Section (1.3): Methods for solving linear and nonlinear partial differential equations of order one

(1.3.1) Lagrange's method of solving $Pp + Qq = R$, when P, Q and R are function of x, y, z.

A quasi-linear partial differential equation of order one is of the form $Pp + Qq = R$, where P, Q and R are function of x, y, z. Such a partial differential equation is known as (Lagrange equation), for example: * $xyp + yzq = zx$

* $(x - y)p + (y - z)q = z - x$

(1.3.2) Working Rule for solving $Pp + Qq = R$ by Lagrange's method

Step 1. Put the given quasi-linear PDE of the first order in the standard form $Pp + Qq = R$ …………(1)

Step 2. Write down Lagrange's auxiliary equations for (1) namely $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ …………(2)

Step 3. Solve (2) by using the method for solving ordinary differential equation of order one. The equation (2) gives three ordinary differential equations. every two of them are independent and give a solution.

Let $u(x, y, z) = a$ and $v(x, y, z) = b$, then the (general solution) is $\emptyset(u, v) = 0$, wher $\emptyset$ is an arbitrary function and the complete solution is $u = \alpha v + \beta$ where α, β are arbitrary constant.

<u>Ex.1:</u> Solve $2\frac{\partial z}{\partial x} - 3\frac{\partial z}{\partial y} = 2x$

Sol. Given $2\frac{\partial z}{\partial x} - 3\frac{\partial z}{\partial y} = 2x$ ....(1)

The Lagrange's auxiliary for (1) are

$\frac{dx}{2} = \frac{dy}{-3} = \frac{dz}{2x}$ ....(2)

Taking the first two fractions of (2), we have

$\frac{dx}{2} = \frac{dy}{-3} \rightarrow -3dx - 2dy = 0$ ....(3)

Integrating (3), $-3x - 2y = a$ ....(4)

a being an arbitrary constant

Next, taking the first and the last fractions of (2), we get

$\frac{dx}{2} = \frac{dz}{2x} \rightarrow xdx = dz \rightarrow xdx - dz = 0$ ......(5)

Integrating (5), $\frac{x^2}{2} - z = b$ ....(6)

b being an arbitrary constant

From (4) and (6) the required general solution is

$$\emptyset(a, b) = 0 \rightarrow \emptyset\left(-3x - 2y, \frac{x^2}{2} - z\right) = 0$$

Where $\emptyset$ is an arbitrary function.

Ex.2: **Solve** $\left(\frac{y^2 z}{x}\right) p + xzq = y^2$

Sol. Given $\left(\frac{y^2 z}{x}\right) p + xzq = y^2$ …………(1)

The Lagrange's auxiliary equation for (1) are

$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2}$ …………(2)

Taking the first two fractions of (2), we have

$x^2 zdx = y^2 zdy \quad \rightarrow \; x^2 dx - y^2 dy = 0$ …………(3)

Integrating (3), $\frac{x^3}{3} - \frac{y^3}{3} = a \;\rightarrow\; x^3 - y^3 = a_1$ ……(4)

a_1 being an arbitrary constant.

Next, taking the first and the last fractions of (2), we get

$xy^2 dx = y^2 zdz \;\rightarrow\; xdx - zdz = 0$ …………(5)

Integrating (5), $\frac{x^2}{2} - \frac{z^2}{2} = b \;\rightarrow x^2 - z^2 = b_1$ …(6)

b_1 being an arbitrary constant

From (4) and (6) the general solution is

$$\emptyset(a_1, b_1) = 0 \;\rightarrow\; \emptyset(x^3 - y^3, x^2 - z^2) = 0$$

Ex.3: **Solve** $\mathbf{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + t\frac{\partial z}{\partial t} = xyt}$

Sol. Given $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + t\frac{\partial z}{\partial t} = xyt$ …………(1)

The Lagrange's auxiliary equation for (1) are

$\frac{dx}{x} = \frac{dy}{y} = \frac{dt}{t} = \frac{dz}{xyt}$ …………(2)

