	$f(1,3) = .1$	$f(1,4) = 0$	$f_1(1) = .1 + .1 = .2$
2	$f(2,1) = .3$	$f(2,2) = 0$	$f(2,3) = .1$	$f(2,4) = .2$	$f_1(2) = .3 + .1 + .2 = .6$
3	$f(3,1) = 0$	$f(3,2) = .2$	$f(3,3) = 0$	$f(3,4) = 0$	$f_1(3) = .2$
$f_2(y)$ بالالوان	$f_2(1) = .4$	$f_2(2) = .2$	$f_2(3) = .2$	$f_2(4) = .2$	$\sum_{x=1}^{3} f_1(x) = \sum_{y=1}^{4} f_2(y) = 1$

$f(x,y) \equiv P(X=x, Y=y)$

a. T.P. $f_1(X) \cdot f_2(Y) = f(X,Y) \quad \forall (x,y)$

Note $\sum_x f_1(x) = 1$, $f_1(x)$ is p.m.f. of X

$\sum_y f_2(y) = 1$, $f_2(y)$ is p.m.f. of Y

Prove that

$f_1(1) \cdot f_2(1) = (.2)(.4) = .08$

$f(1,1) = .1$

$\therefore f_1(1) f_2(1) \neq f(1,1)$ & $f_1(2) f_2(2) \neq f(2,2)$

$\therefore$ X and Y are not s. indep.

b. Find $P(X \geq 2, Y \leq 3)$

$$P(X \geq 2, Y \leq 3) = \sum_{y=1}^{3} \sum_{x=2}^{3} f(x,y) = \sum_{y=1}^{3} [f(2,y) + f(3,y)]$$

$$= [f(2,1) + f(2,2) + f(2,3)] + [f(3,1) + f(3,2) + f(3,3)]$$

$$= .03 + 0 + .1 + 0 + .2 + 0$$

$$= .6$$

c. Find $P(X \geq 2) = \sum_{X=2}^{3} f_1(x)$

$$= f_1(2) + f_1(3)$$

$$= 0.6 + 0.2 = 0.8$$

& $P(Y \leq 3) = \sum_{y=1}^{3} f_2(y)$

$$= f_2(1) + f_2(2) + f_2(3)$$

$$= .4 + .2 + .2 = .8$$

Uniform Distribution of two R.V.'s

الوقت
شرح بـ 3 ساعة
التوزيع

Def. If a c.r.v. X and y havea uniform distribution on a region R, then the joint p.d.f. of X and y is:

$$\boxed{f(x,y) = \frac{1}{\text{Area}(R)} \quad \text{for } (x,y) \in R}$$

(مساحة المنطقة)

Example ⑥ X and y have a uniform dist. on a triangle Δ with vertices (0,0), (0,1) and (1,1). Find f(x,y).

Sol.

R is a triangle Δ

$y \geq x \quad x \geq 0$

$y \leq 1 \quad \Rightarrow 0 \leq x \leq y \leq 1$

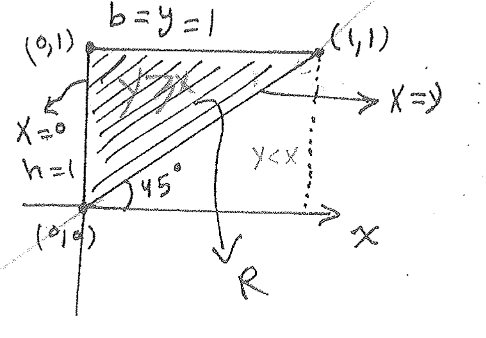
dependent

$R = \{(x,y);\ 0 \leq x \leq y \leq 1\}$

$\text{Area}(\Delta) = \frac{1}{2} \cdot h \cdot b$

$= \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$

X, Y are dep. $\Rightarrow f_1(x) f_2(y) \neq f(x,y)$



$$f(x,y) = \frac{1}{\text{Area}(R)} = \frac{1}{\frac{1}{2}} = \begin{cases} 2 & \text{for } 0 \leq x \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

Example ⑦ A point (x,y) is chosen from a region R which is bounded by a curve $y = x^2$ and a line $y = x$, find $f(x,y)$.

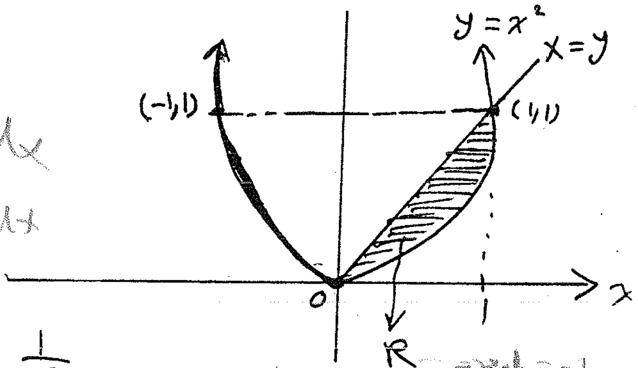
Sol. Sketch $y = x$ and $y = x^2$

A point $(x,y) \in R$ must be a bove the curve $y = x^2$ and under the line $y = x$, then $x^2 \leq y \leq x$. ($x^2 = x$ نقاط التقاطع $(0,0)$, $(1,1)$ $\rightarrow x=0, x=1$ و $y=0, y=1$)

Also that point must be between $0 \leq x \leq 1$, $0 \leq y \leq 1$

$0 \leq x^2 \leq y \leq x \leq 1$

$R = \{(x,y);\ 0 \leq x^2 \leq y \leq x \leq 1\}$



$$\text{Area}(R) = \int_0^1 \int_{x^2}^{x} dy\, dx = \int_0^1 |x - x^2|\, dx = \int_0^1 x - x^2\, dx$$

$x^2 \geq x$ لان [illegible]

$$= \int_0^1 |(x - x^2)|\, dx$$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$f(x,y) = \frac{1}{1/6} = \begin{cases} 6 & \text{for } 0 \leq x^2 \leq y \leq x \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

[illegible]

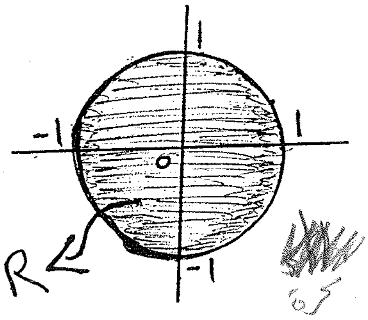
$$\int_0^1 |x^2 - x|\, dx = \int_0^1 (x - x^2)\, dx$$

Example ⑧ A point (x,y) is chosen from a region R where

$R = \{(x,y) ;\ x^2 + y^2 \leq 1\}$

a. find $f(x,y)$

b. find $f_1(x)$ & $f_2(y)$



كرة

Spheir

النطق

Sfery

Sol.

a. Let $x^2 + y^2 = 1 \rightarrow r = 1$ نصف القطر

$\text{Area}(R)| = \pi r^2 = \pi$

radius $\leftarrow r = 1$ نصف القطر

$$\text{then } f(x,y) = \begin{cases} \frac{1}{\pi} & \text{for } x^2 + y^2 \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

b. We must find $f_1(x)$ & $f_2(y)$:

$$f_1(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy \quad \text{for } -1 \le x \le 1$$

$$= \frac{1}{\pi} y \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} = \frac{1}{\pi}\left[\sqrt{1-x^2} + \sqrt{1-x^2}\right] = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} \\ 0 \end{cases}$$

for $-\sqrt{1-y^2} \le x \le$
o.w.
o.w. $\sqrt{1-y^2}$

$$f_2(y) = \int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} \frac{1}{\pi} dx \quad \text{for } -1 \le y \le 1$$

$$= \frac{1}{\pi} x \Big|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} = \begin{cases} \frac{2}{\pi}\sqrt{1-y^2} \\ 0 \end{cases}$$

for $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$
o.w.

$$f_1(x) \cdot f_2(y) = \frac{2}{\pi} \cdot \frac{2}{\pi} \sqrt{(1-y^2)(1-x^2)} = \frac{4}{\pi^2}\sqrt{(1-y^2)(1-x^2)}$$

$$f(x,y) \ne f_1(x) f_2(y) \qquad (\text{since } f(x,y) = \frac{1}{\pi})$$

---

* Conditional Function and Conditional probability

Def. Let X and Y be two r.v.'s , $f(y|x)$ denoted the conditional p.d.f or p.m.f. of y given $X = x$.
$f(x|y)$ denoted the conditional p.d.f. or p.m.f. of x given $Y = y$.

where

$f(y\|x) = \dfrac{f(x,y)}{f_1(x)}$	;	$f_1(x) \ne 0$
$f(x\|y) = \dfrac{f(x,y)}{f_2(y)}$	;	$f_2(y) \ne 0$

دالة كثافة [illegible] y

دالة كثافة [illegible]

[illegible]

$$\left(P(A|B) = \frac{P(AB)}{P(B)} \right)$$

*Note :

$$P(a < x < b \,/\, y = c) = \int_a^b f(x|y=c)\,dx$$

$$= \sum_{x=a}^{b} f(x|y=c)$$

$$P(c<y<d \mid X=a) = \int_c^d f(y \mid x=a)\,dy$$
$$= \sum_{y=c}^{d} f(y \mid x=a)$$

Example ⑧ Given a J.p.d.f.

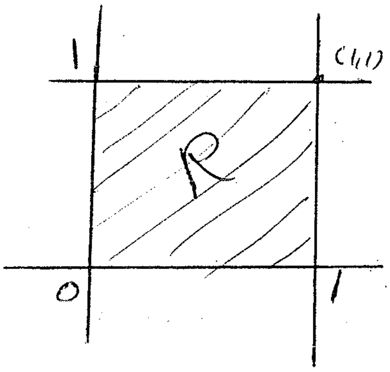
$$f(x,y) = \begin{cases} 8xy & \text{for } 0<x<y<1 \\ 0 & \text{o.w.} \end{cases}$$

(1) find $f(x \mid y)$ & $f(y \mid x)$

(2) find $P(x>\frac{1}{4} \mid y=\frac{1}{2})$, $P(y<\frac{1}{4} \mid x=\frac{1}{2})$.

Sol. $R = \{(x,y) ; 0<x<y<1\}$

$$R = \begin{Bmatrix} 0<x<y & x<y<1 \\ 0<x<1 & 0<y<1 \end{Bmatrix}$$



(1) we must find $f_1(x)$ and $f_2(y)$

$$f_1(x) = \int_x^1 8xy\,dy = 8x \frac{y^2}{2}\Big|_x^1 = 4x(1-x^2)$$

$$= 4x(1-x^2) \text{ for } 0<x<1$$

$$f_2(y) = \int_0^y 8xy\,dx = \frac{8}{2} y x^2 = 4y^3 \text{ for } 0<y<1$$

$$f(x \mid y) = \frac{f(x,y)}{f_2(y)} = \frac{8xy}{4y^3} = \begin{cases} \frac{2x}{y^2} & \text{for } \begin{cases} 0<x<y \\ 0<y<1 \end{cases} \\ 0 & \text{o.w.} \end{cases}$$

$$f(y \mid x) = \frac{f(x,y)}{f_1(x)} = \frac{8xy}{4x(1-x^2)} = \frac{2y}{1-x^2}$$

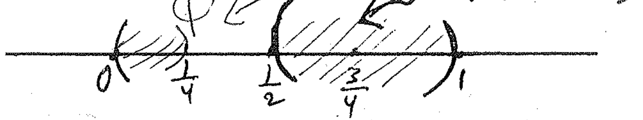
$$\therefore f(y \mid x) = \begin{cases} \frac{2y}{1-x^2} & \text{for } x<y<1 \ \& \ 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

$x \neq 1$ لأن مقام $f(y|x)$ = صفر

[illegible]

$$(2)\ P(y<\frac{1}{4} \mid x=\frac{1}{2}) = \int^{\frac{1}{4}} f(y \mid x=\tfrac{1}{2})\,dy = \int^{\frac{1}{4}} 0\,dy = 0$$

$$x<y<1, \quad x=\frac{1}{2} \implies \frac{1}{2}<y<1$$



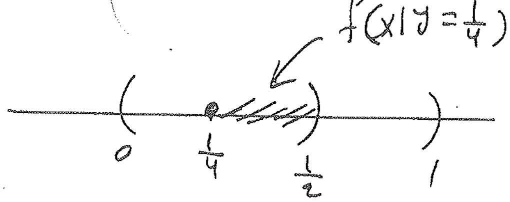
⑭

$f(y|x=\frac{1}{2})$ [illegible]

$$P(X > \tfrac{1}{4} \mid y=\tfrac{1}{2}) = \int_0^y f(x \mid y=\tfrac{1}{2})\,dx$$ ← (نعوض اولاً Y بـ $f(x|y=\frac{1}{2})$ ثم نأخذ التكامل)

$\boxed{\begin{array}{l} P(X > \frac{1}{4} \mid y=\frac{3}{4}) \\ 0<x<y,\ y=\frac{3}{4} \\ \therefore 0<x<\frac{3}{4} \Rightarrow \frac{1}{4}<x<\frac{3}{4} \end{array}}$

$$= \int_{\frac{1}{4}}^{y} \frac{2x}{y}\,dx$$



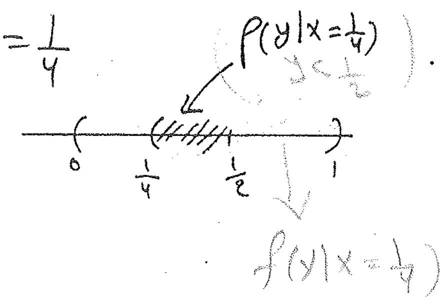
$$0<x<y,\ y=\tfrac{1}{2} \Rightarrow 0<x<\tfrac{1}{2}$$

$$= \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{2x}{(\frac{1}{2})^2}\,dx = 4 \cdot 2 \cdot \frac{x^2}{2}\Big|_{1/4}^{1/2}$$

$$= 4\left[\tfrac{1}{4} - \tfrac{1}{16}\right] = 4 \cdot \left(\tfrac{3}{16}\right) = \tfrac{3}{4}$$

$$P(y<\tfrac{1}{2} \mid X=\tfrac{1}{4}) = \ ?$$

$$x<y<1,\ x=\tfrac{1}{4}$$
$$\tfrac{1}{4}<y<1$$



$$P(y<\tfrac{1}{2} / x=\tfrac{1}{4}) = \int_x^{\frac{1}{2}} f(y/x=\tfrac{1}{4})\,dy$$

$$= \int_x^{\frac{1}{2}} \frac{2y}{1-x^2}\,dy = \int_{\frac{1}{4}}^{1/2} \frac{2y}{1-(\frac{1}{4})^2}\,dy$$

$$= \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{2y}{15/16}\,dy = \frac{16}{15} \cdot \cancel{2} \cdot \frac{y^2}{\cancel{2}}\Big|_{1/4}^{1/2}$$

← **ملاحظة**

نعوض بـ $f(y|x=\frac{1}{4})$

ونعوض بـ $X=\frac{1}{4}$

ثم نأخذ التكامل

$$= \frac{16}{15}\left(\tfrac{1}{4} - \tfrac{1}{16}\right) = \frac{\cancel{16}}{\cancel{15}_{5}} \cdot \frac{\cancel{3}}{\cancel{16}} = \frac{1}{5}$$

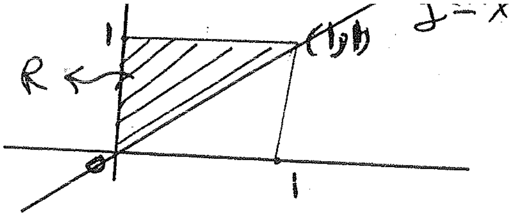
Example ⑥ If $X \sim unif(0,1)$ and

$$f(y|x) = \begin{cases} \frac{1}{1-x} & \text{for } x<y<1,\ 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

(a) find $f(x,y)$?

