Case 2 **when $f(x,y) = \sin(ax+by)$ *or* $\cos(ax+by)$ where a and b are arbitrary constants**

Here, we will find the P.I. of (H.L.P.D.E.) of order 2 only, by the same way that in case 1 we will derive $f(x,y)$ for x and y .

Let $f(x,y) = \sin(ax+by)$

$$D_x \sin(ax+by) = a\cos(ax+by)$$

$$D_x^2 \sin(ax+by) = -a^2 \sin(ax+by)$$

$$D_y \sin(ax+by) = b\cos(ax+by)$$

$$D_y^2 \sin(ax+by) = -b^2 \sin(ax+by)$$

$$D_x D_y \sin(ax+by) = D_x[b\cos(ax+by)]$$

$$= -ab\sin(ax+by)$$

$$F\left(D_x^2, D_x D_y, D_y^2\right)\sin(ax+by) = F(-a^2,-ab,-b^2)\sin(ax+by)$$

Multiplying both sides by $\frac{1}{F\left(D_x^2, D_x D_y, D_y^2\right)}$

$$\sin(ax+by) = \frac{1}{F\left(D_x^2, D_x D_y, D_y^2\right)} F(-a^2,-ab,-b^2)\sin(ax+by)$$

If $\boxed{F(-a^2,-ab,-b^2) \neq 0}$ then we can divide on it

$$\rightarrow z = \frac{1}{F\left(D_x^2, D_x D_y, D_y^2\right)}\sin(ax+by)$$

$$= \frac{1}{F(-a^2,-ab,-b^2)}\sin(ax+by)$$

Which is the particular integral.

and if $F(-a^2,-ab,-b^2) = 0$, then we write

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad , \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

and follow the solution of the exponential function in case1.

Ex.4: Solve $\left(D_x^2 - D_x D_y - 6D_y^2\right)z = \sin(2x - 3y)$

Sol.

1) The general solution z_1 is the same in Ex.2

2) The P.I. z_2

$$a = 2 \ , b = -3$$

$$F(-a^2, -ab, -b^2) = -a^2 + ab + 6b^2$$

$$F(-4, 6, -9) = -4 - 6 + 54 = 44 \neq 0$$

$$z_2 = \frac{1}{44}\sin(2x - 3y)$$

The required general solution

$$\therefore \quad z = z_1 + z_2$$

$$= \emptyset_1(y + 3x) + \emptyset_2(y - 2x) + \frac{1}{44}\sin(2x - 3y)$$

Where $\emptyset_1$ and $\emptyset_2$ are arbitrary functions.

Ex. 5: Solve $\left(D_x^2 - 3D_x D_y + 2D_y^2\right)z = e^{2x+3y} + e^{x+y} + \sin(x - 2y)$

Sol.

1) Finding the general solution z_1

The A.E. is

$$m^2 - 3m + 2 = 0 \implies (m - 2)(m - 1) = 0$$

$$\therefore m_1 = 2 \ , m_2 = 1$$

$\therefore z_1 = \emptyset_1(y + 2x) + \emptyset_2(y + x)$

where $\emptyset_1$ and $\emptyset_2$ are arbitrary functions.

2) The P.I. of the given equation is

P.I. $z_2 = \frac{1}{F(D_x,D_y)} e^{2x+3y} + \frac{1}{F(D_x,D_y)} e^{x+y} + \frac{1}{F(D_x,D_y)} \sin(x-2y)$

Let $u_1 = \frac{1}{F(D_x,D_y)} e^{2x+3y}$ $\quad , a = 2, b = 3$

$F(D_x, D_y) = a^2 - 3ab + 2b^2$

$F(2,3) = 4 - 18 + 18 = 4 \neq 0$

$\boxed{u_1 = \frac{1}{4} e^{2x+3y}}$

$u_2 = \frac{1}{F(D_x,D_y)} e^{x+y}$ $\quad ,a = 1, b = 1$

$F(D_x, D_y) = a^2 - 3ab + 2b^2$

$F(1,1) = 1 - 3 + 2 = 0$

Analyze $F(D_x, D_y)$,

$F(D_x, D_y) = (D_x - 2D_y)(D_x - D_y)$

$$u_2 = \frac{1}{G(a,b)} \frac{x^r}{r!} e^{ax+by}$$

$$= \frac{1}{-1} \frac{x}{1} e^{x+y}$$

$\boxed{u_2 = -x\, e^{x+y}}$

$$u_3 = \frac{1}{F(D_x, D_y)} \sin(x - 2y)$$

$F(-a^2, -ab, -b^2) = -a^2 + 3ab - 2b^2$

$F(-1,2,-4) = -1 - 6 - 8 = -15 \neq 0$

$\boxed{u_3 = \frac{1}{-15} \sin(x - 2y)}$

Then, the required general solution is

$$z = z_1 + z_2 = \emptyset_1(y+2x) + \emptyset_2(y+x) + \frac{1}{4}e^{2x+3y} - xe^{x+y}$$
$$- \frac{1}{15}\sin(x-2y)$$

where $\emptyset_1$ and $\emptyset_2$ are arbitrary functions.

