B) When the equation contains the term (qy) or its' powers we use the hypothesis $\boxed{Y = \ln y}$

as follows:

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial y}.\frac{\partial Y}{\partial Y} = \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y} = \frac{\partial z}{\partial Y}.\frac{1}{y}$ (since $Y = \ln y \implies \frac{\partial Y}{\partial y} = \frac{1}{y}$)

$$\implies qy = \frac{\partial z}{\partial Y}$$

Then solving by the same way in (A).

Example 3: Solve $2p + qy = 4$ by hypotheses

Sol. Given that $2p + qy = 4$ …(10)

from $Y = \ln y$ we have $qy = \frac{\partial z}{\partial Y}$ …(11)

Substituting (11) in (10), we get

$$2p + \frac{\partial z}{\partial Y} = 4$$

Let $\frac{\partial z}{\partial Y} = t$ then,

$2p + t = 4$ …(12)

The equation (12) is of the form $f(p, t) = 0$

Then let $p = a$ and $t = b$, putting in (12) $2a + b = 4$

$\implies b = 4 - 2a$ …(13)

Substituting (13) in $z = ax + bY + c$

$\implies z = ax + (4 - 2a)Y + c$…(14)

where c is an arbitrary constant

replacing Y in (14) to get the complete integral

$$\boxed{z = a\,x + (4 - 2a)\ln y + c}$$

Example 4: Solve $p^2x^2 = z^2 + q^2y^2$ by hypotheses

Sol. Given that $p^2x^2 = z^2 + q^2y^2$ …(15)

from $X = \ln x$ and $Y = \ln y$ we have

$xp = \frac{\partial z}{\partial X}$ and $qy = \frac{\partial z}{\partial Y}$ …(16)

Substituting (16) in (15), we get

$\left(\frac{\partial z}{\partial X}\right)^2 = z^2 + \left(\frac{\partial z}{\partial Y}\right)^2$ …(17)

Let $t = \frac{\partial z}{\partial X}$ and $r = \frac{\partial z}{\partial Y}$ putting in (17)

$t^2 - r^2 = z^2$ …(18)

Note that (18) is of the form $f(t, r, z) = 0$

Taking $u = X + aY$ (a is constant)

Then $t = \frac{\partial z}{\partial X} = \frac{\partial z}{\partial X} . \frac{\partial u}{\partial u} = \frac{\partial z}{\partial u} . \frac{\partial u}{\partial X} = \frac{dz}{du}$

$r = \frac{\partial z}{\partial Y} = \frac{\partial z}{\partial Y} . \frac{\partial u}{\partial u} = \frac{\partial z}{\partial u} . \frac{\partial u}{\partial Y} = a\frac{dz}{du}$ …(19)

because ($\frac{\partial u}{\partial X} = 1$ and $\frac{\partial u}{\partial Y} = a$)

putting (19) in (18)

$$\left(\frac{dz}{du}\right)^2 - a^2\left(\frac{dz}{du}\right)^2 = z^2$$

$$(1 - a^2)\left(\frac{dz}{du}\right)^2 = z^2$$

$\pm\sqrt{1 - a^2}\,\frac{dz}{du} = z$ (taking the square root)

$\pm\sqrt{1 - a^2}\,\frac{dz}{z} = du$ …(20)

Integrating (20),

$\pm\sqrt{1 - a^2}\ln z = u + \ln c$ (c is constant) …(21)

Now, replacing u in (21) to get the complete integral

$\pm\sqrt{1-a^2}\ln z = X + aY + \ln c$ …(22)

Next, replacing X and Y in (22) to get the complete integral

$\pm\sqrt{1-a^2}\ln z = \ln x + a\ln y + \ln c$

$\ln z^b = \ln cx\,y^a$ where $b = \pm\sqrt{1-a^2}$

$\Longrightarrow z^b = cx\,y^a$ …(23)

So, (23) is the complete integral.