$\because X \sim unif(0,1)$

$$f_1(x) = \frac{1}{1-0} = 1 \Longrightarrow f_1(x) = \begin{cases} 1 & \text{for } 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$



$$f(y|x) = \frac{f(x,y)}{f_1(x)}$$

$$f(x,y) = f_1(x)\cdot f(y|x)$$

$$= (1)\left(\frac{1}{1-x}\right) = \begin{cases} \frac{1}{1-x} & \text{for } x<y<1,\ 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

$$\therefore \boxed{f(y|x) = f(x,y)}$$

b. Find $P(X>\frac{1}{2} \mid y=\frac{3}{4}) = \int_0^y f(x|y=\frac{3}{4})\,dx$

$$f(x|y) = \frac{f(x,y)}{f_2(y)}$$

$$f_2(y) = \int_0^y f(x,y)\,dx = \int_0^y \frac{1}{1-x}\,dx = -Ln(1-x)\Big|_0^y$$

$$= -[Ln(1-y) - \underbrace{\cancel{Ln}(1-0)}_{\text{Zero}}]$$

$$= -Ln(1-y) \quad \text{for } 0<y<1$$

$$\therefore f_2(y) = -Ln(1-y) \quad \text{for} \quad 0<y<1$$

$$\therefore f(x|y) = \frac{\frac{1}{1-x}}{-Ln(1-y)}$$

$$f(x|y) = \begin{cases} \frac{1}{-Ln(1-y)}\cdot\frac{1}{1-x} & \text{for } 0<x<y \ \& \ 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

$$P(X>\tfrac{1}{2} \mid y=\tfrac{3}{4}) = \int_{1/2}^{y} f(x|y=\tfrac{3}{4})\,dx$$

$(0<X<y,\ y=\frac{3}{4})$

$\Rightarrow 0<X<\frac{3}{4}$

بشكل عام

$X>\frac{1}{2}$ ثم

$\Rightarrow \frac{1}{2}<X<\frac{3}{4}$

$$= \int_{1/2}^{y} \frac{-1}{Ln(1-y)}\cdot\frac{1}{1-x}\,dx = \int_{1/2}^{3/4} \frac{-1}{Ln(1-\frac{3}{4})}\cdot\left(\frac{1}{1-x}\right)dx$$

$$= \frac{-1}{Ln1 - Ln4}\cdot\Big[-Ln(1-x)\Big]_{1/2}^{3/4}$$

$$= \frac{+1}{-Ln4}\left[Ln(1-\tfrac{3}{4}) - Ln(1-\tfrac{1}{2})\right]$$

$$= \frac{-1}{Ln4}\left[Ln(\tfrac{1}{4}) - Ln(\tfrac{1}{2})\right]$$

$$= \frac{-1}{Ln2^2}\left[(\underbrace{\cancel{Ln}1}_{\text{zero}} - Ln4) - (\underbrace{\cancel{Ln}1}_{\text{zero}} - Ln(2))\right]$$

$$= \frac{-1}{2\ln 2}(-2\ln 2+\ln 2) = \frac{-1}{2\cancel{\ln 2}}(-\cancel{\ln 2}) = \frac{1}{2}$$

H.w Find $P(Y<\frac{3}{4} \mid X=\frac{1}{2})$

التوفيق والنجاح

Expectation of two Random. Variables

Review

① **Def.** $E(XY)$ is the expectation or the mean of both X and Y such that:

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\,f(x,y)\,dx\,dy \quad \text{if } X \text{ and } Y \text{ are c.r.v.s}$$

$$= \sum_{y}\sum_{x} xy\,f(x,y) \quad \text{if } X \text{ and } Y \text{ are d.r.v.s}$$

Note: ① $E(X) \equiv$ mean of $X = \mu_x$

s.t. $E(X) = \int_{-\infty}^{\infty} x\,f_1(x)\,dx$ if X is c.r.v.

$= \sum_{x} x\,f_1(x)$ if X is d.r.v.

② $E(Y) \equiv$ mean of $Y = \mu_y$

s.t. $E(y) = \int_{-\infty}^{\infty} y\,f_2(y)\,dy$ if Y is c.r.v.

$= \sum_{y} y\,f_2(y)$ if Y is d.r.v.

③ (a) $V(X) = E(X^2) - [E(X)]^2$

(b) $V(Y) = E(Y^2) - [E(Y)]^2$

④ Covariance of both X & Y

We use the symbole $Cov(X,y)$ (it's mean Covariance), s.t

$$Cov(X,y) = E\{[X-E(X)][Y-E(y)]\}.$$

Theorem (1) If X and Y are random variables, then:

$$Cov(x,y) = E(XY) - E(X)E(Y)$$

Proof

$$Cov(x,y) = E\{[X-E(x)][Y-E(Y)]\}$$
$$= E\{XY - YE(X) - XE(Y) + E(X)\cdot E(Y)\}$$
$$= E(XY) - E(X)E(Y) - E(X)E(Y) + E(Y)E(X)$$
$$= E(XY) - E(X)E(Y)$$

Theorem (2) If X and Y are independent, then

$$E(XY) = E(X)E(Y)$$

or $Cov(X,Y) = 0$

Proof Case ① If X and y are c.r.v.s

$\because$ X & Y are indep. $\Rightarrow f_1(x)f_2(y) = f(x,y)$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(xy)\, f(x,y)\, dx\, dy$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_1(x) f_2(y)\, dx\, dy$$
$$= \int_{-\infty}^{\infty} x f_1(x)dx \cdot \int_{-\infty}^{\infty} y f_2(y) dy \quad \text{by Comm. Property}$$
$$= E(X)\cdot E(Y)$$

Case ② By the same method.

Note By Th. 2 above, if X & Y are indep., then

$$Cov(X,y) = 0$$

Example: Given a J.p.d.f.

$$f(x,y) = \begin{cases} 2 & \text{for } 0<y<x<1 \\ 0 & \text{o.w.} \end{cases}$$

هناك ٣ طرق للاجابة

Are X & Y indep.? (If $Cov(X,y) = 0$ Then X & Y are indep.)

① By using $Cov(X,y)$

or ② By using $E(X,y)$

③ $f_1(x)f_2(y) = f(x,y)$

$$E(xy) = \int_0^1 \int_0^x xy\,(2)\,dy\,dx$$

$$= \int_0^1 \cancel{2}x \frac{y^2}{\cancel{2}} \Big]_0^x dx$$

$$= \int_0^1 x \cdot x^2\, dx$$

$$= \frac{x^4}{4} \Big]_0^1$$

$$= \frac{1}{4}. \#$$

Rho

$$\rho_{x,y} = \frac{Cov(x,y)}{\sqrt{V(x) \cdot V(y)}}$$

$$\rho = \frac{\sigma_{x,y}}{\sigma_x\ \sigma_y}$$

$\rho_{x,y}$

Correlation coefficient

$$E(x^2) = \int_0^1 x^2 (2x)\, dx = \frac{2}{4} x^4 \Big]_0^1 = \frac{2}{4} = \frac{1}{2}$$

$$V(x) = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{9-8}{18} = \frac{1}{18}$$

$$E(y^2) = \int_0^1 y^2 \left[2(1-y)\right] dy = 2\left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1$$

$$= 2\left[\frac{1}{3} - \frac{1}{4}\right]$$

$$= 2\left[\frac{4-3}{12}\right]$$

$$= \frac{2 * }{12} = \frac{1}{6}$$

$$V(y) = \frac{1}{6} - \left[\frac{2}{6}\right]^2$$

$$= \frac{1}{6} - \frac{4}{36}$$

$$= \frac{6}{36} - \frac{4}{36}$$

$$= \frac{2}{36} = \boxed{\frac{1}{18}}$$

$$r_{x,y} = \frac{n\sum XY - \sum X \sum Y}{\sqrt{[n\sum X^2 - (\sum X)^2] - [n\sum Y^2 - (\sum Y)^2]}}$$

البيانات

$$R = \left\{ \begin{matrix} 0 < y < x & , & y < x < 1 \\ 0 < y < 1 & , & 0 < x < 1 \end{matrix} \right\}$$

$$E(XY) = \int_0^1 \int_y^1 xy \cdot f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_y^1 xy\,(2)\,dx\,dy = \int_0^1 \int_0^X (XY)\, f(x,y)\,dy\,dx$$

$$= \int_0^1 2y\, \frac{x^2}{2} \Big|_y^1 dy = \int_0^1 y\,(1-y^2)\,dy$$

$$= \int_0^1 [y - y^3] = \frac{y^2}{2} - \frac{y^4}{4} \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$E(X) = \int_0^1 x\, f_1(x)\,dx \quad , \quad E(y) = \int_0^1 y\, f_2(y)\,dy$$

$$f_1(x) = \int_0^x 2\,dy = \int_0^x 2\,dy = 2y\Big|_0^x = 2X \Rightarrow f_1(x) = \begin{cases} 2x & \text{for } 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f_2(y) = \int_y^1 f(x,y)\,dx = \int_y^1 2\,dx = 2x\Big|_y^1 = 2[1-y] \Rightarrow f_2(y) = \begin{cases} 2(1-y) & \text{for } 0 \leq y < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$E(X) = \int_0^1 x\, f_1(x)\,dx = \int_0^1 x\,(2x)\,dx = \int_0^1 2x^2\,dx = \frac{2}{3} X^3 \Big|_0^1 = \frac{2}{3}$$

$$E(y) = \int_0^1 y\, f_2(y)\,dy = \int_0^1 2y\,(1-y)\,dy = \int_0^1 (2y - 2y^2)\,dy = y^2 - \frac{2}{3} y^3 \Big|_0^1$$

$$= 1 - \frac{2}{3} = \frac{1}{3}$$

$$\therefore \operatorname{cov}(X,y) = E(Xy) - E(X)\,E(y)$$

$$= \frac{1}{4} - \left(\frac{2}{3}\right) \cdot \left(\frac{1}{3}\right) = \frac{9-8}{36} = \frac{1}{36}$$

$\operatorname{cov}(X,y) \neq 0 \Longrightarrow$ X and y are dependent.

~~$\forall$ $\operatorname{cov}(X,y) \geq 0$ all ways~~

(تصحيح)

$$\left. \begin{matrix} V(X) = 1/18 \\ V(y) = 1/18 \end{matrix} \right] \rho_{X,y} = \frac{\operatorname{cov}(X,y)}{\sqrt{V(X) \cdot V(y)}} = \frac{\frac{1}{36}}{\sqrt{\frac{1}{18} \cdot \frac{1}{18}}} = \frac{\frac{1}{\cancel{36}\,2}}{\frac{1}{\cancel{18}\,1}} = \frac{1}{2}$$

معامل الارتباط Correlation Coefficient

Def. when X and y are dependent and we want to know how much X depends on y we use the correlation to measure it, denoted by "$\rho_{x,y}$" read (Rho) where:

لا وحدات

$$\boxed{\rho_{x,y} = \frac{Cov(x,y)}{\sqrt{V(x)V(y)}} = \frac{E(xy)-E(x)E(y)}{\sqrt{V(x)\,V(y)}}} = \frac{Cov(x,y)}{\sigma_x\,\sigma_y}$$

درجة العلاقة الخطية بين x و y

$-1 \le \rho_{x,y} \le 1$

وصف علاقة طردية أو عكسية بين x و y للبيانات

$Cov(x,y) \neq 0$ if X & Y are dependent where

$-1 \le \rho \le 1$

H.w. Find $\rho_{x,y}$ from above example. $\longrightarrow Cov(x,y) = \frac{1}{2} \neq 0$

② Conditional Mean and Conditional Variance

① $f(x|y) = \frac{f(x,y)}{f_2(y)}$

$f(y|x) = \frac{f(x,y)}{f_1(x)}$

$E(y|x)$ denote to the conditional mean of y given $X = x$.
$E(x|y)$ denote to the Conditional mean of X given $Y = y$.

where $\underline{E(y|x)} = \int_{\square}^{\square} y\,f(y|x)\,dy$, limit of integral follows, limit of y.
$g(x) = $ func. of X

$\underline{E(x|y)} = \int_{\square}^{\square} x\,f(x|y)\,dx$, limit of integral follows, limit of x.
$g(y) = $ func. of y

Note Since $f(y|x)$ defined for interval of y interm of X, then $E(y|x)$ is a function of X. Also $E(x|y)$ is a function of Y.

$V(y|x)$ denote to the conditional variance of y given $X = x$.
$V(x|y)$ denote to the Conditional Variance of X given $Y = y$.
where

$$V(y|X) = E\{[y-E(y|X)]^2|X\}$$

$$= \int_{\boxed{X}} [y-E(y|X)]^2 f(y|X)\,dy \quad \Big\} \text{ by def. of } E(X|x)$$

$$\cancel{= \sum_{\boxed{X}} [y-E(y|x)]^2] f(y|x)}$$

$$V(X|y) = E\{[X-E(X|y)]^2|y\}$$

$$\cancel{= \sum_{\boxed{X}} [X-E(X|y)]^2 f(X|y)}$$

$$= \int_{\boxed{X}} [X-E(X|y)]^2 f(X|y)\,dx \quad \Big\} \text{ by def. of } E(X|x)$$

Example: Given a J.P.d.f $f(x,y)$

$$f(x,y) = \begin{cases} 2 & \text{for } 0 \leq X+y \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

Find $E(X|y), E(y|X), V(X|y), V(y|X)$

Sol.

$$R = \{(x,y);\ 0 \leq x+y \leq 1\} = \begin{Bmatrix} 0 \leq x \leq 1-y & , & 0 \leq y \leq 1-x \\ 0 \leq x \leq 1 & , & 0 \leq y \leq 1 \end{Bmatrix}$$

$$E(X|y) = \int_X x f(x|y)\,dx$$

$$f(X|y) = \frac{f(x,y)}{f_2(y)}$$

$$f_2(y) = \int_0^{1-y} f(x,y)\,dx = \int_0^{1-y} 2\,dx = 2x\Big|_0^{1-y} = 2(1-y)$$

$$f_2(y) = \begin{cases} 2(1-y) & \text{for } 0 \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f(X|y) = \frac{2}{2(1-y)} = \frac{1}{1-y}$$

$$f(X|y) = \begin{cases} \frac{1}{1-y} & \text{for } 0 \leq X \leq 1-y \\ 0 & \text{o.w.} \end{cases}$$ ~~$0 \leq y$~~ ≤ 1

(هي دالة لـ X وقيمة y معلومة)

$$E(X|y) = \int_0^{1-y} x \cdot \frac{1}{(1-y)}\,dx = \frac{1}{(1-y)} \cdot \frac{x^2}{2}\Big|_0^{1-y} = \frac{(1-y)^2}{2(1-y)} = \left(\frac{1-y}{2}\right) \quad 0 \leq y \leq 1$$ (دالة لـ X)

$$V(X|y) = \int_0^{1-y} [X-E(X|y)]^2 \overset{f(X|y)}{dx} = \int_0^{1-y} [x - \frac{(1-y)}{2}]^2 dx$$

$$= \int_0^{1-y} [x^2 - (1-y)x + \frac{(1-y)^2}{4}]\,dx = \frac{1}{12}(1-y)^2 \text{ for } 0 \leq y \leq 1$$

طريقة ثانية للحل

Find $E(y|X)$ & $V(y|X)$ H.w

or

$$V(X|Y) = E(X^2|Y) - [E(X|Y)]^2$$

$$= \frac{(1-y)^2}{3} - \frac{(1-y)}{2}$$

$$E(X^2|Y) = \int_0^{1-y} x^2 f(X|y)\,dx$$

$$= \frac{x^3}{3(1-y)}\Big]_0^{1-y} = \frac{(1-y)^3}{3(1-y)} = \frac{(1-y)^2}{3}$$

Theorem (3) If X and y are two random variables, then:

a. $E[E(Y|X)] = E(Y)$, b. $E[E(X|Y)] = E(X)$

Proof Case (1): If X and y are c.r.v. with [redacted] p.d.f $f(x)$.

$E(Y|X)$ is a function of X.

suppose that $E(Y|X) = g(X)$ = as a func of X

$$E[E(Y|X)] = E[g(x)] = \int_{-\infty}^{\infty} g(x)\cdot f_1(x)\,dx = \int_{-\infty}^{\infty}[E(Y|X)]f_1(x)\,dx$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} y f(y|x)\,dy\right] f_1(x)\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(y|x) f_1(x)\,dy\,dx$$

$$f(y|x) = \frac{f(x,y)}{f_1(x)} \Rightarrow f(y|x)\cdot f_1(x) = f(x,y)$$

$$\int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_{-\infty}^{\infty} y\, f_2(y)\,dy = E(Y)$$

Case (2): If X and y are d.r.v. with [redacted] p.m.f. $f(x)$.

$E(Y|X)$ is a function of X.

Suppose that $E(Y|X) = g(X)$

$$E(E(Y|X)) = E(g(X)) = \sum_x g(x)\cdot f_1(x) = \sum_x [E(Y|X)]f_1(x)$$

$$= \sum_x\left[\sum_y y f(y|x)\right] f_1(x) = \sum_x\sum_y y\, f(y|x) f_1(x)$$

$$f(y|x) = \frac{f(x,y)}{f_1(x)} \Rightarrow f(y|x)\cdot f_1(x) = f(x,y)$$

$$\sum_y y \sum_x f(x,y) = \sum_y y\, f_2(y) = E(Y)$$

b. By the same method.

Example: Given a J. p.d.f. $f(x,y)$

$$f(x,y) = \begin{cases} 2 & \text{for } 0<x<y<1 \\ 0 & \text{o.w.} \end{cases}$$

Find $E[E(Y|X)]$.

Sol.

