❖ Particular integral (P.I.) of homogeneous linear partial differential equation

When $f(x,y) \neq 0$in the equation (3) which it's$F(D_x,D_y)z = f(x,y)$ multiplying (3) by the inverse operator$\frac{1}{F(D_x,D_y)}$ of the operator $F(D_x,D_y)$ to have

$$\frac{1}{F(D_x,D_y)} \cdot F(D_x,D_y)\, z = \frac{1}{F(D_x,D_y)}\, f(x,y)$$

$$\rightarrow \boxed{z = \frac{1}{F(D_x,D_y)}\, f(x,y)} \quad \ldots\ldots\ldots (11)$$

Which it's the particular integral (P.I.)

The operator $F(D_x,D_y)$ can be written as

$$F(D_x,D_y) = (D_x - m_1 D_y)(D_x - m_2 D_y) \ldots (D_x - m_n D_y) \quad \ldots.(12)$$

Substituting (12) in (11) :

$$z = \frac{1}{(D_x - m_1 D_y)(D_x - m_2 D_y) \ldots (D_x - m_n D_y)}\, f(x,y) \quad \ldots\ldots\ldots\ldots(13)$$

Taking $u_1 = \frac{1}{(D_x - m_n D_y)}\, f(x,y)$

$$\therefore\ (D_x - m_n D_y)u_1 = f(x,y)$$

This equation can be solved by Lagrange's method .

The Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-m_n} = \frac{du_1}{f(x,y)} \quad \ldots\ldots\ldots..(14)$$

Taking the first two fractions of (14)

$$m_n dx + dy = 0 \quad \rightarrow \boxed{m_n x + y = a} \quad \ldots\ldots\ldots.(15)$$

Taking the first and third fractions of (14)

$$dx = \frac{du_1}{f(x,y)} \quad \rightarrow \boxed{f(x,y)dx = du_1} \quad \ldots\ldots\ldots.(16)$$

Substituting (15) in (16) we have

$$f(x, a - m_n x)dx = du_1$$

Integrating the last one we have

$$u_1 = \int f(x, a - m_n x)dx + b$$

Let $b = 0$, then we have u_1

By the same way , we take

$$u_2 = \frac{1}{D_x - m_{n-1} D_y} u_1$$

And solve it by Lagrange's method to get u_2 , then continue in this way until we get to

$$z = u_n = \frac{1}{D_x - m_1 D_y} u_{n-1}$$

And by solving this equation we get the particular integral (P.I.)

Ex.1: solve $\left(\boldsymbol{D_x^2 - D_y^2}\right)\boldsymbol{z} = \mathbf{sec}^2(\boldsymbol{x + y})$

Sol. Firstly, we will find the general solution of

$\left(D_x^2 - D_y^2\right)z = 0$ ……….(1)

The A. E. is $m^2 - 1 = 0 \rightarrow m^2 = 1 \rightarrow m = \pm 1$

$$\therefore \quad m_1 = 1 \,, \; m_2 = -1$$

$\therefore \; z = \emptyset_1(y + x) + \emptyset_2(y - x)$ ………….(2)

Where $\emptyset_1 , \emptyset_2$ are arbitrary functions.

Second, we will find the particular integral as follows

$$z_2 = \frac{1}{D_x^2 - D_y^2} \sec^2(x + y)$$

$$= \frac{1}{(D_x - D_y)(D_x + D_y)} \sec^2(x+y)$$

Let $u_1 = \frac{1}{(D_x + D_y)} \sec^2(x+y)$

$$(D_x + D_y)u_1 = \sec^2(x+y)$$

The Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du_1}{\sec^2(x+y)}$$

Taking the first two fractions

$dx = dy \quad \rightarrow \quad x - y = a$ ……….(3)

Taking the first and third fractions

$dx = \frac{du_1}{\sec^2(x+y)} \quad \rightarrow \quad \sec^2(x+y)\, dx = du_1$ ……….(4)

Substituting (3) in (4), we have

$\sec^2(2x - a)\, dx = du_1$ …………(5)

Integrating (5), we have

$$u_1 = \frac{1}{2}\tan(2x - a) + b$$

Let $b = 0$ and replacing a , we get

$u_1 = \frac{1}{2}\tan(x+y)$ ……..(6)

Putting (6) in z_2

$$z_2 = \frac{1}{(D_x - D_y)} \cdot \frac{1}{2}\tan(x+y)$$

$$\rightarrow (D_x - D_y)z_2 = \frac{1}{2}\tan(x+y)$$

The Lagrange's auxiliary equation are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz_2}{\frac{1}{2}\tan(x+y)}$$

Taking the first two fractions

$dx = -dy \;\; \rightarrow \;\; x + y = a \quad$ ……….(7)

Taking the first and third fractions

$$dx = \frac{dz_2}{\frac{1}{2}\tan(x+y)}$$

$\frac{1}{2}\tan(x+y)\, dx = dz_2 \qquad\qquad$ ……..(8)

Substituting (7) in (8)

$\frac{1}{2}\tan a \;\, dx = dz_2 \qquad\qquad$ ………(9)

Integrating (9) , we get

$$\frac{1}{2}x\tan a \;= z_2 + b$$

Let $b = 0$, and replacing a from (7) we get the particular integral

$z_2 = \frac{1}{2}x\tan(x+y)$ ……………(10)

