(1.3.2) The equation of the form $f(p, q) = 0$

Here we consider equations in which p and q occur other than in the first degree, that is nonlinear equations. To solve the equation

$f(p, q) = 0$ ………(1)

Taking $p = constant = a$ …………(2)

$q = constant = b$ …………(3)

Substituting (2),(3) in (1), we get

$F(a, b) = 0 \rightarrow b = F_1(a) \quad or \quad a = F_2(b)$………..(4)

From $dz = pdx + qdy$ …………(5)

Using (2),(3) $\rightarrow dz = adx + bdy$ ………..(6)

Integrating (6), $z = ax + by + c$ ………..(7)

Where c is an arbitrary constant

Substituting (4) in (7) to obtain the complete integral (complete solution)

$z = ax + F_1(a)y + c \quad or \quad z = F_2(b)x + by + c$ ……….(8)

Ex.1: Solve $p^2 + p = q^2$

Sol. $p^2 + p - q^2 = 0$ ………..(1)

The equation (1) of the form $f(p, q) = 0$

Let $p = a \ , q = b$

Substituting in (1)

$$a^2 + a - b^2 = 0 \rightarrow b^2 = a^2 + a \rightarrow b = \pm\sqrt{a^2 + a}$$

The complete integral is

$$z = ax + by + c$$

$$= ax \pm \sqrt{a^2 + a}y + c$$

Where c is an arbitrary constant.

Ex.2: Solve $\mathbf{pq = k}$, where k is a constant.

Sol. Given that $pq = k$ …………(1)

Since (1) is of the form $f(p,q) = 0$, it's solution is

$z = ax + by + c$ …………(2)

Let $p = a$, $q = b$, substituting in (1) , then $ab = k \rightarrow b = \frac{k}{a}$ ...(3)

Putting (3) in (2), to get the complete solution

$z = ax + \frac{k}{a}y + c$; c is an arbitrary constant .

Ex.3: Solve $\frac{\partial z}{\partial x} - 3\frac{\partial z}{\partial y} = (\frac{\partial z}{\partial y})^3$

Sol. Given that $p - 3q = q^3$ ……….. (1)

Since (1) is of the form $f(p,q) = 0$, then

Let $p = a$, $q = b$

Substituting in (1) , $a - 3b = b^3 \rightarrow a = b^3 + 3b$ ………(2)

Putting (2) in the equation $z = ax + by + c$, we get

$$z = (b^3 + 3b)x + by + c$$

Where c is an arbitrary constant

The equation (3) is the complete integral.

(1.3.3) The Equation of the form $z = px + qy + f(p,q)$

A first order partial differential equation is said to be of **Clariaut** form if it can be written in the form

$$z = px + qy + f(p,q) \qquad \ldots(1)$$

to solve this equation taking $p = a$, $q = b$ and substituting in (1), so the complete integral is

$$z = ax + by + f(a,b) \qquad \ldots(2)$$

Example 1: Solve $z = px + qy + pq$

Sol. The given equation is of the form $z = px + qy + f(p,q)$

let $p = a$ and $q = b$ substituting in the given equation, so the complete integral is

$$\boxed{z = ax + by + ab}$$

where a, b being arbitrary constant.

Example 2: Solve $x\,\frac{\partial z}{\partial x} + y\,\frac{\partial z}{\partial y} = z - 5\frac{\partial z}{\partial x} + \frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}$

Sol. Rearrange the given equation, we have

$$x\,p + y\,q = z - 5p + pq$$

$$z = x\,p + y\,q + 5p - pq \qquad \ldots(3)$$

Equation (3) is of Clariaut form

let $p = a$ and $q = b$ substituting in (3), then the complete integral is $\boxed{z = ax + by + 5a - ab}$

where a, b being arbitrary constant.

Example 3: Solve $px + qy = z - p^3 - q^3$

Sol. Rearrange the given equation, we have

$$z = px + qy + p^3 + q^3 \qquad \ldots(4)$$

let $p = a$ and $q = b$ substituting in (4)

$\boxed{z = ax + by + a^3 + b^3}$ that is the complete integral and a, b being arbitrary constants.

(1.3.4) The Equation of the form $f(z, p, q) = 0$

To solve the equation of the form

$f(z, p, q) = 0 \quad \ldots(1)$

1. Let $\quad u = x + ay \qquad \ldots(2)$

   where a is an arbitrary constant

2. Replace p and q by $\frac{dz}{du}$ and $a\frac{dz}{du}$ respectively in (1) as follows,

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x}.\frac{\partial u}{\partial u} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial y}.\frac{\partial u}{\partial u} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} = a\frac{dz}{du} \qquad \ldots(3)$$

   from (2) $\quad \frac{\partial u}{\partial x} = 1 \quad$ and $\quad \frac{\partial u}{\partial y} = a$

3. Substituting (3) in (1) and solve the resulting ordinary differential equation of first order by usual methods.

4. Next, replace u by $x + ay$ in the solution obtained in step 3 to get the complete solution.

Example 1: Solve $\boldsymbol{z = p + q}$

Sol. Given equation is $\quad z = p + q \qquad \ldots(4)$

which is of the form $f(z, p, q) = 0$. Let $u = x + ay$ where a is an arbitrary constant.