Taking the first two fractions of (2), we have

$\frac{dx}{x} = \frac{dy}{y} \;\rightarrow\; \frac{dx}{x} - \frac{dy}{y} = 0$ …………(3)

Integrating (3), $\ln x - \ln y = \ln a \rightarrow \frac{x}{y} = a$ ……(4)

Taking the second and the third fractions of (2), we get

$\frac{dy}{y} = \frac{dt}{t} \rightarrow \frac{dy}{y} - \frac{dt}{t} = 0$ …………(5)

Integrating (5), $\ln y - \ln t = \ln b \rightarrow \frac{y}{t} = b$ ……(6)

Next, taking the second and the last fractions of (2), we get

$\frac{dy}{y} = \frac{dz}{xyt} \rightarrow xtdy - dz = 0$ …………(7)

Substituting (4) and (6) in (7), we get

$\frac{a}{b}y^2dy - dz = 0$ …………(8)

Integrating (8), $\frac{a}{3b}y^3 - z = c$

Using (4) and (6), $\frac{1}{3}xyt - z = c$ …………(9)

Where a, b and c are an arbitrary constant

The general solution is

$$\emptyset(a, b, c) = 0 \rightarrow \emptyset\left(\frac{x}{y}, \frac{y}{t}, \frac{1}{3}xyt - z\right) = 0$$

$\emptyset$ being an arbitrary function.

<u>Rule</u>: for any equal fractions, if the sum of the denominators equal to zero, then the sum of the numerators equal to zero also.

Now, Return to the last example depending on the Rule above we will find the constant c.

Multiplying each fraction in Lagrange's auxiliary (2) by $yt, xt, xy, -3$ respectively, we get the sum of the denominators is

$xyt + xyt + xyt - 3xyt = 0$ ………..(10)

Then the sum of the numerators equal to zero also:

$ytdx + xtdy + xydt - 3dz = 0 \rightarrow \quad d(xyt) - 3dz = 0$……(11)

Integrating (11), $xyt - 3z = c$ ………..(12)

Note that (12) and (9) are the same.

<u>Ex.4:</u> Solve $\mathbf{(y - z)p + (z - x)q = x - y}$

Sol. Given$(y - z)p + (z - x)q = x - y$ ………..(1)

The Lagrange's auxiliary equations for (1) are

$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}$ ………..(2)

The sum of the denominators is

$$y - z + z - x + x - y = 0$$

Then, the sum of the numerators is equal to zero also, (by Rule)

$dx + dy + dz = 0$ ………..(3)

Integrating (3), $x + y + z = a$ ………..(4)

To find b, multiplying (2) by x, y, z resp. the sum of the denominators is

$x(y - z) + y(z - x) + z(x - y) = xy - xz + yz - xy + zx - yz = 0$

Then, the sum of the numerators is equal to zero

$xdx + ydy + zdz = 0$ ………..(5)

Integrating (5), $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = b$ ………..(6)

Where, a and b are arbitrary constants.

The general solution is

$$\emptyset(\mathrm{a},\mathrm{b}) = 0 \;\; \rightarrow \;\; \emptyset\left(\mathrm{x}+\mathrm{y}+\mathrm{z}, \frac{\mathrm{x}^2}{2}+\frac{\mathrm{y}^2}{2}+\frac{\mathrm{z}^2}{2}\right) = 0$$

… Exercises …

Solve the following partial differential equation:

1. $p\ tanx + q\ tany = tanz$.

2. $y^2p - xyq = x(z-2y)$.

3. $(x^2+2y^2)p - xyq = xz$.

4. $xp + yq = z$.

5. $(-a+x)p + (-b+y)q = (-c+z)$.

6. $x^2p + y^2q + z^2 = 0$.

7. $yzp + zxq = xy$.

8. $y^2p + x^2q = x^2y^2z^2$.

9. $p - q = \frac{z}{(x+y)}$