Hence the required general solution is

$$z = z_1 + z_2$$

$$= \emptyset_1(y+x) + \emptyset_2(y-x) + \frac{x}{2}\tan(x+y) \quad \text{……….(11)}$$

Short methods of finding the P.I. in certain cases :

Case 1 **When $\boldsymbol{f(x,y) = e^{ax+by}}$ where $\boldsymbol{a}$ and $\boldsymbol{b}$ are arbitrary constants**

To find the P.I. when $F(a,b) \neq 0$, we derive $f(x,y)$ for x any y n times:

$$D_x e^{ax+by} = ae^{ax+by}$$

$$D_x^2 e^{ax+by} = a^2 e^{ax+by}$$

$$\vdots$$

$$D_x^n e^{ax+by} = a^n e^{ax+by}$$

$$D_y e^{ax+by} = be^{ax+by}$$

$$D_y^2 e^{ax+by} = b^2 e^{ax+by}$$

$$\vdots$$

$$D_y^n e^{ax+by} = b^n e^{ax+by}$$

$D_x^r D_y^s e^{ax+by} = a^r b^s e^{ax+by}$ where $r + s = n$

So

$$F(D_x, D_y)e^{ax+by} = F(a,b)e^{ax+by}$$

Multiplying both sides by $\frac{1}{F(D_x,D_y)}$, we get

$$e^{ax+by} = \frac{1}{F(D_x, D_y)} F(a,b)e^{ax+by}$$

Since $F(a,b) \neq 0$, then we can divide on it :

$\frac{1}{F(a,b)} e^{ax+by} = \frac{1}{F(D_x,D_y)} e^{ax+by}$ ....*

Which it is equal to z , then the P. I. is

$\boxed{z = \frac{1}{F(D_x,D_y)} e^{ax+by} = \frac{1}{F(a,b)} e^{ax+by}}$, where $F(a,b) \neq 0$

when $F(a,b) = 0$, then analyze $F(D_x, D_y)$ as follows

$$F(D_x, D_y) = (D_x - \frac{a}{b}D_y)^r G(D_x, D_y)$$

Where $G(a,b) \neq 0$, we get

$$z = \frac{1}{F(D_x, D_y)} e^{ax+by} = \frac{1}{(D_x - \frac{a}{b}D_y)^r G(D_x, D_y)} e^{ax+by}$$

$$= \frac{1}{(D_x - \frac{a}{b}D_y)^r} \cdot \frac{1}{G(a,b)} e^{ax+by} \quad \text{from } *$$

Since $G(a,b) \neq 0$

$$= \frac{1}{G(a,b)} \cdot \frac{1}{(D_x - \frac{a}{b}D_y)^r} e^{ax+by}$$

Then by Lagrange's method r times , we get

$$\boxed{z = \frac{1}{F(D_x, D_y)} e^{ax+by} = \frac{1}{G(a,b)} \cdot \frac{x^r}{r!} e^{ax+by}}$$

Which it's the P.I. where $F(a,b) = 0$, $G(a,b) \neq 0$

<u>Ex.2</u>: Solve $\boldsymbol{(D_x^2 - D_x D_y - 6D_y^2)z = e^{2x-3y}}$

Sol.

1) To find the general solution

The A.E. of the given equation is

$$m^2 - m - 6 = 0 \quad \rightarrow \quad (m-3)(m+2) = 0$$

$$\therefore \quad m_1 = 3 \ , \qquad m_2 = -2$$

$\therefore \quad z_1 = \emptyset_1(y+3x) + \emptyset_2(y-2x)$

Where $\emptyset_1$and$\emptyset_2$are arbitrary functions

2) To find the particular Integral (P.I.)

$$a = 2 \ , b = -3$$

$$F(a,b) = a^2 - ab - 6b^2$$

$F(2,-3) = 4 + 6 - 54 = -44 \neq 0$

$$z_2 = \frac{1}{F(a,b)} e^{ax+by} = \frac{1}{-44} e^{2x-3y}$$

$$\therefore \ z = z_1 + z_2$$

$$= \emptyset_1(y+3x) + \emptyset_2(y-2x) - \frac{1}{44} e^{2x-3y}$$

<u>Ex.3</u>: Solve $\left(D_x^2 - D_x D_y - 6D_y^2\right) z = e^{3x+y}$

Sol.

1) The general solution is similar to that in Ex.2
2) To find P.I.

$$a = 3 \ , b = 1$$

$$F(a,b) = a^2 - ab - 6b^2$$

$F(3,1) = 9 - 3 - 6 = 0$,

analyze $F(D_x, D_y)$,$F(D_x, D_y) = D_x^2 - D_x D_y - 6D_y^2$

$$= (D_x - 3D_y)(D_x + 2D_y)$$

$(D_x - \frac{a}{b} D_y)^r \rightarrow \therefore r = 1$, $3 + 2 = 5 \neq 0 = G$

$$z_2 = \frac{1}{G(a,b)} \cdot \frac{x^r}{r!} e^{ax+by} = \frac{1}{5} \cdot \frac{x}{1} e^{3x+y} = \frac{x}{5} e^{3x+y}$$

$$\therefore \ z = z_1 + z_2$$

$$= \emptyset_1(y+3x) + \emptyset_2(y-2x) + \frac{x}{5} e^{3x+y}$$

Where $\emptyset_1$and$\emptyset_2$are arbitrary functions