<u>Ex. 6:</u> Find the P.I. of the equation

$$\left(D_x^2 - 4D_xD_y + 3D_y^2\right)z = \cos(x+y)$$

Sol. $a = 1, b = 1$

$F(-a^2, -ab, -b^2) = -a^2 + 4ab - 3b^2$

$F(-1,-1,-1) = -1 + 4 - 3 = 0$

Taking $\cos(x+y) = \frac{e^{ix+iy} + e^{-ix-iy}}{2}$

$$z = \frac{1}{2}\left[\frac{1}{D_x^2 - 4D_xD_y + 3D_y^2}e^{ix+iy} + \frac{1}{D_x^2 - 4D_xD_y + 3D_y^2}e^{-ix-iy}\right]$$

Let $u_1 = \frac{1}{D_x^2 - 4D_xD_y + 3D_y^2}e^{ix+iy}$

To find $u_1, a = i, b = i$

$F(a,b) = a^2 - 4ab + 3b^2$

$F(i,i) = i^2 - 4i^2 + 3i^2 = 0$

Analyze $F(D_x, D_y)$,

$F(D_x, D_y) = (D_x - D_y)(D_x - 3D_y)$

$$u_1 = \frac{1}{-2i}xe^{ix+iy}$$

By the same way $u_2 = \frac{1}{2i}xe^{-ix-iy}$

$$\therefore z = \frac{1}{2}\left[\frac{1}{-2i}xe^{ix+iy} + \frac{1}{2i}xe^{-ix-iy}\right]$$

$= \frac{-x}{2}\left[\frac{e^{ix+iy} - e^{-ix-iy}}{2i}\right] = \frac{-x}{2}sin(x+y)$ which is the P.I.

Case 3 When $f(x,y) = x^a y^b$ where a and b are Non- Negative Integer Numbers

The particular integral (P.I.) is evaluated by expanding the function $\frac{1}{F(D_x,D_y)}$ in an infinite series of ascending powers of D_x or D_y (i.e.) by transfer the function $\frac{1}{F(D_x,D_y)}$ according to the following

$$\frac{1}{1-\theta} = 1 + \theta + \theta^2 + \cdots$$

<u>Ex.7</u>: Find P.I. of the equation $(D_x^2 - 2D_xD_y)z = x^3y$

Sol. P.I. $= \frac{1}{D_x^2 - 2D_xD_y} x^3y$

$= \frac{1}{D_x^2(1-2\frac{D_y}{D_x})} x^3y,$ $\qquad D_y^n y^m = 0 \; if \; n > m$

$= \frac{1}{D_x^2}\left[1 + 2\frac{D_y}{D_x} + \frac{4D_y^2}{D_x^2} + \cdots\right] x^3y$ $\qquad , \frac{4D_y^2}{D_x^2} = 0$

$$= \frac{1}{D_x^2}\left[x^3y + \frac{1}{2}x^4\right]$$

$$= \frac{1}{D_x}\left[\frac{x^4y}{4} + \frac{x^5}{10}\right] = \frac{x^5y}{20} + \frac{x^6}{60}$$

<u>Ex.8</u>: Find P.I. of the equation $(D_x^3 - 7D_xD_y^2 - 6D_y^3)z = x^2y$

Sol. P.I. $= \frac{1}{D_x^3 - 7D_xD_y^2 - 6D_y^3} x^2y$

$= \frac{1}{D_x^3\left[1 - \left(\frac{7D_y^2}{D_x^2} + \frac{6D_y^3}{D_x^3}\right)\right]} x^2y$

$$=\frac{1}{D_x^3}\left[1+\left(\frac{7D_y^2}{D_x^2}+\frac{6D_y^3}{D_x^3}\right)+\left(\frac{7D_y^2}{D_x^2}+\frac{6D_y^3}{D_x^3}\right)^2+\cdots\right]x^2y$$

$=\frac{1}{D_x^3}[x^2y]$ since $\left(\frac{7D_y^2}{D_x^2}+\frac{6D_y^3}{D_x^3}\right)=0\,,\left(\frac{7D_y^2}{D_x^2}+\frac{6D_y^3}{D_x^3}\right)^2=0$

$$=\frac{1}{D_x^2}\frac{x^3y}{3}=\frac{1}{D_x}\frac{x^4y}{12}=\frac{x^5y}{60}$$

<u>Ex.9</u>: Solve $(\boldsymbol{D_x^3-a^2D_xD_y^2})\boldsymbol{z=x}$, where $\boldsymbol{a}\in\boldsymbol{R}$

Sol.

1) the general solution z_1

The A.E. of the given equation is

$m^3-a^2m=0\ \Rightarrow\ m(m^2-a^2)=0$

$$\Rightarrow m(m-a)(m+a)=0$$

$\therefore m_1=0\,,m_2=a\ \ ,m_3=-a$ (different roots) $\therefore z_1=\emptyset_1(y)+\emptyset_2(y+ax)+\emptyset_3(y-ax)$

where $\emptyset_1,\emptyset_2$ and $\emptyset_3$ are arbitrary functions.

2) The P.I. of the given equation is

P.I. $=z_2=\frac{1}{D_x^3-a^2D_xD_y^2}x$

$=\frac{1}{D_x^3\left[1-\frac{a^2D_y^2}{D_x^2}\right]}x$

$=\frac{1}{D_x^3}\left[1+\frac{a^2D_y^2}{D_x^2}+\left(\frac{a^2D_y^2}{D_x^2}\right)^2+\cdots\right]x$, where $\frac{a^2D_y^2}{D_x^2}=0$ and $\left(\frac{a^2D_y^2}{D_x^2}\right)^2=0,$

$$=\frac{1}{D_x^3}[x]$$

$$= \frac{1}{D_x^2}\left[\frac{x^2}{2}\right]$$

$$= \frac{1}{D_x}\left[\frac{x^3}{6}\right] = \frac{x^4}{24}$$

then, the required general solution is

$$z = z_1 + z_2 = \emptyset_1(y) + \emptyset_2(y + ax) + \emptyset_3(y - ax) + \frac{x^4}{24}$$

where $\emptyset_1, \emptyset_2$ and $\emptyset_3$ are arbitrary functions.