$E(E(Y|X)) = E(Y)$ (by Theorem 3)

$E(Y) = \int_{-\infty}^{\infty} y\, f_2(y)\,dy$ [illegible] $\int_0^y f(x,y)\,dx = \int_0^y 2\,dx = 2x\Big|_0^y = 2y$

$$f_2(y) = \begin{cases} 2y & \text{for } 0<y<1, \\ 0 & \text{o.w.} \end{cases}$$

$$\therefore E(Y) = \int_0^1 y(2y)\,dy = 2\int_0^1 y^2\,dy = \frac{2}{3}y^3\Big|_0^1 = \boxed{\frac{2}{3}}$$

Joint Distribution function (J.d.f) of two aR.v.s X and y

Def. The J.d.f of X and y is defined to a function such ③ that for all values of X and y $(-\infty < x < \infty, -\infty < y < \infty)$ then:

$$F(x,y) = P(X \le x, Y \le y)$$

$F(X) = P(X \le x)$ R.V.

① $= \int_{-\infty}^{x} f(t)\,dt$ if X is c.r.v.

② $= \sum_{x_j \le x} f(x_j)$ if X is d.r.v.

Remarks

1. $P(a < X \le b, c < Y \le d) = [F(b,d) - F(a,d)] - [F(b,c) - F(a,c)]$.

نقارن اعلاه للمتغير العشوائي الواحد (X) :

$$\left\{ \begin{array}{l} 1.\ F(X) = P(X \le x) \\ 2.\ P(a < X \le b) = F(b) - F(a) \end{array} \right\}$$

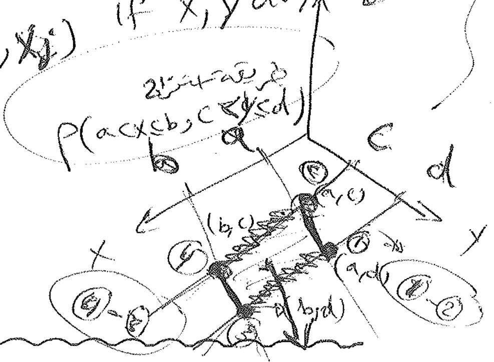
2. (i) $F_1(x) = P(X \le x) = \lim_{y \to \infty} P(X \le x \text{ and } Y \le y) = \lim_{y \to \infty} F(X,Y)$

(ii) $F_2(y) = \lim_{x \to \infty} F(X,Y)$

3. If X and Y have Cont. J.d.f $F(X,Y)$ with J.P.d.f $f(X,Y)$, then for any values of X and Y the J.d.f. is

① $F(X,Y) = \sum_{x_j \le x} \sum_{y_j \le y} f(x_j, y_j)$ if x, y d.r.v

② $F(X,Y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(r,s)\,dr\,ds$ therefore



Note

$$* \; f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \, \partial y}, \quad f(x,y) \text{ diff.}$$

Example: Given a J.d.f $F(x,y)$

$F(x,y) = P(X \le x, Y \le y)$

$$F(x,y) = \begin{cases} 0 & \text{for } x < 0, y < 0 \\ \frac{1}{16} xy(x+y) & \text{for } 0 \le x \le 2, 0 \le y \le 2 \\ 1 & \text{for } x > 2, y > 2 \end{cases}$$

Find $F_1(X)$, $F_2(y)$, $f(X,y)$.

Sol. $F_1(X) = \lim_{y \to \infty} F(x,y) = \lim_{y > 2} F(x,y) = F(x,2)$

$$= \frac{1}{16} X(2)(X+2) = \frac{X(X+2)}{8}$$

$$F_1(x) = \begin{cases} 0 & \text{for} \quad x<0 \\ \frac{x(x+2)}{8} & \text{for} \quad 0 \leq x \leq 2 \\ 1 & \text{for} \quad x>2 \end{cases}$$

$$F_2(y) = \lim_{x \to \infty} F(x,y) = \lim_{x>2} F(x,y) = F(2,y)$$

$$= \frac{1}{16}(2)\,y(2+y) = \frac{y(2+y)}{8}$$

$$F_2(y) = \begin{cases} 0 & \text{for} \quad y<0 \\ \frac{y(2+y)}{8} & \text{for} \quad 0 \leq y \leq 2 \\ 1 & \text{for} \quad y>2 \end{cases}$$

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial F(x,y)}{\partial y}\right]$$

$$F(x,y) = \frac{1}{16}(x^2 y + x y^2)$$

$$f(x,y) = \frac{\partial}{\partial x}\left[\frac{1}{16}(x^2 + 2xy)\right] = \left[\frac{1}{16}(2x+2y)\right]$$

$$= \frac{1}{8}(x+y)$$

$$f(x,y) = \begin{cases} \frac{x+y}{8} & \text{for} \quad 0 \leq y \leq 2, \; 0<x<2 \\ 0 & \text{o.w.} \end{cases}$$

H.W. ① Prove that : $E(X+Y) = E(X) + E(Y)$

② = = : $E(XY) = E(X)E(Y)$

Proof ① If X&Y are independent d.r.v.s ;

$$E(x_1+x_2) = \sum_{x_1}\sum_{x_2}(x_1+x_2)\,f(x_1,x_2)$$

$$= \sum_{x_1}\sum_{x_2}\left(x_1 f(x_1,x_2) + x_2 f(x_1,x_2)\right)$$

$$= \sum_{x_1}\sum_{x_2} x_1 f(x_1,x_2) + \sum_{x_1}\sum_{x_2} x_2 f(x_1,x_2)$$

$$= \left(\sum_{x_1} x_1 f_1(x_1)\right)\left(\sum_{x_2} f_2(x_2)\right) + \left(\sum_{x_2} x_2 f_2(x_2)\right)\left(\sum_{x_1} f_1(x_1)\right)$$

$$= E(x_1)\cdot(1) + E(x_2)\,(1)\,.$$

$$\rho_{x,y} = \frac{\mathrm{Cov}(x,y)}{\sigma_x \sigma_y} = \frac{E(XY) - E(X)E(Y)}{\sqrt{\sigma_x^2}\ \sqrt{\sigma_y^2}}$$

$$\sigma_x^2 = E(X^2) - [E(X)]^2$$

$$\sigma_y^2 = E(Y^2) - [E(Y)]^2$$

$$E(X^2) = \left.\frac{\partial^2 M(t_1,0)}{\partial t_1^2}\right|_{t_1=0} = -2(1-t_1)^{-3}(-1)\Big|_{t_1=0} = \boxed{2}$$

$$E(Y^2) = \left.\frac{\partial^2 M(0,t_2)}{\partial t_2^2}\right|_{t_2=0} = -2(1-t_2)^{-3}(-1)\Big|_{t_2=0} = \boxed{2}$$

$$\sigma_x^2 = 2-(1)^2 = \boxed{1} \Rightarrow \boxed{\sigma_x = 1}$$

$$\sigma_y^2 = 2-(1)^2 = \boxed{1} \Rightarrow \boxed{\sigma_y = 1}$$

$$\rho_{x,y} = \frac{1-(1)(1)}{(1)(1)} = 0$$

$\therefore$ X & Y are independent (since $\mathrm{Cov}(x,y) = 0$).

H.W. Given a J.p.f of X & Y:

$$f(x,y) = \begin{cases} e^{-y} & \text{for } 0<x<y<\infty \\ 0 & \text{o.w.} \end{cases}$$

Find:
① the M.g.f. of X, Y $[M_{x,y}(t_1,t_2)]$.
② $E(XY)$
③ $f_1(x)$ & $f_2(y)$
④ $\mu_x = E(X)$ & $\mu_y = E(Y)$ (by two methods).
⑤ $\mathrm{Cov}(X,Y)$.

Note: $R = \left\{ \begin{matrix} 0<x<y & , x<y<\infty \\ 0<x<\infty & , 0<y<\infty \end{matrix} \right\}$

$$\text{M.g.f of } X = M_{X,Y}(t_1, 0) = M_X(t_1) = \frac{1}{1-t_1}$$

$$\mu_x = E(X) = \left.\frac{\partial M(t_1,0)}{\partial t_1}\right|_{t_1=0} = \left.\frac{\partial M_X(t_1)}{\partial t_1}\right|_{t_1=0}$$

$$= \left.\frac{1}{(1-t_1)^2}\right|_{t_1=0} = \boxed{1}$$

$$\text{M.g.f of } y = M_{X,Y}(0, t_2) = M_Y(t_2) = \frac{1}{1-t_2}$$

$$\mu_y = E(Y) = \left.\frac{\partial M(0,t_2)}{\partial t_2}\right|_{t_2=0} = \left.\frac{\partial M_Y(t_2)}{\partial t_2}\right|_{t_2=0}$$

$$= \left.\frac{1}{(1-t_2)^2}\right|_{t_2=0} = \boxed{1}$$

$$\therefore \text{M.g.f of } X, Y = M_{X,Y}(t_1,t_2) = \frac{1}{(1-t_1)(1-t_2)}$$

$$E(XY) = \left.\frac{\partial^2 M(t_1,t_2)}{\partial t_1 \partial t_2}\right|_{t_1=t_2=0}$$

$$= \left.\frac{\partial}{\partial t_1}\left(\frac{\partial M(t_1,t_2)}{\partial t_2}\right)\right|_{t_1=t_2=0}$$

$$= \left.\frac{\partial}{\partial t_1}\left(\frac{1}{(1-t_1)(1-t_2)^2}\right)\right|_{t_1=t_2=0}$$

$$= \left.\frac{1}{(1-t_1)^2(1-t_2)^2}\right|_{t_1=t_2=0}$$

$$= \boxed{1}$$

The variance of X or Y by using $M_{x,y}(t_1,t_2)$:

$$Var(X) = \sigma_x^2 = \frac{\partial^2 M(0,0)}{\partial t_1^2} - \left(\frac{\partial M(0,0)}{\partial t_1}\right)^2$$
$$= E(X^2) - [E(X)]^2$$

$$Var(Y) = \sigma_y^2 = \frac{\partial^2 M(0,0)}{\partial t_2^2} - \left(\frac{\partial M(0,0)}{\partial t_2}\right)^2$$
$$= E(Y^2) - [E(Y)]^2$$

Example : Let $f(x,y)$ be the J.p.d.f of X, Y:

$$f(x,y) = \begin{cases} e^{-(x+y)} & \text{for } 0 \quad x>0, y>0 \\ 0 & \text{o.w.} \end{cases}$$

Find the M.g.f of X, Y ; also, find μ_x, μ_y, $E(XY)$, and $\rho_{x,y}$.

Sol.

$$M_{x,y}(t_1,t_2) = E[e^{t_1x+t_2y}]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1x+t_2y} f(x,y)\, dy\, dx$$
$$= \int_0^{\infty}\int_0^{\infty} e^{t_1x+t_2y} e^{-(x+y)} dy\, dx$$
$$= \left(\int_0^{\infty} e^{-x(1-t_1)} dx\right)\left(\int_0^{\infty} e^{-y(1-t_2)} dy\right)$$
$$= \frac{1}{(1-t_1)} \cdot \frac{1}{(1-t_2)} = \frac{1}{(1-t_1)(1-t_2)}$$

Moment Generating Function of two R.V.'s X & y

② Let $f(x,y)$ be the J.p.d.f of X, y ; and consider that:

① $E[e^{t_1 x + t_2 y}]$ exists for $|t_1| < h_1$, $|t_2| < h_2$; h_1, h_2 are positive Contants, then:

$E[e^{t_1 x + t_2 y}]$ is the m.g.f of X & y and is deflotted by $M_{x,y}(t_1, t_2)$ and is defined as follows:

$$M_{x,y}(t_1,t_2) = E[e^{t_1 x + t_2 y}] = \begin{cases} \sum_{\forall x} \sum_{\forall y} e^{t_1 x + t_2 y} f(x,y) & \text{if } x,y \text{ are d.r.v.'s} \\ \int_x \int_y e^{t_1 x + t_2 y} f(x,y)\, dy\, dx & \text{if } x,y \text{ are c.r.v.'s} \end{cases}$$

and

② $$M_x(t_1) = M_x(t_1, 0) = E[e^{t_1 x}] = \begin{cases} \sum_{\forall x} e^{t_1 x} f_1(x) & \text{if } x \text{ is d.r.v.} \\ \int_{-\infty}^{\infty} e^{t_1 x} f_1(x)\, dx & \text{if } x \text{ is c.r.v.} \end{cases}$$

$$M_y(t_2) = M_y(0, t_2) = E[e^{t_2 y}] = \begin{cases} \sum_{\forall y} e^{t_2 y} f_2(y) & \text{if } y \text{ is d.r.v.} \\ \int_{-\infty}^{\infty} e^{t_2 y} f_2(y)\, dy & \text{if } y \text{ is c.r.v.} \end{cases}$$

and

③ $$\frac{\partial M(0,0)}{\partial t_1} = \mu_x = E(X) \; ; \quad \frac{\partial M(0,0)}{\partial t_2} = \mu_y = E(y)$$

$$\frac{\partial^2 M(0,0)}{\partial t_1^2} = E(X^2) \; ; \quad \frac{\partial^2 M(0,0)}{\partial t_2^2} = E(y^2)$$

$$\frac{\partial^2 M(0,0)}{\partial t_1 \partial t_2} = E(XY)$$

Exersices About Ch.5

Q1: Given a J.P.f. :

$$f(x,y) = \begin{cases} Cy^2 & \text{for } 0 \leq X \leq 2 \ \& \ 0 \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

$$c \int_0^2 \int_0^1 y^2 \, dy \, dx = c \int_0^2 \frac{y^3}{3} \Big|_0^1 dx = \frac{c}{3} x \Big|_0^2 = \frac{2}{3} c$$

Find :

① The value of (C)

② $P(X+Y > 2)$

③ $P(Y < \frac{1}{2})$, ④ $P(X \leq 1)$

Q2: Let the J.P.m.f of X & Y be:

$$f(x,y) = \begin{cases} \frac{xy^2}{30} & \text{for } x = 1,2,3 \ \& \ y = 1,2 \\ 0 & \text{o.w.} \end{cases}$$

(i) Find the Marginal P.f. of X ?

(ii) Find the Marginal P.f. of Y ?

(iii) Are X & Y independent ?

Q3: Let X & Y have the J.P.m.f. be :

$$f(x,y) = \begin{cases} \frac{x+2y}{18} , & x = 1,2, \ y = 1,2 \\ 0 & \text{o.w.} \end{cases}$$

Find: $\mu_x = E(X)$, $\mu_y = E(Y)$, & $Cov(X,Y)$.

Q4: Suppose that X, Y are C.r.v.s with J.P.d.f. :

$$f(x,y) = \begin{cases} e^{-(x+y)} & \text{for } X > 0, \ Y > 0 \\ 0 & \text{o.w.} \end{cases}$$

Find : $M_{X,Y}(t_1,t_2)$, $M_X(t_1)$, $M_Y(t_2)$.

* $F(x,y)$, [illegible] , $f_1(x)$, $f_2(y)$.
[illegible] X and Y indep[illegible] . Check [illegible] by [illegible]

Q5: Suppose X and y are two r.v.s having the following J.p.d.f :

$$f(x,y) = \begin{cases} x+y & \text{for } 0<x<1,\ 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

(i) Find the conditional p.d.f $f(x|y)$ and $f(y|x)$.

(ii) Are these Conditional function $f(x|y)$ and $f(y|x)$ Pr. d.f.? (prove that).

Q6: Consider the following J.p.M.f. of X and y whose values are given by:

$y \backslash X$	2	3	4	5	6	$f_2(y)$
0	1/9	0	4/27	0	2/27	1/3
1	2/27	0	8/81	0	4/81	2/9
2	4/27	0	16/81	0	8/81	4/9
$f_1(x)$	1/3	0	4/9	0	2/9	1

Find: ① $f(x|y)$ & $f(y|x)$

② $p(x|y=1)$ & $p(y|x=2)$

Q7: If J.p.m.f :

$p(1,1)=\frac{1}{9}$, $p(2,1)=\frac{1}{3}$, $p(3,1)=\frac{1}{9}$

$p(1,2)=\frac{1}{9}$, $p(2,2)=0$, $p(3,2)=\frac{1}{18}$

$p(1,3)=0$, $p(2,3)=\frac{1}{6}$, $p(3,3)=\frac{1}{9}$

Are X & y independent ?

Q8: If X and y are independent with M.p.d.f :

$$f_1(x)=\begin{cases} 3x^2 & \text{for } 0\le x\le 1 \\ 0 & \text{o.w.} \end{cases}, \quad f_2(y)=\begin{cases} 2y & \text{for } 0\le y\le 1 \\ 0 & \text{o.w.} \end{cases}$$

Find $E(XY)$.

Q9: Suppose that J.p.d.f of X & Y is:

$$f(x,y) = \begin{cases} c\sin x & \text{for } 0 \leq x \leq \pi,\ 0 \leq y \leq [illegible] \\ 0 & \text{o.w.} \end{cases}$$

Find the Value of C. $\left(C = \frac{1}{2\pi}\right)$

Q10: Let X and Y have the joint p.m.f. be:

(X,Y)	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)
P(X,Y)	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{4}{15}$

Find the Correlation Coefficient $\rho_{x,y}$.