Now, replacing p and q by $\frac{dz}{du}$ and $a\frac{dz}{du}$ respectively in (4), we get

$$z = \frac{dz}{du} + a\,\frac{dz}{du}$$

$$\Rightarrow z = (1+a)\,\frac{dz}{du}$$

$$\Rightarrow du = (1+a)\frac{dz}{z} \qquad \ldots(5)$$

Integrating (5), $u + c = (1+a)\ln z$

where c is an arbitrary constant

Replacing u,

$$x + ay + c = \ln z^{(1+a)}$$

$$\Rightarrow\ e^{x+ay+c} = z^{(1+a)}$$

$$\Rightarrow\ z = e^{\frac{x+ay+c}{1+a}} \qquad \ldots(6)$$

and that is the complete integral.

Example 2: Solve $\boldsymbol{\left(\frac{\partial z}{\partial x}\right)^2 z - \left(\frac{\partial z}{\partial y}\right)^2 = 1}$

Sol. Rearrange the given equation, we have

$p^2 z - q^2 = 1\ldots(7)$

This equation is of the form $\mathrm{f(z,p,q)} = 0$

Let $u = x + ay$, where a is an arbitrary constant

Now, replacing p and q by $\frac{dz}{du}$ and $a\frac{dz}{du}$ respectively in (7), we get

$$\left(\frac{dz}{du}\right)^2 z - \left(a\,\frac{dz}{du}\right)^2 = 1$$

$$\Rightarrow (z-a^2)\left(\frac{dz}{du}\right)^2 = 1$$

$\Rightarrow \pm\sqrt{z-a^2}\,\frac{dz}{du} = 1$ by taking the square root

$\Rightarrow \pm\sqrt{z-a^2}\,dz = du$…(8)

Integrating (8),

$\pm\frac{2}{3}(z-a^2)^{3/2} = u + c$…(9)

Replacing u in (9) to get the complete integral

$$\boxed{\pm\frac{2}{3}(z-a^2)^{\frac{3}{2}} = x + ay + c}$$

(1.3.5) The Equation of the form $\boldsymbol{f_1(x,p) = f_2(y,q) = 0}$

In this form z does not appear and the terms containing x and p are on one side and those containing y and q on the other side.

To solve this equation putting

$f_1(x,p) = f_2(y,q) = a$…(1)

where a is an arbitrary constant

$\therefore\ f_1(x,p) = a \quad \Rightarrow \quad p = g_1(x,a)$…(2)

$f_2(y,q) = a \quad \Rightarrow \quad q = g_2(y,a)$…(3)

Substituting (2) and (3) in $dz = pdx + qdy$, we get

$dz = g_1(x,a)dx + g_2(y,a)dy$…(4)

Integrating (4),

$$\boxed{z = \int g_1(x,a)dx + \int g_2(y,a)dy + b}$$

which is a complete integral containing two arbitrary constants a and b.

Example 1: Solve $\boldsymbol{p = 2xq^2}$

Sol. Separating p and x from q and y, the given equation reduces to $\frac{p}{x} = 2q^2$…(5)

Equating each side to an arbitrary constant a, we have

$$\frac{p}{x} = a \qquad \Longrightarrow p = ax$$

$$2q^2 = a \quad \Longrightarrow q = \pm\sqrt{\frac{a}{2}}$$

Putting these values of p and q in

$dz = pdx + qdy$, we get

$$dz = axdx \pm \sqrt{\frac{a}{2}}dy \qquad \text{…(6)}$$

Integrating (6), $z = \frac{a}{2}x^2 \pm \sqrt{\frac{a}{2}}\, y + b$

where a and b are two arbitrary constants.

Example 2: Solve $\boldsymbol{xq - y^2p - x^2y^2 = 0}$

Sol. Separating p and x from q and y, the given equation reduces to $\frac{p+x^2}{x} = \frac{q}{y^2}$…(7)

Equating each side to an arbitrary constant a, we have

$$\frac{p+x^2}{x} = a \qquad \Longrightarrow \quad p = ax - x^2 \qquad \text{…(8)}$$

$$\frac{q}{y^2} = a \qquad \Longrightarrow \quad q = a\, y^2 \qquad \text{…(9)}$$

Putting (8) and (9) in $dz = pdx + qdy$, we get

$$dz = (ax - x^2)dx + ay^2dy \qquad \ldots(10)$$

Integrating (10) , $\boxed{z = \frac{ax^2}{2} - \frac{x^3}{3} + a\frac{y^3}{3} + b}$

which is a complete integral containing two arbitrary constants a and b.

Example 3: Solve $\boldsymbol{p - 3x^2 = q^2 - y}$

Sol. Equating each side to an arbitrary constant a, we get

$$p - 3x^2 = a \quad \Longrightarrow \quad p = a + 3x^2 \qquad \ldots(11)$$

$$q^2 - y = a \quad \Longrightarrow \quad q = \pm\sqrt{a + y} \qquad \ldots(12)$$

Putting these values of p and q in $dz = pdx + qdy$, we get

$$dz = (a + 3x^2)dx \pm \sqrt{a + y}dy \qquad \ldots(13)$$

Integrating (13) , $\boxed{z = ax + x^3 \pm \frac{2}{3}(a + y)^{3/2} + b}$

which is a complete integral containing two arbitrary constant a and b.