is an invertible.

Example:

1. $(\mathbb{Q},+,\cdot)$, $(\mathbb{R},+,\cdot)$, $(\mathbb{C},+,\cdot)$ are fields.

2. $(\mathbb{Z},+,\cdot)$, $(Ze,+,\cdot)$ is not a field.

3. direct sum of $(\mathbb{R},+,\cdot)$ with $(\mathbb{R},+,\cdot)$ is not a field.

Since all elements of the form $(a,0)$ and $(0,b)$ $a \neq 0$, $b \neq 0$ are not invertible elements.

problems:

1. let $S = \mathbb{R} \times \mathbb{R}$, Define $\oplus$, $\boxdot$

$(a,b) \oplus (c,d) = (a+c, b+d)$
$(a,b) \boxdot (c,d) = (ac-bd, ad+bc)$.

Show that $(S, \oplus, \boxdot)$ is field.

2. IS $(\mathbb{Z}[\sqrt{3}],+,\cdot)$ field?

let $x = 1+3\sqrt{3} \in \mathbb{Z}[\sqrt{3}]$

$$\frac{1}{x} = \frac{1}{1+3\sqrt{3}} = \frac{1-3\sqrt{3}}{(1+3\sqrt{3})\cdot(1-3\sqrt{3})} = \frac{1-3\sqrt{3}}{-26} = \frac{-1}{26} + \frac{3}{26}\sqrt{3} \notin \mathbb{Z}[\sqrt{3}]$$

$\therefore$ $\mathbb{Z}[\sqrt{3}]$ is not field.

<u>Theorem</u>: Every field is an integral domain. But the Converse is not true ingeneral for example $(\mathbb{Z},+,\cdot)$ is an integral domain but it is not field.

Theorem

Every finite integral domain is a field.

Proof:

let $(R, +, \cdot)$ be a finite integral domain and let $R = \{a_1, a_2, \ldots, a_n\}$ n من العناصر المختلفة
since, R is comm. ring with unity 1

let a_k be a non zero element of R;

$S = \{a_k a_1, a_k a_2, \ldots a_k a_k, \ldots a_k a_n\} \subseteq R$

Now, if $a_k a_t = a_k a_s$, $t \neq s$

$\Rightarrow a_t = a_s$ (since R is an integral domain)

إذن S ليس فيها عناصر متكررة

Then S has exactly n elements and hence

$S \subseteq R$, also R has n elements. Then

$S = R$.

$1 \in R \Rightarrow 1 \in S$. Thus $1 \in S \Rightarrow 1 = a_k a_r$

for some r, $1 \leq r \leq n$

$\Rightarrow a_k$ is an invertible element; $a_k^{-1} = a_r$

Therefore R is field.

□

Definition

A division ring is a ring with identity in which every non zero element has invertible.

Remark

Every field is a division ring, but the Converse is not true, ingeneral. For example:

let $R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$ with usual addition and multiplication operadions on matrices. Then $(R, +, \cdot)$ Is a ring with identity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Since, for example:

$$\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$$

That is $(R, +, \cdot)$ is not Comm. ring, but every element in R has inverse

Therefore, R is division ring but not field.

Definition

Let $(F, +, \cdot)$, $(K, +, \cdot)$ be two fields s.t. $F \subseteq K$. Then F is Called a subfield of K.

Remark

Let $(F, +, \cdot)$ be a field and let $\emptyset \neq S \subseteq F$. Then $(S, +, \cdot)$ is a subfield of $F \Longleftrightarrow$

1. $a - b \in S \quad \forall a, b \in S$
2. $a \cdot b^{-1} \in S \quad \forall a, b \in S$

Example

$\mathbb{Q}[\sqrt{2}] = \{ a + b\sqrt{2} : a, b \in \mathbb{Q} \}$

Then $(Q[\sqrt{2}],+,\cdot)$ is a subfield of $\mathbb{R}$.

let $a+b\sqrt{2}$, $c+d\sqrt{2} \in Q[\sqrt{2}]$. Then

1. $(a+b\sqrt{2})-(c+d\sqrt{2}) = (a-c)+(b-d)\sqrt{2} \in Q[\sqrt{2}]$

2. $(a+b\sqrt{2})\cdot(c+d\sqrt{2})^{-1} = (a+b\sqrt{2})\cdot\dfrac{1}{c+d\sqrt{2}}$

$$= \frac{a+b\sqrt{2}}{c+d\sqrt{2}} = \frac{a+b\sqrt{2}}{c+d\sqrt{2}}\cdot\frac{c-d\sqrt{2}}{c-d\sqrt{2}}$$

$$= \frac{ac+ad\sqrt{2}+cb\sqrt{2}-2bd}{c^2-2d^2} = \frac{(ac-2bd)+(ad+cb)\sqrt{2}}{c^2-2d^2}$$

$$= \underbrace{\frac{ac-2bd}{c^2-2d^2}}_{\in Q} + \underbrace{\frac{ad+cb}{c^2-2d^2}}_{\in Q}\sqrt{2} \in Q[\sqrt{2}].$$

Remark:

Let F be a fied and let $\{S_\alpha\}$ be any Collection of subfield of F. Then $\cap S_\alpha$ is a subfield of F.

Proof:

Since, $0 \in S_\alpha \ \forall \alpha \Rightarrow \cap S_\alpha \neq \emptyset$

let $a,b \in \cap S_\alpha$

$\Rightarrow a,b \in S_\alpha \ \forall \alpha$

But S_α is a subfield of $F \ \forall \alpha$

$\therefore a-b \in S_\alpha$, $ab^{-1} \in S_\alpha \ \forall \alpha$

$\therefore\ a-b \in \bigcap S_\alpha \ , \ ab^{-1} \in \bigcap S_\alpha$

$\therefore \bigcap S_\alpha$ is a subfield of F.