[illegible]

Q11: If $f(x,y) = \begin{cases} e^{-x-y} & \text{for } 0 < x < \infty \ \& \ 0 < y < \infty \\ 0 & \text{o.w.} \end{cases}$

is the J. p.d.f of two r.v.s X & Y.
Show that X & Y are stochastically independent.

Q12: Let X & Y be independent r.v.'s with the following distribution:

X	1	2
$f_1(x)$	.6	.4

y	10	15	5
$f_2(y)$	.5	.3	.2

① Find the Joint pr. f of X & Y

② also, find E(XY).

Q13: Let $f(x|y) = \begin{cases} \frac{c_1 x}{y^2} & \text{for } 0 < x < y,\ 0 < y < 1 \\ 0 & \text{o.w.} \end{cases}$

& $f_2(y) = \begin{cases} c_2 y^4 & \text{for } 0 < y < 1 \\ 0 & \text{o.w.} \end{cases}$; then:

$(f(x,y) = \qquad 0 < x < y < 1)$

Find:

(1) Determine the Constants C_1 and C_2.

(2) the $f_1(X)$ p.d.f. of X.

(3) the J.p.d.f. of X & Y ; $f(x,y)$

(4) $P(-\frac{1}{2}<X<\frac{1}{2} \mid Y=\frac{5}{8})$

(5) $P(\frac{1}{4}<X<\frac{1}{2})$.

Q14: Let $f(x,y) = \begin{cases} \frac{1}{16} & \text{for } X=1,2,3,4 \\ & \quad y=1,2,3,4 \\ 0 & \text{o.w.} \end{cases}$

be the J.p.f of X & Y.

Show that X & Y are stochastically independent.

Q15: Given a J.p.f of X & Y:

$f(x,y) = \begin{cases} x+y & \text{for } 0<x<1 \text{ \& } 0<y<1 \\ 0 & \text{o.w.} \end{cases}$

Find the Correlation Coefficient of X & Y ($\rho_{x,y}$).

Solution of Q_1:

① by cond. (2) $\Longrightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

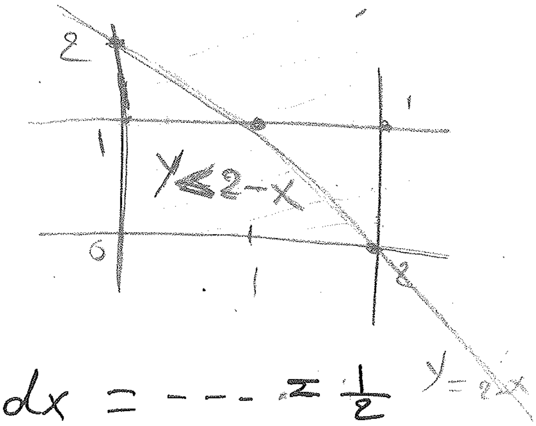
$$C\int_0^2\int_0^1 y^2\,dy\,dx = 1 \Longrightarrow \boxed{C=\frac{3}{2}}$$

$P(X+Y>2)$

② $P(X+Y\cancel{<}2) = 1-P(X+Y\cancel{\geq}2)$

Let $x+y=2 \Longrightarrow y=2-x$

$x+y\leq 2 \Longrightarrow y\leq 2-x$



$$R=\left\{\begin{matrix} 0\leq y\leq 2-x \\ 0\leq x\leq 2\end{matrix}\right\}$$

$$P(x+y\cancel{\leq}2) = \int_0^2\int_0^{2-x}\frac{3}{2}y^2\,dy\,dx = \dots = \frac{1}{2}$$

$\int_0^1\int_0^{2-y}\frac{3}{2}y^2\,\cancel{dx}\,dy$

③ $P(Y<\frac{1}{2}) = ?$

To find $f_2(y)$ in the first.

$$f_2(y) = \int_0^2 f(x,y)\,dx = \frac{3}{2}\int_0^2 y^2\,dx = \dots = \begin{cases} 3y^2 & \text{for } 0\leq y\leq 1 \\ 0 & \text{o.w.}\end{cases}$$

$$P(Y<\tfrac{1}{2}) = 3\int_0^{1/2} y^2\,dy = \dots = \frac{1}{8}$$

④ $P(X\leq 1) = ?$

To find $f_1(x)$ in the begining.

$$f_1(x) = \int_0^1 f(x,y)\,dy = \frac{3}{2}\int_0^1 y^2\,dy = \dots = \begin{cases} \frac{1}{2} & \text{for } 0\leq x\leq 2 \\ 0 & \text{o.w.}\end{cases}$$

$$P(X\leq 1) = \int_0^1 f_1(x)\,dx$$

$$= \int_0^1 \frac{1}{2}\,dx = \dots = \frac{1}{2}.$$

⑤ $P(X=3Y) = \int_0^2\int_0^{\frac{1}{3}x} f(x,y)\,dy\,dx = \int_0^2\int_0^{\frac{1}{3}x}\frac{2}{3}y^2\,dy\,dx.$

$\Longrightarrow \boxed{y=\frac{1}{3}x}$

$= \dots = \frac{2}{27}$

Q2: If $f_1(x) \cdot f_2(y) = f(x,y)$, then X & Y are indep.

$$f_1(x) = \sum_{y=0}^{3} f(x,y) = \sum_{y=0}^{3} \frac{1}{30}(x+y)$$

$$= \frac{1}{30}\left[x + (x+1) + (x+2) + (x+3)\right]$$

$$= \frac{1}{30}(4x+6) = \begin{cases} \frac{1}{15}(2x+3) & \text{for } x=0,1,2 \\ 0 & \text{o.w.} \end{cases}$$

$$f_2(y) = \sum_{x=0}^{2} f(x,y) = \sum_{x=0}^{2} \frac{1}{30}(x+y)$$

$$= \frac{1}{30}\left[y + (1+y) + (2+y)\right]$$

$$= \frac{1}{30}(3+3y) = \frac{1}{10}(y+1)$$

$$f_2(y) = \begin{cases} \frac{1}{10}(y+1) & \text{for } y=0,1,2,3 \\ 0 & \text{o.w.} \end{cases}$$

سؤال آخر

$$f(x,y) = \begin{cases} \frac{x+y}{30} & y=0,1,2,3 \\ & x=0,1,2,3 \\ 0 & \text{o.w.} \end{cases}$$

$$f_1(x) f_2(y) = \frac{1}{15}(2x+3) \cdot \frac{1}{10}(y+1)$$

$$\neq \frac{1}{30}(x+y) = f(x,y)$$

$$\therefore f_1(x) \cdot f_2(y) \neq f(x,y)$$

Q3: To find f_1 & f_2 in the first time.

$$f_1(x) = \sum_{x_2=1}^{2} \frac{1}{18}(x_1+12x_2)$$

$$= \frac{1}{18} \sum_{x=1}^{2} x_1 + \frac{1}{18} \sum_{x_2=1}^{2} 2x_2$$

$$= \dots = \begin{cases} \frac{x_1+3}{9} & \text{for } x_1=1,2 \\ 0 & \text{o.w.} \end{cases}$$

$$\mu_{x_1} = E(x_1) = \sum_{x_1=1}^{2} x_1 f_1(x_1) = \sum_{x_1=1}^{2} x_1\left(\frac{x_1+3}{9}\right) = \dots = \frac{14}{19}$$

$$f_2(x_2) = \sum_{x_1=1}^{2} \frac{1}{18}(x_1+2x_2) = \begin{cases} \frac{1}{18}(3+4x_2) & \text{for } x_2=1,2 \\ 0 & \text{o.w.} \end{cases}$$

$$\mu_{x_2} = E(x_2) = \sum_{x_2=1}^{2} x_2 f_2(x_2) = \sum_{x_2=1}^{2} x_2 \frac{(3+4x_2)}{18} = \dots$$

$$= \frac{29}{18}$$

$$Cov(X,Y) = E(X_1X_2) - E(X_1)E(X_2)$$
$$= \mu_{X_1X_2} - \mu_{X_1}\mu_{X_2}$$

$$\mu_{X_1X_2} = \sum_{X_1=1}^{2}\sum_{X_2=1}^{2} f(x_1,x_2)\cdot X_1X_2$$
$$= \frac{1}{18}\sum_{X_1=1}^{2}\sum_{X_2=1}^{2} X_1X_2(X_1+2X_2)$$
$$= \frac{1}{18}\Big[(1)(1)(1+2(1)) + (1)(2)(1+2(2)) + (2)(1)(2+2(1))$$
$$+ (2)(2)(2+2(2))\Big] = \frac{9}{2}$$

$$\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X\ \sigma_Y} = \frac{E(XY)-E(X)E(Y)}{\sqrt{V(X)}\ \sqrt{V(Y)}}$$

$$\sigma_{X_1} = \sqrt{\sigma_{X_1}^2}\ ;\ \sigma_{X_1}^2 = E(X_1^2) - [E(X_1)]^2$$
$$\sigma_{X_2} = \sqrt{\sigma_{X_2}^2}\ ;\ \sigma_{X_2}^2 = E(X_2^2) - [E(X_2)]^2$$

$$E(X_1^2) = \sum_{X_1=1}^{2} X_1^2 f_1(x_1) = \sum_{X_1=1}^{2} X_1^2\left(\frac{x_1+3}{9}\right) = \frac{24}{9}$$
$$E(X_2^2) = \sum_{X_2=1}^{2} X_2^2 f_2(X_2) = \sum_{X_2=1}^{2} X_2^2\left(\frac{1}{18}(3+4X_2)\right) = \frac{51}{18}$$

$$\mu_{X_1}^2 = [E(X_1)]^2 \quad \& \quad \mu_{X_2}^2 = [E(X_2)]^2$$

$$\sigma_{X_1}^2 = E(X_1^2) - [E(X_1)]^2$$
$$= \frac{24}{9} - \left(\frac{14}{9}\right)^2 = \sim$$
$$\sigma_{X_2}^2 = \frac{51}{18} - \left(\frac{29}{18}\right)^2 = \sim$$

(اكمل الحل)

Q6 : ① $f(X|Y) = \frac{f(x,y)}{f_2(y)}$ ، $f(Y|X) = \frac{f(x,y)}{f(X)}$

② $f(X|Y=1) = \frac{f(x,y=1)}{f_2(y=1)} = \frac{f(x_i,y=1)}{2/9}$ ، $x_i = 2,3,\dots,6$.

$$= \begin{cases} \frac{2/27}{2/9} = \frac{1}{3} & \text{for } x=2 \\ \frac{0}{2/9} = 0 & \text{for } x=3 \\ \frac{8/81}{2/9} = \frac{4}{9} & \text{for } x=4 \\ & \text{for } x=5 \\ \frac{0}{2/9} = 0 & \\ & \text{for } x=6 \\ \frac{4/81}{2/9} & \end{cases}$$

$$f(y|x=2) = \frac{f(y|x=2)}{f(x=2)} = \frac{f(y_i \,|\, x=2)}{1/3}, \; y=0,1,2$$

$$= \begin{cases} 1/9 \,/\, 1/3 & \text{for } y=0 \\ \frac{2/27}{1/3} = \frac{2}{9} & \text{for } y=1 \\ \frac{4/27}{1/3} = \frac{4}{9} & \text{for } y=2 \end{cases}$$

Hence $f(y) = f(y|x=2)$; therefore X & Y are independent.

Q7:

y \ x	1	2	3	$f_2(y)$
1	$1/9$	$1/3$	$1/9$	$5/9 = f_2(1)$
2	$1/9$	0	$1/18$	$3/18 = f_2(2)$
3	0	$1/6$	$1/9$	$15/54 = f_2(3)$
$f_1(x)$	$2/9$ $f_1(1)$	$3/9$ $f_1(2)$	$\overset{5}{3}/18$ $f_1(3)$	1

$f(1,1) = \frac{1}{9} \neq f_1(1) f_2(1) = (\frac{2}{9})(\frac{5}{9})$

$f(2,1) = \frac{1}{3} \neq f_1(2) f_2(1) = (\frac{3}{9})(\frac{5}{9})$

$f(3,1) = \frac{1}{9} \neq f_1(3) f_2(1) = (\frac{5}{18})(\frac{5}{9})$

$f(1,2) = \frac{1}{9} \neq f_1(1) f_2(2) =$

$\therefore$ X & Y are dependent.

Q8: Since X & Y are independent; then:

$f_1(x) f_2(y) = f(x,y)$

$$f_1(x) \cdot f_2(y) = \begin{cases} 6yx^2 & \text{for } 0 \le x \le 1, \; 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$$E(xy) = \int_x \int_y xy f(x,y)\, dy\, dx$$

$$= \int_0^1 \int_0^1 (6yx^2)\, xy \, dy\, dx = \cdots = \frac{1}{2}$$

sol.
Q10

$f_1(x) = \sum_{y=1}^{3} f(x,y)$

$= \begin{cases} 9/15 & \text{for } x=1 \\ 6/15 & \text{for } x=2 \\ 0 & \text{o.w.} \end{cases}$

J.P.f.
$f(x,y)$

y \ X	1	2	$f_2(y)$	
1	2/15	1/15	3/15	$f_2(1)$
2	4/15	1/15	5/15	$f_2(2)$
3	$\textcircled{3/15}$	4/15	7/15	$f_2(3)$
$f_1(x)$	9/15	6/15	1	
	$f_1(1)$	$f_1(2)$		

$f_2(y) = \sum_{x=1}^{2} f(x,y)$

$= \begin{cases} 3/15 & \text{for } y=1 \\ 5/15 & \text{for } y=2 \\ 7/15 & \text{for } y=3 \\ 0 & \text{o.w.} \end{cases}$

$E(X) = \sum_{x=1}^{2} X f_1(x)$

$= (1)(9/15) + (2)(\frac{6}{15})$

$= \frac{21}{15}$

$E(Y) = \sum_{y=1}^{3} y f_2(y)$

$= (1)(\frac{3}{15}) + (2)(\frac{5}{15}) + (3)(\frac{7}{15})$

$= \frac{34}{15}$

$\rho_{xy} = Cov(x,y) / \sigma_x \sigma_y = ---$ اكمل الحل

Q12: ① $f(1,5) = f_1(1) f_2(5) = (.6)(.2) = .12$

$f(1,10) = f_1(1) f_2(10) = (.6)(.5) = .30$

$f(1,15) = f_1(1) f_2(15) = (.6)(.3) = .18$

$f(2,10) = f_1(2) f_2(10) = (.4)(.5) = .20$

$f(2,5) = f_1(2) f_2(5) = (.4)(.2) = .08$

$f(2,15) = f_1(2) f_2(15) = (.4)(.3) = .12$

Since (X & Y are independent r.v.s)

$\Rightarrow f_1(x) f_2(y) = f(x,y)$

$\longrightarrow$

$$\therefore f(x,y) = \begin{cases} .12 & \text{for } x=1, y=5 \\ .08 & \text{for } x=2, y=5 \\ .30 & \text{for } x=1, y=10 \\ .20 & \text{for } x=2, y=10 \\ .18 & \text{for } x=1, y=15 \\ .12 & \text{for } x=2, y=15 \\ 0 & \text{o.w.} \end{cases}$$

$$② E(XY) = \sum_X \sum_Y XY f(x,y)$$

أكمل الحل

إعادة حل السؤال مرة أخرى

Q13: Sol.

$(*_1)$ ⓐ $\int_0^y f(x|y)\,dx = 1$ (By cond. ② of Pr.f)

$$C_1 \int_0^{1\,(y)} \frac{x}{y^2}\,dx = \frac{C_1}{2y^2} x^2 \Big|_0^y = \text{---} = 1 \Longrightarrow \boxed{C_1 = 2y^2} \rightarrow \boxed{C_1 = 2}$$

$\frac{C_1}{2y^2}(y^2) = 1$

$$\therefore f(x|y) = \begin{cases} \frac{2x}{y^2} & \text{for } 0<x<y<1 \quad \cancel{0<x<1} \\ 0 & \end{cases}$$

$0<y<1$

$\boxed{0<x<y}$ $0<x<1$

$(*_2)$ ⓐ $\int_0^1 f_2(y)\,dy = 1$ (by cond. ② of Pr.f.)

$$C_2 \int_0^1 y^4\,dy = 1 \Longrightarrow C_2 = 5$$

$$\therefore f_2(y) = \begin{cases} 5y^4 & \text{for } 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

ⓒ $f(x|y) = \frac{f(x,y)}{f_2(y)} \Longrightarrow f(x,y) = f(x|y)\, f_2(y)$

$$= (2x)(5y^4) = \begin{cases} 10xy^4 & \text{for } 0<x<y<1 \\ 0 & \text{o.w.} \end{cases}$$

ⓑ $f_1(x) = \int_y f(x,y)\,dy = 10\int_0^1 xy^4\,dy = \begin{cases} 2x & \text{for } 0<x<1 \\ 0 & \text{o.w.} \end{cases}$

ⓓ $P(-\frac{1}{2} < x < \frac{1}{2} \mid y = \frac{5}{8}) = \int_0^y f(x|y=\frac{5}{8})\,dx$.

$$= \int_0^{5/8} 10x\left(\frac{5}{8}\right)^4 dx = \text{---}$$

ⓔ $P(\frac{1}{4} < x < \frac{1}{2}) = \int_{1/4}^{1/2} f_1(x)\,dx = \int_{1/4}^{1/2} 2x\,dx = \text{--} = \frac{3}{16}$

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Chapter (6)
Some Special Distributions

بعض التوزيعات الخاصة

① Binomial Distribution: توزيع ذات الحدين

Bernoulli experiment: تجربة برنولي
is an experiment that has only two possible outcomes, event A and event A^c

for examples:-
Toss a coin once.
we get H or T
H, T are only two possible outcomes.
let A: to get H
A^c: to get T
suppose that $P(A) = p$, $0 < p < 1$
Then $P(A^c) = 1 - p$
If we repeat the Bernoulli exper. n- times. ($n \in I^+$) and if the r.v. X
$X \equiv$ number of times that A happens.
for Example:-
Toss a coine. 3- times:-
A: to get H
$X \equiv$ number of H.
$X = 0, 1, 2, 3$.