C) When the equation contains the terms $\frac{p}{z}$ or $\frac{q}{z}$ or its' powers we use the hypothesis $\boxed{Z = \ln z}$

as follows:

$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x}.\frac{\partial Z}{\partial Z} = \frac{\partial z}{\partial Z}.\frac{\partial Z}{\partial x} = z\frac{\partial Z}{\partial x}$ (since $\frac{\partial z}{\partial Z} = z$)

hence $\frac{p}{z} = \frac{\partial Z}{\partial x}$

by the same way we have $\frac{q}{z} = \frac{\partial Z}{\partial y}$

then substituting this terms in the given equation and solve it by the same way in (A) and (B).

Example 5: Solve $\boldsymbol{p^2 + q^2 = z^2}$ by $\boxed{\boldsymbol{Z = \ln z}}$

Sol. Given that $\boldsymbol{p^2 + q^2 = z^2}$ …(24)

Dividing on z^2, $\frac{p^2}{z^2} + \frac{q^2}{z^2} = 1$ …(25)

using $Z = \ln z$ we have $\frac{p}{z} = \frac{\partial Z}{\partial x}$ and $\frac{q}{z} = \frac{\partial Z}{\partial y}$,substituting in (25)

$\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 = 1$ …(26)

Let $t = \frac{\partial Z}{\partial x}$ and $r = \frac{\partial Z}{\partial y}$ thus, (26) would be

$t^2 + r^2 = 1$ ...(27)

Clear that (27) is of the form $f(t, r) = 0$

Let $t = a$, $r = b$ (a, b are constant)

Then $a^2 + b^2 = 1$ ...(28)

$a = \pm\sqrt{1 - b^2}$ ...(29)

Substituting a, b in $Z = ax + by + c$

$\Longrightarrow Z = \pm\sqrt{1 - b^2}x + by + c$ ...(30)

Replacing Z from the hypothesis to get the complete integral

$\therefore \ln z = \pm\sqrt{1 - b^2}x + by + c$

$\Longrightarrow \boxed{z = e^{\pm\sqrt{1-b^2}x+by+c}}$ ...(31)

Then (31) is the complete integral.

Example 6: Solve $p^2 + q^2 = z^2(x + y)$ by hypotheses

Sol. Dividing on z^2,

$\frac{p^2}{z^2} + \frac{q^2}{z^2} = x + y$ ...(32)

using $Z = \ln z$ we have $\frac{p}{z} = \frac{\partial Z}{\partial x}$ and $\frac{q}{z} = \frac{\partial Z}{\partial y}$, substituting in (32)

$\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 = x + y$ ...(33)

Let $t = \frac{\partial Z}{\partial x}$ and $r = \frac{\partial Z}{\partial y}$ putting in (33)

$t^2 + r^2 = x + y$ ...(34)

Then $t^2 - x = a \rightarrow t = \pm\sqrt{a + x}$

$y - r^2 = a \rightarrow r = \pm\sqrt{y - a}$

Substituting in $dZ = tdx + rdy$

$$\Longrightarrow dZ = \pm\sqrt{a+x}dx + \pm\sqrt{y-a}dy \qquad \ldots(35)$$

Integrating (35), we get

$Z = \pm\frac{2}{3}(a+x)^{3/2} \pm \frac{2}{3}(y-a)^{3/2} + c$ (where c is constant)

Replacing Z from the hypothesis to get the complete integral

$$\Longrightarrow \boxed{\ln z = \pm\frac{2}{3}(a+x)^{3/2} \pm \frac{2}{3}(y-a)^{3/2} + c}$$

… Exercises …

1. $p^2x^2 = z(z-qy)$
2. $pq = z^2y\sec x$
3. $p + q = z\,e^{x+y}$
4. $p^2 + zq = z^2(x-y)$
5. $p^2 + zp = z^2(x-y)$
6. $xp + 4q = \cos y$
7. $p^2 + q^2 = z^2y$