Know:-

∴ $X \equiv$ number of times that event A happens, then the function of x is

$$f(x) = \begin{cases} \binom{n}{x} p^{x} (1-p)^{n-x} & \text{for } x = 0, 1, 2, \to n \\ 0 & \text{o.w} \end{cases}$$

Know:- To show that $f(x)$ is a p.m.f.

since (x is d.r.v.)

cond ① :- T.P $f(x) \geq 0$

since $n \in I^{+}$

$x \equiv 0, 1, 2, 3, \_ \_ , n \quad ; \quad x \leq n$

$n - x \geq 0$

then $\binom{n}{x} \geq 1 \quad , p^{x} > 0 \quad$ for $0 < p < 1$

$(1-p)^{n-x} > 0$

Then $f(x) > 0 \quad$ for $x = 0, 1, \_\_, n$

$= 0 \quad$ o.w

* Notes:

p احتمال النجاح

$1-p$ احتمال الفشل

n عدد مرات تكرار التجربة

X عدد مرات حدوث النجاح

IV: Gamma Distribution توزيع كاما

Def: " Gamma function Γ "

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} \quad ; \quad \alpha > 0$$

by use integration by parts we get

$$\boxed{\Gamma(\alpha) = (\alpha-1)!}$$

$$f(x, \alpha, \beta) = \begin{cases} \frac{X^{\alpha-1} e^{-\frac{X}{\beta}}}{\Gamma(\alpha)\beta^\alpha} & ; \quad 0 < X < \infty \\ 0 & \text{o.w} \end{cases}$$

Note: $f(x; \alpha, \beta)$ above is called p.d.f. for Gamma dist.

i.e

If $X \sim G(\alpha, \beta)$ then

$$f(x, \alpha, \beta) = \begin{cases} \frac{X^{\alpha-1} e^{-\frac{X}{\beta}}}{\Gamma(\alpha)\beta^\alpha} & ; \quad 0 < X < \infty \\ 0 & \text{o.w} \end{cases}$$

Theorem (8) If $X \sim G(\alpha, \beta)$, then the m.g.f. of x is
$$M_x(t) = (1-\beta t)^{-\alpha}$$

Proof:

$$M_x(t) = E(e^{tx})$$

$$M_x(t) = \int_0^\infty e^{tx}\left(\frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^\alpha}\right)dx$$

$$= \int_0^\infty \frac{x^{\alpha-1} e^{-x\left(\frac{1-\beta t}{\beta}\right)}}{\Gamma(\alpha)\beta^\alpha}\,dx$$

let $y = \frac{x(1-\beta t)}{\beta}$; $\Rightarrow x = \frac{\beta}{1-\beta t}y$

$$dy = \frac{1-\beta t}{\beta}dx$$

$$d\alpha = \frac{\beta}{1-\beta t}dy$$

$$\therefore M_x(t) = \int_0^\infty \frac{\left(\frac{(\beta y)}{1-\beta t}\right)^{\alpha-1}e^{-y}}{\Gamma(\alpha)\beta^\alpha}\left(\frac{\beta}{1-\beta t}\right)dy$$

$$= \int_0^\infty \overset{=\Gamma(\alpha)}{\boxed{\frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)}dy}}\;\frac{\cancel{\beta^\alpha}}{\cancel{\beta^\alpha}(1-\beta t)^\alpha}$$

$$\therefore M_x(t) = (1-\beta t)^{-\alpha}$$

H.W Theorem ⑨ ; If $X \sim G(\alpha , \beta)$, then H.W

$\mu_x = E(X) = \alpha\beta$; $\sigma_x^2 = V(x) = \alpha\beta^2$

proof :-

$\because X \sim G(\alpha , \beta)$

$\therefore M_x(t) = (1-\beta t)^{-\alpha}$

$M'_x(t) = +\alpha\beta(1-\beta t)^{-\alpha-1}$

$M'_x(t) = \alpha\beta(1-\beta t)^{-(\alpha+1)}$

$M'_\alpha(0) = \alpha\beta(1-0)^{-(\alpha+1)} = \alpha\beta$

$\therefore \mu_x = E(x) = M'_x(0) = \alpha\beta$

$M''_x = -\alpha\beta(\alpha+1)\cdot(1-\beta t)^{-(\alpha+2)}\cdot(-\beta)$

$= \alpha\beta^2(\alpha+1)(1-\beta t)^{-\alpha+2}$

$E(X^2) = M''_x(0) = \alpha\beta^2(\alpha+1)\cdot 1$

$= \alpha^2\beta^2 + \alpha\beta^2$

$\therefore V(x) = E(X^2) - [E(x)]^2$

$= \alpha^2\beta^2 + \alpha\beta^2 - (\alpha\beta)^2$

$= \cancel{\alpha^2\beta^2} - \cancel{\alpha^2\beta^2} + \alpha\beta^2$

$\therefore V(x) = \alpha\beta^2$

EX⑧: find the p.d.f. of a c.r.v. X whose m.g.f. is

$M_\alpha(t) = (1-3t)^{-2}$

sol.

$\therefore X \sim G(2,3)$

$$\therefore f(x,2,3) = \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\,\beta^{\alpha}}$$

$$= \frac{x e^{-\frac{x}{3}}}{1 \cdot (3)^2} = \frac{x e^{-\frac{x}{3}}}{9}$$

EX: find the m.g.f. of x has a p.d.f.

$$f(\alpha) = \begin{cases} \dfrac{x e^{-\frac{x}{2}}}{4} & ; \ 0 < x < \infty \\ 0 & \text{o.w} \end{cases}$$

$\therefore X \sim G(2,2)$

$\alpha - 1 = 1$

$\therefore \alpha = 2$

$\beta = 2$

$\therefore M_x(t) = (1-\beta t)^{-\alpha}$

$= (1-2t)^{-2}$

H.W EX:

$$f(\alpha) = \begin{cases} \dfrac{x^2 e^{-\frac{x}{3}}}{\Gamma(3) \cdot (27)} & ; \ 0 < x < \infty \\ 0 & \text{o.w} \end{cases}$$

sol.

$M_x(t) = (1-\beta t)^{-\alpha}$; $\therefore X \sim G(3,3)$

$M_x(t) = (1-3t)^{-3}$

$-\frac{r}{2}-1-$

$r-2$

Lemma: If $X \sim \chi^2(\frac{x}{2})$, $\alpha = \frac{r}{2}$, $\beta > 0$, Then the r.v. $Y = \frac{2X}{\beta} \sim \chi^2(r)$

Ex: find $p(3.28 < X < 25.2)$

$\alpha = 3$, $\beta = 4$

let $y = \frac{2X}{\beta}$, $r = 2\alpha = 6$

$$X = \frac{y\beta}{2} = \frac{y \cdot 4}{2} = 2y$$

$$\begin{aligned} p(3.28 < X < 25.2) &= p(3.28 < 2y < 25.2) \\ &= p(1.64 < y < 12.6) \\ &= p(y < 12.6) - p(y < 1.64) \\ &= 0.95 - 0.05 \\ &= 0.9. \end{aligned}$$

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(V) Chi - Square Dist.

Chi - Square dist. is aspecial Case of Gamma dist. where $\alpha = \frac{r}{2}$, r is positive integer and $\beta = 2$.

The new dist. is called chi - Square dist. (denoted by $\chi^2(r)$) درجات الحرية where (r) is called degree of fredom (d. f.).

Therefore, if $X \sim \chi^2(r)$, then the p.d.f. of X is as followings:-

$$f(x;r) = \begin{cases} \dfrac{x^{\frac{r}{2}-1} e^{-\frac{x}{2}}}{\Gamma(\frac{r}{2}) \, 2^{\frac{r}{2}}} & ; \text{ for } 0 < x < \infty \\ 0 & \text{o.w} \end{cases}$$

[illegible]

(مطلوب حفظها ٢)

Also; the M.g.f. of X is

$$M_X(t) = (1-2t)^{-\frac{r}{2}}$$

$$E(x) = \mu_X = \alpha\beta = (\tfrac{r}{2}) \cdot 2 = r$$

$$V(x) = \sigma^2 = \alpha\beta^2 = (\tfrac{r}{2}) \, 4 = 2r$$

Ex: If a is v. X has a p.d.f.

$$f(x) = \begin{cases} \dfrac{x \, e^{-\frac{x}{2}}}{4} & \text{for } 0 < x < \infty \\ 0 & \text{o.w} \end{cases}$$

find M.g.f. of X :

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sol.
$$f(x) = \begin{cases} \dfrac{x^{\alpha-1} \cdot e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\, \beta^{\alpha}} & 0 < x < \infty \\ 0 & o.w \end{cases}$$

$\therefore X \sim G(2, 2)$

$\therefore \frac{r}{2} = \alpha \rightarrow 2 = \frac{r}{2} \Rightarrow r = 4$

$\therefore M_x(t) = (1-2t)^{-\frac{r}{2}} = (1-2t)^{-\frac{4}{2}}$
$= (1-2t)^{-2}$

$\therefore X \sim X^2(4)$

EX: If a r.v. X has a m.g.f. H.W
$M_X(t) = (1-2t)^{-7}$, $t < \frac{1}{2}$
then find the p.f. of X.

sol.
$\frac{r}{2} = 7 \Rightarrow r = 14$, $\beta = 2$
$\therefore X \sim G(7, 2)$

$f(x) = \Big\{$

Table of ch. Square dist.

$$p(X \le x) = \int_0^x \frac{x^{\frac{r}{2}-1} e^{-\frac{r}{2}}}{\Gamma(\frac{r}{2})\, 2^{\frac{r}{2}}}\, dx$$

$p(X \le x)$

dif (r)	(.001)	0.025	0.05	0.95	0.975	.99
1		.001	.004	.38		
2						
3	.115		.352		9.35	11.3
4	.554	.484	.711	.49	11.1	(13.3)
⋮						
10	2.56	3.25	3.44	18.3	20.3	23.2

Note, $p(a \le x \le b) = p(x \le b) - p(x \le a)$

Ex) If $X \sim \chi^2(10)$, then find $p(3.25 \le X \le 20.5)$

sol.

$p(3.25 \le X \le 20.5) = p(X \le 20.5) - p(X \le 3.25)$

$= 0.975 - 0.025$

$= 0.950.$

Exercises :

① let $y \sim \chi^2(3)$, $r = 3$

find $p(0.352 \leq y \leq 11.3)$

② If $(1-2t)$, $t < \frac{1}{2}$ $= \frac{r}{2} = 1 \Rightarrow r = 2$

is the m.g.f. of the c.r.v. X Find

① $p(X \leq 4.4)$

② $p(5.23 \leq X \leq 26.2)$

③ If X has a p.d.f.

$$f(x) = \begin{cases} \frac{x}{16} e^{-\frac{x}{2}} & 0 < x < \infty \\ 0 & \text{o.w} \end{cases}$$

, $\beta = 2$

$\alpha - 1 = 4$

$\Rightarrow \alpha = 2$

$\frac{r}{2} = \alpha$

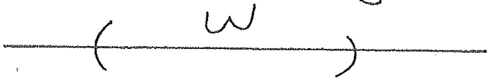
$r = 4$

find ① $p(x \leq 0.872)$

② $p(1.24 \leq x \leq 16.8)$.

③ Poisson Distribution توزيع بواسون

Given an interval of length W, $W>0$

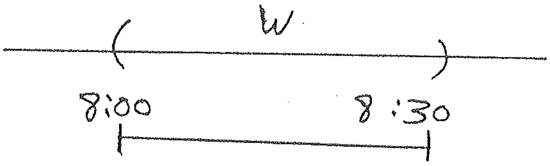


Let A be an event happen in W.

Note: W is interval of time or space or distance ... etc.

Example ①

A : Car accident that happens in interval (8:00, 8:30)



W : 30 mintes = interval of times.

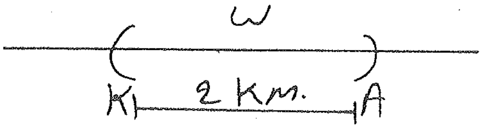
X ≡ number of car accident in W = 30 minutes.

Example ②

A : Car accident between Kadmia and Adamia.

X ≡ number of car accident in W = 2Km.

W = interval of distance = 2Km.



Example ③

A : typing errors in one page.

X ≡ number of typing errors in one page (W)

W ≡ interval of space

Know;

If X = number of events A that happens in interval of length, then the function of X is:

$$f(x) = \begin{cases} \dfrac{e^{-\lambda w}(\lambda w)^x}{x!} & \text{for } x = 0, 1, 2, \ldots \\ 0 & \text{o.w.} \end{cases}$$

where λ = number of event A, when $\boxed{w=1}$ & $w>0, \lambda>0$

Let $m = \lambda \cdot w$; $m>0$, $w>0$

λ = عدد تكرار الحادثة A
(A تكرار)
w=1 ع

$$f(x;m) = \begin{cases} \dfrac{e^{-m}(m)^x}{x!} & \text{for } x=0,1,2,\ldots \\ 0 & \text{o.w.} \end{cases}$$

where (m) is a parameter.

Know, to show that $f(x;m)$ is a p.m.f.

Cond. ① T.P $f(x;m) \geq 0 \quad \forall x \in R_X$

$m>0$, $X \geq 0$, $X! > 0$

$(m)^x > 0$, $e^{-m} = \dfrac{1}{e^m} > 0$

$$\therefore f(x;m) = \begin{cases} \dfrac{e^{-m}(m)^x}{x!} > 0 & \text{for } x=0,1,2,\ldots \\ 0 & \text{o.w.} \end{cases}$$

Cond. ② T.P $\sum_{x=0}^{\infty} f(x;m) = 1$

$$\sum_{x=0}^{\infty} f(x;m) = \sum_{x=0}^{\infty} \frac{e^{-m} m^x}{x!} = e^{-m} \sum_{x=0}^{\infty} \frac{m^x}{x!}$$

$$\because e^x = \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right] = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\therefore \sum_{x=0}^{\infty} f(x;m) = e^{-m} \cdot e^{m} = 1$$

Cond. ① & Cond. ② are satisfied; then $f(x;m)$ is p.m.f.

Note : $f(x;m)$ is called Poisson p.m.f.

Theorem (6): If X has a poisson dist. with parameter (m), then the M.g.f. of X is:

$$M_X(t) = e^{m(e^t - 1)} \quad ; \quad -\infty < t < \infty$$

proof: $\because X \sim P(m)$

$$\therefore f(x;m) = \begin{cases} \dfrac{e^{-m} m^x}{x!} & \text{for } x=0,1,2,\ldots \\ 0 & \text{o.w.} \end{cases}$$

القتلى

Example ① The average number of homicides per day in (Los Angeles) Contry is (2). Use the poisson dist. to determine the pr. that on a given days:

a. There will be three or less homicides in (Los Angeles) Country.

b. There will be exactly three homicides.

Sol.

عدد القتلى في كل يوم

$m = 2$; $w = 1$ day ; $X \equiv$ no. of homicides in a day.

$\therefore X \sim P(2)$

$$f(x;2) = \begin{cases} \dfrac{e^{-2}\, 2^x}{x!} & \text{for } x = 0, 1, 2, \ldots \\ 0 & \text{o.w.} \end{cases}$$

a. $P(X \leqslant 3) = \sum_{x=0}^{3} f(x) = f(0) + f(1) + f(2) + f(3)$

b. $P(X = 3) = f(3) = \dfrac{e^{-2}\, 2^3}{3!} = \dfrac{8\, e^{-2}}{6} = \dfrac{4}{3} e^{-2}$.

الزبائن الداخلين معرض

Example ② The average number of customers entering a store is (4) per minte. Assuming (أفرض) the number of customers has a poisson dist. Find the pr. that at least (2) customers enter the store in $(\frac{1}{2})$ minte.

Sol.

عدد الزبائن الذين يدخلون المعرض

$X \equiv$ number of customers enter the store.

Given $m = 4$, $w = 1$ in one minte.

$m = \lambda \cdot w$

$4 = \lambda \cdot 1 \Rightarrow \lambda = 4$

if $w = \frac{1}{2}$; find $P(X \geqslant 2)$?

$m = \lambda \cdot w$

$= 4 \cdot (\frac{1}{2}) = 2 \Rightarrow m = 2$

$X \sim P(2)$

$$\therefore f(x;2) = \begin{cases} \dfrac{e^{-2}\, 2^x}{x!} & \text{for } x = 0, 1, 2, \ldots \\ 0 & \text{o.w.} \end{cases}$$

$$\begin{aligned} P(X \geqslant 2) = 1 - P(X < 2) &= 1 - P[(X=0) \cup (X=1)] \\ &= 1 - [P(X=0) + P(X=1)] \\ &= 1 - [f(0) + f(1)] = 1 - [e^{-2} + 2e^{-2}] \\ &= 1 - 3e^{-2}. \end{aligned}$$

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} f(x;m)$$

$$= \sum_{x=0}^{\infty} e^{tx} \cdot e^{-m} \frac{m^x}{x!} = e^{-m}\sum_{x=0}^{\infty} \frac{(me^t)^x}{x!}$$

By $e^X = 1 + X + \frac{X^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{X^k}{k!}$

$$M_X(t) = e^{-m} \cdot \sum_{x=0}^{\infty} \frac{(me^t)^x}{x!} = e^{-m} \cdot e^{me^t} = e^{m(e^t-1)} \quad -\infty<t<\infty$$

Theorem (7): If X has apoisson dist. with parameter (m); then $\mu_X = E(X) = m$ & $\sigma_X^2 = Var(X) = m$.

Proof: ∵ $X \sim P(m)$

∴ $M_X(t) = e^{m(e^t-1)}$; $-\infty<t<\infty$

$M'_X(t) = e^{m(e^t-1)} \cdot me^t$

$M'_X(0) = E(X) = 1 . m . 1 = m \Rightarrow$ ∴ $E(X) = \mu_X = m$

$M''_X(t) = e^{m(e^t-1)} \cdot (me^t) + (me^t)\, e^{m(e^t-1)} \cdot (me^t)$

$M''_X(0) = E(X^2) = m + m^2$

$\sigma_X^2 = Var(X) = E(X^2) - (E(X))^2$

$= m + \cancel{m^2} - \cancel{m^2} = m \Rightarrow \sigma_X^2 = m$

H.W.

① If $\mu_X = \sigma_X^2 = 2$; Find $P(X \geq 1)$, $M_X(t)$, $P(X=0)$

② Given $M_X(t) = e^{4(e^t-1)}$ for $-\infty<t<\infty$

Show that:

$P(\mu_X - 2\sigma_X < X < \mu_X + 2\sigma_X) = 0.93$

Examples

→

about Poisson distribution

العيوب شريط

Example ③ The average number of defects on atape is (4) per (1000) fect.
Assuming that the number of defects has apoisson dist.
a. Find the pr. that a tape of long (2400) ft. has (2) defects.
b. Find the pr. that a tape of long (1200) ft. has <u>no</u> defects.

Sol. X = number of defects on a tape.

Given $m = 4$, $w = 1000$

$$m = \lambda \cdot w$$
$$4 = \lambda \cdot (1000) \Rightarrow \lambda = \frac{1}{250} = .004$$

a. if $w = 2400$; find $p(X=2)$

$$m = \lambda \cdot w$$
$$= \frac{1}{250} \cdot (2400) = 9.6 \Rightarrow \therefore m = 9.6$$

$\therefore X \sim p(9.6)$

$$f(x; 9.6) = \begin{cases} \dfrac{e^{-9.6}(9.6)^x}{x!} & ; x = 0, 1, 2, \ldots \\ 0 & \text{o.w.} \end{cases}$$

$$p(x=2) = f(2) = \frac{e^{-9.6}(9.6)^2}{2!}$$

b. if $w = 1200$; find $p(X=0)$

$$m = \lambda \cdot w$$
$$= \frac{1}{250} \cdot (1200) = 4.8 \Rightarrow \therefore m = 4.8$$

$\therefore X \sim p(4.8)$

$$f(x; 4.8) = \begin{cases} \dfrac{e^{-4.8}(4.8)^x}{x!} & ; x = 0, 1, 2, \ldots \\ 0 & \text{o.w.} \end{cases}$$

$$p(x=0) = f(0) = \frac{e^{-4.8}(4.8)^0}{0!} = e^{-4.8} \quad (\text{since } 0! = 1)$$

مكالمات هاتفية

Example ④ The average number of phone calls is (5) per minte, find the pr. that there are (3) calls (15) per seconds.

ثانية

عدد المكالمات الهاتفية في الدقيقة الواحدة

Sol. X = number of phone calls in one minte.

Given $m = 5$; $w = 1$ minute.

$$m = \lambda . w$$

$$5 = \lambda . 1 = 5 \Rightarrow \lambda = 5$$

if $w = 15$ seconds ; find $P(X=3)$?

$$w = \frac{15}{60} = \frac{1}{4} \text{ minute}$$ ⟵ (تحول الثواني الى دقائق)

($w = 1$ minute $\Rightarrow w = 45$ sec.)

$$m = \lambda . w$$

$$= 5 . (\tfrac{1}{4}) = \frac{5}{4} = 1.25 \Rightarrow m = 1.25$$

$\therefore X \sim P(1.25)$

$$f(x; 1.25) = \begin{cases} \dfrac{e^{-1.25}(1.25)^x}{x!} & \text{for } x = 0, 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

$$P(X=3) = f(3) = \frac{e^{-1.25}(1.25)^3}{3!}$$

Example ⑤ The average number of defects in a roll of a certain type of wall paper is 2.5. Use the poisson dist. to determine the pr. that a roll will have (4) or more defects.

العيوب رولة تيب ورق الحائط

Sol. X = number of defects in a roll of a type of wall paper.

Given $m = 2.5$

$\therefore X \sim P(2.5)$

$$f(x; 2.5) = \begin{cases} \dfrac{(2.5)^x e^{-2.5}}{x!} & \text{for } x = 0, 1, 2, \dots \\ 0 & \text{o.w.} \end{cases}$$

$$P(X \geq 4) = 1 - P(X < 4)$$

$$= 1 - \sum_{x=0}^{3} f(x) = 1 - \sum_{x=0}^{3} \frac{(2.5)^x e^{-2.5}}{x!}$$

$$= 1 - [f(0) + f(1) + f(3)]$$

الحوادث المرورية

Example ⑥ The average number of traffic accidents that take place on the Baghdad high way on a week day between 7:00 A.M. and 8:00 A.m. is (.7) accident per hour. Use poisson dist. to determine the pr. that at least (2) traffic accidents would the Baghdad high way on Tuesday morning between 7:00 A.m. & 8:00 A.m.

تحدث الطريق السريع يوم بالأسبوع

Sol.

$X \equiv$ number of traffic accidents that take place on the Baghdad high way on a week day between (7:00 – 8:00) A.m.

Given $m = .7$

$\therefore X \sim P(.7)$

$$f(x; .7) = \begin{cases} \dfrac{e^{-.7}(.7)^x}{x!} & \text{for } x = 0, 1, 2, \ldots \\ 0 & \text{o.w.} \end{cases}$$

$$P(X \leqslant 2) = \sum_{x=0}^{2} f(x) = \sum_{x=0}^{2} \frac{e^{-.7}(.7)^x}{x!} = f(0) + f(1) + f(2)$$

في يوم الثلاثاء

$m = .7 , w = 1 (week)$

$m = w.\lambda$

$.7 = 1.\lambda \Rightarrow \lambda = .7$

$w =$ Tuesday ([illegible])

$w = \frac{1}{7} \rightarrow$

$m = \frac{1}{7} \cdot \frac{7}{10} = .7 = \frac{1}{}$

$X \sim P(.1)$

Example ⑦ A machine produces a certain item with (0.05) defectives. We select a sample of (100) item. Let the r.v. X be the number of defective. What is the pr. that no defective?

ماكنة تنتج معين عيوب

sol.

$X \equiv$ number of defective in w.

عدد المعيوب

$p = 0.05$; $n = 100$

$m = \lambda . w$
$= (100).(0.05) = 5$

$\therefore X \sim P(5)$

$$f(x; 5) = \begin{cases} \dfrac{e^{-5}5^x}{x!} & \text{for } x = 0, 1, 2, \ldots \\ 0 & \text{o.w.} \end{cases}$$

$$P(X = 0) = f(0) = \frac{e^{-5}5^0}{0!} = e^{-5}$$

* يمكن حله بطريقة أخرى

$b(n, p)$
$= b(100, 0.05)$

Example ⑧ A car salesman sells, on the average (2.5) cars perday. Use poisson dist. to determine the pr. that on a given day the salesman would sell:

بائع سيارات يبيع

a. at least (4) cars, b. exactly (4) cars.

sol. X = number of cars that salesman would sell.

Given $m = 2.5$

$\therefore X \sim P(2.5)$

$$f(x;2.5) = \begin{cases} \dfrac{e^{-2.5}(2.5)^x}{x!} & \text{for } x=0,1,2,\dots \\ 0 & \text{o.w.} \end{cases}$$

a. $P(X \geq 4) = 1 - P(X < 4) = 1 - \sum_{x=0}^{3} f(x)$

$= 1 - \sum_{x=0}^{3} \dfrac{e^{-2.5}(2.5)^x}{x!}$

$= 1 - (f(0) + f(1) + f(3))$

b. $P(X=4) = f(4) = \dfrac{e^{-2.5}(2.5)^4}{4!}$

Example ⑨ The average number of major fires per month in a certain city is (1.5). Use the Poisson dist. to determine the pr. that there will be exactly one major fire in a period of two months.

حرائق كبيرة

Sol.

عدد الحرائق الكبيرة كل شهر

$X \equiv$ number of major fires per month.

Given $m = 1.5$, $w = 1$ month

$m = \lambda \cdot w$

$1.5 = \lambda \cdot 1 \Rightarrow \lambda = 1.5$

if $w = 2$; find $P(X=1)$?

التحويل من شهر الى شهرين

$w=1 \longrightarrow w=2$

$m = \lambda \cdot w$

$= (1.5)(2) = 3 \Rightarrow m = 3$

$\therefore X \sim P(3)$

$$f(x;3) = \begin{cases} \dfrac{e^{-3}3^x}{x!} & \text{for } x=0,1,2,\dots \\ 0 & \text{o.w.} \end{cases}$$

$P(x=1) = f(1) = 3e^{-3}$.

عباس نجم سلمان
استاذ مساعد

التوزيع الأسي "Exponential Dist"

يعتبر التوزيع الأسي حالة خاصة من توزيع جاما (!!) عندما
$\alpha = 1$ و $\beta = \frac{1}{\lambda}$ حيث $(\lambda > 0)$

لذلك فان المتغير العشوائي X الذي يتبع التوزيع الأسي له دالة الكثافة الاحتمالية التالية

$$f(X|\lambda) = \begin{cases} \lambda e^{-\lambda X} & \text{for } 0 < X < \infty \\ 0 & \text{o.w} \end{cases}$$

ويرمز له بالرمز $X \sim EXP.(\lambda)$

وايضا فان $E(x) = \mu = \frac{1}{\lambda}$

$V(x) = \sigma^2 = \frac{1}{\lambda^2}$

مثال اذا كانت

$$f(x) = \begin{cases} 4 e^{-4X} & \text{for } 0 < x < \infty \\ 0 & \text{o.w} \end{cases}$$

حدد نوع التوزيع ثم أوجد بعد ذلك $E(x)$ ، $V(x)$

Soln: اكمل من خلال معادلة هذه الدالة مع دالة التوزيع الأسي نلاحظ
انها تتبع التوزيع الأسي مع المعلمة $(\lambda = 4)$

$\therefore E(x) = \frac{1}{4}$

$V(x) = \frac{1}{4^2} = \frac{1}{16}$.

t - distribution

let w be a c.r.v. s.t. $w \sim N(0,1)$

i.e $f(w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}$ for $-\infty < w < \infty$

let v be a c.r.v. st. $v \sim \chi^2(r)$

i.e $f(v) = \frac{1}{\Gamma(\frac{r}{2}) \cdot 2^{r/2}} v^{\frac{r}{2}-1} e^{-\frac{v}{2}}$ $0 < v < \infty$

$\because$ w and v are s. Independent.

i.e $f(w,v) = f(w) \cdot f(v)$

$$\therefore \quad f(w,v) = \frac{1}{\sqrt{2\pi}\,\Gamma(\frac{r}{2})\, 2^{r/2}} v^{\frac{r}{2}-1} e^{-\left(\frac{w^2+v}{2}\right)} \quad \begin{matrix} -\infty < w < \infty \\ 0 < v < \infty \end{matrix}$$

Define $T = \frac{w}{\sqrt{\frac{v}{r}}}$

To find $g(t)$ (use transf———)

$t = \frac{w}{\sqrt{\frac{v}{r}}} = u_1(w,v)$

let $u = v = u_2(w,v)$

$w = t\sqrt{\frac{u}{r}} = u_1^{-1}(t,u)$

$v = u = u_2^{-1}(t,u)$

$$J = \begin{vmatrix} \frac{\partial w}{\partial t} & \frac{\partial w}{u} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{u} \end{vmatrix} = \begin{vmatrix} \sqrt{\frac{u}{r}} & \frac{t}{2\sqrt{ur}} \\ 0 & 1 \end{vmatrix} = \sqrt{u/r}$$

$$g(t,u) = f\left[u_1^{-1}(t,u),\ u_2^{-1}(t,u)\right] \cdot |J|$$

$$= \frac{1}{\sqrt{2\pi}\ \Gamma(\frac{r}{2})\, 2^{r/2}}\ u^{\frac{r}{2}-1}\ e^{-\left[\frac{\frac{t^2u}{r}+u}{2}\right]} \left(\sqrt{\frac{u}{r}}\right)$$

②

$$\therefore g(t,u) = \frac{1}{\sqrt{2\pi}\,\Gamma(\frac{r}{2})\,2^{r/2}}\, u^{\frac{r}{2}-1}\, e^{-\frac{\left[t^2\frac{u}{r}+u\right]}{2}} \left(\sqrt{\frac{u}{r}}\right)$$

for $-\infty < t < \infty$
$0 \leq u < \infty$

$$g_1(t) = \frac{1}{\sqrt{2\pi r}\,\Gamma(\frac{r}{2})\,2^{r/2}} \int_0^\infty u^{\frac{r-1}{2}}\, e^{-\frac{u\left[\frac{t^2}{r}+1\right]}{2}}\, du$$

let $z = \frac{u\left[\frac{t^2}{r}+1\right]}{2}$, $2z = u(\frac{t^2}{r}+1)$

$u = \frac{2z}{\frac{t^2}{r}+1}$, $du = \frac{2}{\frac{t^2}{r}+1}\,dz$

$$\therefore g_1(t) = \frac{1}{\sqrt{2\pi r}\,\Gamma(\frac{r}{2})\,2^{r/2}} \int_0^\infty \left[\frac{2z}{(\frac{t^2}{r}+1)}\right]^{\frac{r-1}{2}} e^{-z}\, \frac{2}{(\frac{t^2}{r}+1)}\,dz$$

$$\boxed{2^{-1/2}\,2^{-1/2}\,2^{1} = 1}$$

$$= \frac{2^{\cancel{\frac{r-1}{2}}}\cdot\cancel{2}}{\sqrt{\cancel{2}\pi r}\,\Gamma(\frac{r}{2})\,(\frac{t^2}{r}+1)^{\frac{r+1}{2}}\cdot \cancel{2^{r/2}}} \int_0^\infty z^{\frac{r-1}{2}}\, e^{-z}\, dz$$

$$= \frac{1}{\sqrt{\pi r}\,\Gamma(\frac{r}{2})\,(\frac{t^2}{r}+1)^{\frac{r+1}{2}}}\;\Gamma\left(\frac{r+1}{2}\right)$$

$\alpha - 1 = \frac{r-1}{2}$
$\alpha = \frac{r-1}{2} + 1$
$\alpha = \frac{r+1}{2}$
$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\, dy$

مهم

$\therefore$ if $X \sim t(r)$ [t-distribution with degree of freedom r]

then
$$f(t) = \begin{cases} \dfrac{\Gamma(\frac{r+1}{2})}{\sqrt{\pi r}\,\Gamma(\frac{r}{2})\,(\frac{t^2}{r}+1)^{\frac{r+1}{2}}} & \text{for } -\infty < t < \infty \\ 0 & \text{o.w} \end{cases}$$

$$P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$$

<u>Note</u> (r) is called degree of freedom of t-dist.

⑧ Sample of Table of t-dist. with (r) d.f.

$$P(T \leq t) = \int_{-\infty}^{t} \frac{\Gamma(\frac{r+1}{2})}{\sqrt{\pi r}\, \Gamma(\frac{r}{2})} \left[1 + \frac{x^2}{r}\right]^{-\frac{(r+1)}{2}} dx$$

d.f (r)	$P(T \leq t)$				
	0.90	0.95	0.975	0.99	0.995
1	3.078	6.314	12.706		
⋮					
3	1.638	2.353	3.182	4.541	5.841
⋮					
7	1.415	1.895	2.365	2.998	3.499
14	1.345	1.761	2.145	2.629	2.977

Example ① using the table of T-dist. Find

(i) $P(T \leq 2.353)$, $r = 3$ ANS 0.95

(ii) $P(1.415 \leq T \leq 2.998)$, $r = 7$

$P(1.415 \leq T \leq 2.998) = P(T \leq 2.998) - P(1.415)$
$= 0.99 - 0.9$
$= 0.09$ (ANS)

(iii) $P(T > 2.145)$, $r = 14$

$P(T > 2.145) = 1 - P(T \leq 2.145) = 1 - 0.975$
$= 0.025$ (ANS)

(iii) $P(T \leq -5.84)$, $r = 3$

$P(T \leq -5.84) = 1 - P(T \leq 5.84) = 1 - 0.995 = 0.005$ (ANS)

(11)

Example ② $T \sim t(r=10)$

Find $P(|T| > 2.228)$

Sol

$$P(|T| > 2.228) = 1 - P(|T| \leq 2.228)$$

$$= 1 - P(-2.228 \leq T \leq 2.228)$$

$$= 1 - [P(T \leq 2.228) - P(T \leq -2.228)]$$

$$= 1 - [P(T \leq 2.228) - (1 - P(T \leq 2.228)]$$

$$= 2 - 2P(T \leq 2.228)$$

$$= 2 - 2[.975] \longrightarrow$$ من جدول T, $r=10$

$$= 2 - [1.950]$$

$$= 0.05$$

Example ③ If $T \sim T(14)$, then find the value of b s.t. $P[-b < T < b] = 0.90$

Sol.

$$P[-b < T < b] = P[T < b] - P(T < -b)$$

$$= P(T < b) - [1 - P(T < b)]$$

$$0.90 = 2P(T < b) - 1$$

$$2P(T < b) = 1.90$$

$$P(T < b) = \frac{1.90}{2} = 0.95$$

$$\therefore b = 1.761$$ من جدول T, $r=14$

⑤ F- distribution

let u be c.r.v. s.t $u \sim \chi^2(r_1)$.
let v be c.r.v. s.t $v \sim \chi^2(r_2)$.

Suppose u and v are S. Indep.

$$\therefore \quad f(u,v) = \frac{1}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})\, 2^{\frac{r_1+r_2}{2}}} U^{\frac{r_1}{2}-1} V^{\frac{r_2}{2}-1} e^{-(\frac{u+v}{2})}$$

$$0 < u < \infty$$
$$0 < v < \infty$$

Define a r.v F s.t

$$F = \frac{U/r_1}{V/r_2}$$

وبنفس طريقة التحويلات الخطيه نجد $g(F)$

بعد ان نفرض ما يأتي

$$f = \frac{u/r_1}{v/r_2} = u_1(u,v)$$

$$Z = v = u_2(u,v)$$

$$u = \frac{r_1}{r_2} f\, z = u_1^{-1}(f,z)$$

$$v = z = u_2^{-1}(f,z)$$

$$g(f,z) = F\left[u_1^{-1}, u_2^{-1}\right] \cdot |J| \leftarrow \text{Jacobian deter.}$$

وبنفس الاسلوب السابق نستنتج

$$g_1(f) = \begin{cases} \dfrac{\left(\frac{r_1}{r_2}\right)^{r_1/2} f^{\frac{r_1}{2}-1}\, \Gamma\left(\frac{r_1+r_2}{2}\right)}{\Gamma(\frac{r_1}{2})\,\Gamma(\frac{r_2}{2})\left(1+\frac{r_1}{r_2} f\right)^{\frac{r_1+r_2}{2}}} & 0 < f < \infty \\ 0 & \text{o.w.} \end{cases}$$

$$f \sim F(r_1, r_2)$$

- أكمل الاشتقاق رجاءً -

(5)

⑥ Sample of F-dist table

$$P(F \le f) = \int_0^f f(w)\,dw$$

		r_1							
$P(F\le f)$	r_2	1	2	3	4	5	10	12	15
0.95	1	161		216		230	242		246
0.975		648		864		922	964		985
0.99		4052		5625		5764	6056		6157
	2								
		①	②		④			⑫	
0.95	3	10.1	9.55		9.12			8.74	
0.975		17.4	16.0		15.1			14.3	
0.99		34.1	30.8		28.7			27.1	

Example Find ① $P(F \le 9.12)$, $r_1 = 4$, $r_2 = 3$

② $P(F \le -14.3)$, $r_1 = 12$, $r_2 = 3$

③ $P(242 \le F \le 6056)$, $r_1 = 10$, $r_2 = 1$

Sol.

(1) $P(F \le 9.12) = 0.95$

(2) $P(F \le -14.3) = 1 - P(F \le 14.3)$, $r_1 = 12$, $r_2 = 3$

$= 1 - 0.975$

$= 0.25$

(3) $= P(242 \le F \le 6056) = P(F \le 6056) - P(F \le 242)$

$= 0.99 - 0.95$

$= 0.04$

⑦ **Exersies**

① Show that t-dist. with $r=1$ (d.f) and the Cauchy dist. are the Same

Sol.

$$g(t) = \frac{\Gamma(\frac{r+1}{2})}{\sqrt{\pi r}\ \Gamma(\frac{r}{2})\,(\frac{t^2}{r}+1)^{\frac{r+1}{2}}} \qquad -\infty < t < \infty$$

when $r=1$

$$g(t) = \frac{\Gamma(1)}{\sqrt{\pi}\ \sqrt{\pi}\ (t^2+1)} = \frac{1}{\pi(t^2+1)} \qquad -\infty < t < \infty$$

Cauchy dist.

② let $T = \frac{W}{\sqrt{V/r}}$, where $W \sim N(0,1)$, $V \sim \chi^2(r)$

Show that $T^2 \sim F(1,r)$

Sol.

$$T = \frac{W}{\sqrt{\frac{V}{r}}}$$

$$T^2 = \frac{W^2/1}{V/r} \quad , \text{ where } W^2 \sim \chi^2(1),\ V \sim \chi^2(r)$$

$$\therefore T^2 \sim F(1,r)$$

⑧

The Distribution of $\bar{X}$ and $\frac{nS^2}{\sigma^2}$

let $x_1, x_2, -, x_n$ be Random Sample of size $n \geq 2$ from a dist. have a p.d.f $f(x)$ with mean (μ) and var (σ^2), then

$E(\bar{X}) = \mu$, when $\bar{X}$ is the sample mean

$\underline{\underline{s.t}}$ $\bar{X} = \frac{\sum x_i}{n}$

$var(\bar{x}) = \frac{\sigma^2}{n}$

<u>Thm.1</u> let $x_1, x_2, -, x_n$ be a R.S. of size $n \geq 2$ from $n(\mu, \sigma^2)$, then $\bar{X} \sim n(\mu, \frac{\sigma^2}{n})$

<u>proof</u>

$$M_{\bar{X}}(t) = E[e^{t\bar{X}}] = E[e^{t(\frac{\sum x_i}{n})}]$$

$$= E[e^{\frac{t}{n}[x_1 + \cdots + x_n]}]$$

$$= E[e^{\frac{t}{n}x_1} \cdot e^{\frac{tx_2}{n}} \cdots e^{\frac{t}{n}x_n}]$$

$$= e^{n(\frac{\mu t}{n} + \frac{\sigma^2 t^2}{2n^2})}$$

$$= e^{\mu t + \frac{\sigma^2}{n}\frac{t^2}{2}}$$

$$\therefore \bar{X} \sim N(\mu, \frac{\sigma^2}{n})$$

توضيح

$X_i \sim N(\mu, \sigma^2)$

$M_{X_i}(t) = E[e^{tX_i}]$
$= e^{\mu t + \sigma^2 \frac{t^2}{2}}$

$M_{X_i}(\frac{t}{n}) = E[e^{\frac{t}{n}X_i}]$
$= e^{\frac{\mu t}{n} + \frac{\sigma^2}{n^2}\frac{t^2}{2}}$
$\uparrow$ $\boxed{n^2}$

<u>Note</u> If $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

then $Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \sim N(0,1)$

ex. let $x_1, \text{---}, x_{25}$ be a r.s from $N(\overset{\mu}{71}, \overset{\sigma^2}{100})$, then

Find $p(70 < \bar{x} < 72)$

Sol

$$p(70 < \bar{x} < 72) = p\left(\frac{70-71}{10/\sqrt{25}} < Z < \frac{72-71}{10/\sqrt{25}}\right)$$

$$= P(-0.5 < Z < 0.5)$$

$$= P(Z \leq 0.5) - P(Z \leq -0.5)$$

$$= 2N(0.5) - 1$$

$$= 2(0.6911) - 1$$

$$= 0.382$$

Thm-2 let $x_1, x_2, \text{---}, x_n$ be a r.s. of Size $n \geq 2$ from $N(\mu, \sigma^2)$, then

$$\frac{nS^2}{\sigma^2} \sim \chi^2_{(n-1)} \quad , \text{ where } S^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}$$

proof

$$\sum_{i=1}^{n}(x_i-\mu)^2 = \sum [(x_i-\bar{x}) + (\bar{x}-\mu)]^2$$

$$= \sum_{i=1}^{n}[(x_i-\bar{x})^2 + 2(\bar{x}-\mu)(x_i-\bar{x}) + (\bar{x}-\mu)^2]$$

$$= \sum_{i=1}^{n}(x_i-\bar{x})^2 + 2(\bar{x}-\mu)\boxed{\sum_{i=1}^{n}(x_i-\bar{x})} + n(\bar{x}-\mu)^2$$

$$= \sum_{i=1}^{n}(x_i-\bar{x})^2 + n(\bar{x}-\mu)^2 \qquad \div(n)$$

$$\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n} = S^2 + (\bar{x}-\mu)^2 \qquad \times\left(\frac{n}{\sigma^2}\right)$$

$$\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{\sigma^2} = \frac{nS^2}{\sigma^2} + \frac{(\bar{x}-\mu)^2}{\sigma^2/n}$$

Since $\bar{x}$ and S^2 are S. Indep.

and $\frac{\sum(x_i-\mu)^2}{\sigma^2} \sim \chi^2_{(n)}$, $\frac{(\bar{x}-\mu)^2}{\sigma^2/n} \sim \chi^2_{(1)}$

then $\swarrow$

$$\underbrace{E\left[e^{\frac{t\Sigma(x_i-\mu)^2}{\sigma^2}}\right]}_{\substack{m.g.f.f \\ \frac{\Sigma(x_i-\mu)^2}{\sigma^2} \\ \parallel \\ (1-2t)^{-\frac{n}{2}} \\ \sim\chi^2(n)}} = E\left[e^{t\left[\frac{(\bar{x}-\mu)^2}{\sigma^2/n}+\frac{ns^2}{\sigma^2}\right]}\right]$$

$$= \underbrace{E\left[e^{\frac{t(\bar{x}-\mu)^2}{\sigma^2/n}}\right]}_{\substack{m.g.f.f\,\frac{(\bar{x}-\mu)^2}{\sigma^2/n} \\ \parallel \\ (1-2t)^{-\frac{1}{2}} \\ \sim\chi^2(1)}} \cdot \underbrace{E\left[e^{t\left(\frac{ns^2}{\sigma^2}\right)}\right]}_{m.g.f.f\left(\frac{ns^2}{\sigma^2}\right)}$$

$$\therefore m.g.f.\left(\frac{ns^2}{\sigma^2}\right) = \frac{(1-2t)^{-\frac{n}{2}}}{(1-2t)^{-1/2}} = (1-2t)^{-\frac{(n-1)}{2}}$$

or $M_{\frac{ns^2}{\sigma^2}}(t)$

$$\therefore \frac{ns^2}{\sigma^2} \sim \chi^2(n-1)$$

ex. Let S^2 be the variance of the r.s. of size(6) from $N(\mu,12)$, Find $P(2.30 < S^2 < 22.2)$

Sol.

$$\frac{ns^2}{\sigma^2} \sim \chi^2(n-1)$$

$$\frac{6s^2}{12} \sim \chi^2(5)$$

i.e $\frac{s^2}{2} \sim \chi^2(5)$

$$P\left(\frac{2.30}{2} < \frac{s^2}{2} < \frac{22.2}{2}\right) = P(1.15 < w < 11.1)$$

$\boxed{W \sim \chi^2(5)}$

$$= P(w<11.1) - P(w<1.15)$$
$$= 0.95 - 0.05$$
$$= 0.90$$

من جدول chi-squar with r=5

⑪ <u>Thm</u>, $\because \frac{ns^2}{\sigma^2} \sim \chi^2_{(n-1)}$

then

① $E(\frac{ns^2}{\sigma^2}) = (n-1)$

$\therefore \frac{n}{\sigma^2} E(s^2) = (n-1)$

$\therefore E(s^2) = \frac{(n-1)}{n} \sigma^2$

② $Var(\frac{ns^2}{\sigma^2}) = 2(n-1)$

$\therefore \frac{n^2}{\sigma^4} Var(s^2) = 2(n-1)$

$Var(s^2) = \frac{2(n-1)}{n^2} \sigma^4$

Random Sampling
(Ch: 7)

Remark:

① If $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ then $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

② If $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ then $\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2 \sim \chi^2(1)$

③ If $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ then $\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma/\sqrt{n}}\right)^2 \sim \chi^2(n)$

④ $W = \frac{nS^2}{\sigma^2} \sim \chi^2(n-1)$

⑤ If $W \sim N(0,1)$, $V \sim \chi^2(r)$

$$T = \frac{W}{\sqrt{\frac{V}{r}}} \sim t(r)$$

⑥ $U \sim \chi^2(r_1)$, $V \sim \chi^2(r_2)$

$$F = \frac{U/r_1}{V/r_2} \sim f(r_1, r_2)$$

⑦ $T^2 = \frac{W^2}{V/r} \sim F(1, r)$

⑧ $t(r=1) \sim$ Cauchy dist.

Example (1) Let $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$ be the mean of a R.S. of size $n=16$ from $N(\mu, \sigma^2)$ and $\sigma^2 = 25$, $\mu = 0$; find:

Sol. $P(\bar{X} < 0)$; $\bar{X} \sim N(0, \frac{25}{16})$.

$$P(\bar{X} < 0) = P\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < \frac{0-\mu}{5/\sqrt{16}}\right) \quad ; \mu = 0$$

$$= P(Z < 0) = N(0) = 0.5$$

Example (2): Let $\overline{X}$ be the mean of a R.S. of size 5 from a normal distribution with $\mu=0$ and $\sigma^2=125$. Determine C so that $P(\overline{X}<C)=0.90$.

Sol.

$$\overline{X}\sim N\left(0,\frac{125}{5}\right) \quad ; n=5 , \mu=0, \sigma^2=125$$

$$Z=\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\sim N(0,1)$$

$$Z=\frac{\overline{X}-0}{\sqrt{125}/\sqrt{5}}=\frac{\overline{X}-0}{5\sqrt{5}/\sqrt{5}}$$

$$P(\overline{X}<C)=0.90$$

$$P\left(\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}<\frac{C-\mu}{\sigma/\sqrt{n}}\right)=0.90$$

$$P\left(\frac{\overline{X}-0}{5\sqrt{5}/\sqrt{5}}<\frac{C-0}{5\sqrt{5}/\sqrt{5}}\right)=0.90$$

$$P\left(Z<\frac{C}{5}\right)=0.90$$

$$N\left(\frac{C}{5}\right)=0.90$$

$$\frac{C}{5}=1.282 \Rightarrow C=(1.282)(5)=6.41$$

Example (3):

If $\overline{X}$ is the mean of a R.S of size n from $N(\mu,100)$ find n s.t. $P(\mu-5<\overline{X}<\mu+5)=0.954$.

Sol. $P(\mu-5<\overline{X}<\mu+5)=0.954$

$$P\left(\frac{-5}{10/\sqrt{n}}<\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}<\frac{5}{10/\sqrt{n}}\right)=0.954 \quad ; Z\sim N(0,1)$$

$$N\left(\frac{5}{10/\sqrt{n}}\right)-\left[1-N\left(\frac{5}{10/\sqrt{n}}\right)\right]=0.954$$

$$2N\left(\frac{5}{10/\sqrt{n}}\right)-1=0.954$$

$$N\left(\frac{5}{10/\sqrt{n}}\right)=\frac{0.954+1}{2}=\frac{1.954}{2}=0.977$$

$\therefore n = 16$; since :

$$N\left(\frac{5}{10/\sqrt{n}}\right) = 0.977 \Rightarrow \frac{5}{10/\sqrt{n}} = 2.00 \quad \left(\text{by Table}\right)$$

$$\Rightarrow \frac{\sqrt{n}}{2} = 2 \Rightarrow \sqrt{n} = 4 \Rightarrow \boxed{n=16}$$

Examp(4): Let S^2 be the variance of a R.S of size $n=6$ from the normal $N(\mu, 12)$. Find $P(2.30 < S^2 < 22.2)$.

Sol.

$$P(2.30 < S^2 < 22.2) = P\left(\frac{2.30n}{\sigma^2} < \frac{nS^2}{\sigma^2} < \frac{22.2n}{\sigma^2}\right)$$

$$\boxed{\begin{matrix} n = 6 \\ \sigma^2 = 12 \end{matrix}}$$

$$= P\left(\frac{2.30(6)}{12} < W < \frac{22.2(6)}{12}\right)$$

$$= P(1.15 < W < 11.1)$$

$$\overset{\text{by table}}{=} P(W < 11.1) - P(W < 1.15)$$

$$= 0.950 - 0.050$$

$$= 0.9$$

$$\boxed{\begin{matrix} W \sim \chi^2(n-1) \\ W \sim \chi^2(5) \end{matrix}}$$

Example(5): Let $\overline{X}$ and S^2 be the mean and the variance of a R.S of size (25) from $N(3, 100)$; find :

① $P(0 < \overline{X} < 6)$ ② $P(55.2 < S^2 < 145.6)$

$$\boxed{\begin{matrix} \mu = 3 \\ \sigma^2 = 100 \\ n = 25 ,\ \sqrt{n} = 5 \\ Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1) \end{matrix}}$$

Sol.

① $$P(0 < \overline{X} < 6) = P\left(\frac{0-3}{10/5} < \frac{\overline{X}-3}{10/5} < \frac{6-3}{10/5}\right)$$

$$= P(-0.6 < Z < 0.6)$$

$$= P(Z < 0.6) - P(Z < -0.6)$$

$$= P(Z < 0.6) - [1 - P(Z < 0.6)]$$

$$= 2P(Z < 0.6) - 1$$

$$= 2N(0.6) - 1$$

$$= 2(0.726) - 1 = \quad = 0.452$$

by table of $N(0,1)$ ← 0.726

② $P(55.2 < S^2 < 145.6)$

$$= P\left(\frac{55.2n}{\sigma^2} < \frac{nS^2}{\sigma^2} < \frac{145.6n}{\sigma^2}\right)$$

$$= P\left(\frac{55.2(25)}{100} < W < \frac{145.6(25)}{100}\right)$$

$$= P(13.8 < W < 36.4)$$

$$= P(W < 36.4) - P(W < 13.8)$$

$$= 0.95 - 0.05$$

$$= 0.9$$

Table of Chi-square

by $W \sim \chi^2(n-1)$

$\Rightarrow W \sim \chi^2(24)$

(by Table)

Note

Tables of Distributions

$Z = \frac{X-\mu}{\sigma} \sim N(0,1) \Rightarrow$ table of S.N. $N(0,1)$

$Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1) \Rightarrow$ table of S.N. $N(0,1)$

$W = \frac{nS^2}{\sigma^2} \sim \chi^2(n-1) \Rightarrow$ table of Chi-square with d.f. $r = n-1$

$T = \frac{W}{\sqrt{\frac{V}{r}}} \sim t(r) \Rightarrow$ table of T-dist. with d.f. r

$F = \frac{W/r_1}{V/r_2} \sim f(r_1, r_2) \Rightarrow$ table of F-dist with d.f. r_1, r_2

$Z^2 = \left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2 \sim \chi^2(1)$

$\Rightarrow$ table of chi-square with d.f $r=1$

$\sum_{i=1}^{n}\left(\frac{\bar{X}_i-\mu}{\sigma/\sqrt{n}}\right)^2 \sim \chi^2(n) \Rightarrow$ table of Chi-square with d.f. $r=n$

TABLE IV

*The t Distribution**

$$\Pr(T \leq t) = \int_{-\infty}^{t} \frac{\Gamma[(r+1)/2]}{\sqrt{\pi r}\,\Gamma(r/2)(1 + w^2/r)^{(r+1)/2}}\,dw$$

$$[\Pr(T \leq -t) = 1 - \Pr(T \leq t)]$$

	Pr $(T \leq t)$				
r	0.90	0.95	0.975	0.99	0.995
1	3.078	6.314	12.706	31.821	63.657
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947
16	1.337	1.746	2.120	2.583	2.921
17	1.333	1.740	2.110	2.567	2.898
18	1.330	1.734	2.101	2.552	2.878
19	1.328	1.729	2.093	2.539	2.861
20	1.325	1.725	2.086	2.528	2.845
21	1.323	1.721	2.080	2.518	2.831
22	1.321	1.717	2.074	2.508	2.819
23	1.319	1.714	2.069	2.500	2.807
24	1.318	1.711	2.064	2.492	2.797
25	1.316	1.708	2.060	2.485	2.787
26	1.315	1.706	2.056	2.479	2.779
27	1.314	1.703	2.052	2.473	2.771
28	1.313	1.701	2.048	2.467	2.763
29	1.311	1.699	2.045	2.462	2.756
30	1.310	1.697	2.042	2.457	2.750

---

* This table is abridged from Table III of Fisher and Yates; *Statistical Tables for Biological, Agricultural, and Medical Research,* published by Oliver and Boyd, Ltd., Edinburgh, by permission of the authors and publishers.

TABLE V

The F Distribution*

$$\Pr(F \le f) = \int_0^f \frac{\Gamma[(r_1 + r_2)/2](r_1/r_2)^{r_1/2} w^{r_1/2-1}}{\Gamma(r_1/2)\Gamma(r_2/2)(1 + r_1 w/r_2)^{(r_1+r_2)/2}}\, dw$$

Pr(F≤f)	r_2	r_1											
		1	2	3	4	5	6	7	8	9	10	12	15
0.95	1	161	200	216	225	230	234	237	239	241	242	244	246
0.975		648	800	864	900	922	937	948	957	963	969	977	985
0.99		4052	4999	5403	5625	5764	5859	5928	5982	6023	6056	6106	6157
0.95	2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4	19.4	19.4
0.975		38.5	39.0	39.2	39.2	39.3	39.3	39.4	39.4	39.4	39.4	39.4	39.4
0.99		98.5	99.0	99.2	99.2	99.3	99.3	99.4	99.4	99.4	99.4	99.4	99.4
0.95	3	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70
0.975		17.4	16.0	15.4	15.1	14.9	14.7	14.6	14.5	14.5	14.4	14.3	14.3
0.99		34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.3	27.2	27.1	26.9
0.95	4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86
0.975		12.2	10.6	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.75	8.66
0.99		21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.7	14.5	14.4	14.2
0.95	5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62
0.975		10.0	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.52	6.43
0.99		16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2	10.1	9.89	9.72
0.95	6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94
0.975		8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.37	5.27
0.99		13.7	10.9	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56
0.95	7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51
0.975		8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.67	4.57
0.99		12.2	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31
0.95	8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22
0.975		7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.20	4.10
0.99		11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52
0.95	9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01
0.975		7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.87	3.77
0.99		10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96
0.95	10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85
0.975		6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.62	3.52
0.99		10.0	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56
0.95	12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62
0.975		6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.28	3.18
0.99		9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01
0.95	15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40
0.975		6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	2.96	2.86
0.99		8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52

---

* This table is abridged and adapted from "Tables of Percentage Points of the Inverted Beta Distribution," *Biometrika*, 33 (1943). It is published here with the kind permission of Professor E. S. Pearson on behalf of the authors, Maxine Merrington and Catherine M. Thompson, and of the *Biometrika* Trustees.

TABLE III

The Normal Distribution

$$\Pr(X \leq x) = N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw$$

$$[N(-x) = 1 - N(x)]$$

x	$N(x)$	x	$N(x)$	x	$N(x)$
0.00	0.500	1.10	0.864	2.05	0.980
0.05	0.520	1.15	0.875	2.10	0.982
0.10	0.540	1.20	0.885	2.15	0.984
0.15	0.560	1.25	0.894	2.20	0.986
0.20	0.579	1.282	0.900	2.25	0.988
0.25	0.599	1.30	0.903	2.30	0.989
0.30	0.618	1.35	0.911	2.326	0.990
0.35	0.637	1.40	0.919	2.35	0.991
0.40	0.655	1.45	0.926	2.40	0.992
0.45	0.674	1.50	0.933	2.45	0.993
0.50	0.691	1.55	0.939	2.50	0.994
0.55	0.709	1.60	0.945	2.55	0.995
0.60	0.726	1.645	0.950	2.576	0.995
0.65	0.742	1.65	0.951	2.60	0.995
0.70	0.758	1.70	0.955	2.65	0.996
0.75	0.773	1.75	0.960	2.70	0.997
0.80	0.788	1.80	0.964	2.75	0.997
0.85	0.802	1.85	0.968	2.80	0.997
0.90	0.816	1.90	0.971	2.85	0.998
0.95	0.829	1.95	0.974	2.90	0.998
1.00	0.841	1.960	0.975	2.95	0.998
1.05	0.853	2.00	0.977	3.00	0.999

TABLE II

*The Chi-Square Distribution**

$$\Pr(X \leq x) = \int_0^x \frac{1}{\Gamma(r/2)2^{r/2}} w^{r/2-1} e^{-w/2} \, dw$$

r	Pr $(X \leq x)$ 0.01	0.025	0.050	0.95	0.975	0.99
1	0.000	0.001	0.004	3.84	5.02	6.63
2	0.020	0.051	0.103	5.99	7.38	9.21
3	0.115	0.216	0.352	7.81	9.35	11.3
4	0.297	0.484	0.711	9.49	11.1	13.3
5	0.554	0.831	1.15	11.1	12.8	15.1
6	0.872	1.24	1.64	12.6	14.4	16.8
7	1.24	1.69	2.17	14.1	16.0	18.5
8	1.65	2.18	2.73	15.5	17.5	20.1
9	2.09	2.70	3.33	16.9	19.0	21.7
10	2.56	3.25	3.94	18.3	20.5	23.2
11	3.05	3.82	4.57	19.7	21.9	24.7
12	3.57	4.40	5.23	21.0	23.3	26.2
13	4.11	5.01	5.89	22.4	24.7	27.7
14	4.66	5.63	6.57	23.7	26.1	29.1
15	5.23	6.26	7.26	25.0	27.5	30.6
16	5.81	6.91	7.96	26.3	28.8	32.0
17	6.41	7.56	8.67	27.6	30.2	33.4
18	7.01	8.23	9.39	28.9	31.5	34.8
19	7.63	8.91	10.1	30.1	32.9	36.2
20	8.26	9.59	10.9	31.4	34.2	37.6
21	8.90	10.3	11.6	32.7	35.5	38.9
22	9.54	11.0	12.3	33.9	36.8	40.3
23	10.2	11.7	13.1	35.2	38.1	41.6
24	10.9	12.4	13.8	36.4	39.4	43.0
25	11.5	13.1	14.6	37.7	40.6	44.3
26	12.2	13.8	15.4	38.9	41.9	45.6
27	12.9	14.6	16.2	40.1	43.2	47.0
28	13.6	15.3	16.9	41.3	44.5	48.3
29	14.3	16.0	17.7	42.6	45.7	49.6
30	15.0	16.8	18.5	43.8	47.0	50.9

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* This table is abridged and adapted from "Tables of Percentage Points of the Incomplete Beta Function and of the Chi-Square Distribution," *Biometrika*, 32 (1941). It is published here with the kind permission of Professor E. S. Pearson on behalf of the author, Catherine M. Thompson, and of the Biometrika Trustees.

④ Beta Distribution (توزيع بيتا)

A c.r.v. X have a Beta dist. it's p.d.f. has the following form:

$$f(x;a,b) = \begin{cases} \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} & \text{for } 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

where $a>0$ & $b>0$; denoted by $X \sim B(a,b)$

And $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

The M.g.f. of $X \sim B(a,b)$ is:

$$M_X(t) = \sum_{x=0}^{\infty} \frac{t^n}{n!} \cdot \frac{B(a+n,b)}{B(a,b)}$$

$$E(X) = \frac{a}{a+b}, \quad Var(x) = \frac{ab}{(a+b)^2(a+b+1)}$$

In particular:

1. If $a=b=1$, then Beta dist. reduces to uniform dist. on $(0,1)$:

$$f(x) = \begin{cases} 1 & \text{for } 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

Since $f(x, a=1, b=1) = \begin{cases} \frac{1}{B(1,1)} x^{1-1}(1-x)^{1-1} & 0<x<1 \\ 0 & \text{o.w.} \end{cases}$

$$= \begin{cases} 1 & 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

& $B(1,1) = \frac{\Gamma(1)\Gamma(1)}{\Gamma(2)} = 1 \Rightarrow \therefore X \sim B(1,1)$

$\Rightarrow X \sim \text{unif.}(0,1)$

2. If $a=1$ & $b=2$ then Beta dist. reduces to dist. which is known as triangular dist.:

$$f(x) = \begin{cases} 2(1-x) & 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

Theorem: If $X \sim P(m)$ & if $x_0 =$ mode of X, then

$$x_0 = \begin{cases} m, m-1 & \text{if } m \in I^+ \\ [m] & \text{if } m \notin I^+ \end{cases}$$

Ex. $X \sim P(5)$ then mode $= X = 5, 4$.

$X \sim P(2.6)$ then mode $= X = 2$.

Example The نسبة percentage of betters who make money نقود of the race on a given day has Beta distribution with parameters $a=1$ and $b=5$. Find the pr. that fewer اقل than ten percent نسبة مئوية comment winners الفائزين on given day?

Sol. $P(X < .1)$?

$$P(X < .1) = \int_0^{.1} \frac{\Gamma(1+5)}{\Gamma(1)\Gamma(5)} \cdot x^0 (1-x)^4 \, dx$$

$$= \frac{5!}{0!\,4!} \int_0^{.1} (1-x)^4 \, dx$$

$$= -5 \frac{(1-x)^5}{5} \Big|_0^{.1} = (.9)^5 .$$

عباس نجم سلمان
استاذ مساعد

Chapter Six

⑦ Uniform Distribution

when X is d.r.v.

$X \sim unif(K)$

$x = 1, 2, \dots, K$

$$f(x) = \begin{cases} \frac{1}{K} & \text{for } x = 1, 2, \dots, K \\ 0 & \text{o.w.} \end{cases}$$

$$E(X) = \sum_{\forall x} x \, f(x)$$

$$= \sum_{x=1}^{K} x \left(\frac{1}{K}\right)$$

$$= \frac{1}{\cancel{K}} \left[\frac{\cancel{K}(K+1)}{2} \right] \quad \text{when} \left(\sum_{X=1}^{K} x = \frac{K(K+1)}{2} \right)$$

$$= \left(\frac{K+1}{2} \right)$$

$$V(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = \sum_{x=1}^{K} x^2 f(x)$$

$$= \left(\sum_{x=1}^{K} x^2\right)\left(\frac{1}{K}\right)$$

$$= \frac{1}{\cancel{K}} \left[\frac{\cancel{K}(K+1)(2K+1)}{6} \right] ; \left[\sum_{x=1}^{K} x^2 = \frac{K(K+1)(2K+1)}{6} \right]$$

$$= \frac{(K+1)(2K+1)}{6},$$

$$Var(X) = E(X^2) - (E(X))^2 \geq 0$$

$$Var(X) = \frac{(K+1)(2K+1)}{6} - \frac{(K+1)}{2}$$

$$= \frac{(K+1)(K-1)}{3}$$

Ch.6 ملحق

تحويل التوزيع ال<u>طبيعي</u> القياسي الى $\chi^2(1)$

<u>Th.</u> $X \sim N(0,1)$

$y = X^2 \sim \chi^2(1)$

(Using Moment Teqnique)

<u>Proof.</u> $X \sim N(0,1)$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$M_y(t) = E(e^{ty})$$

$$= E(e^{tx^2})$$

$$= \int_{-\infty}^{\infty} e^{tx^2} f(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx^2} e^{-\frac{x^2}{2}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2(1-2t)}{2}}\,dx$$

Let $w = \sqrt{1-2t}\; X$

$dw = \sqrt{1-2t}\; dX$

$$\therefore\ M_y(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{w^2}{2}} \frac{dw}{\sqrt{1-2t}}$$

$$= \frac{1}{\sqrt{1-2t}} \boxed{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{w^2}{2}} dw}$$

since $(w \sim N(0,1))$ $= 1$

$$M_y(t) = (1-2t)^{-1}$$

$$\therefore\ Y \sim \chi^2(1)$$

Note $X \sim N(\mu, \sigma^2)$

Then $\Rightarrow$ ① $Z \sim N(0,1)$

② $Z^2 \sim \chi^2(1)$

③ $\sum_{i=1}^{n} Z^2 \sim \chi^2(n$