4. $(-a)(-b) = -(a(-b))$ by (2)
$= -(-(ab))$ by (2)
$= ab$ by (3)

8. $(-1)a = -(1.a)$ by (2)
$= -a$

## Direct Sum of Rings and Some Remarks

الجمع المباشر للحلقات مع بعض الخواص

Definition:

let $(R,+,.)$ and $(R',+',.')$ be any two rings.
let $X = R \times R' = \{(a,b) : a \in R \wedge b \in R'\}$
Define $\oplus, \odot$ on $X$ by:
$(a,b) \oplus (c,d) = (a+c, b+'d)$

$(a,b) \odot (c,d) = (a.c, b.'d)$.

Then $(X, \oplus, \odot)$ is a ring.

This ring is called the direct sum of $R$ with $R'$

From above example, we have the following remarks:

Remarks:

1. If $(R,+,.)$, $(R',+',.')$ are Comm. ring, then the ring of direct sum of $R, R'$ is Comm. ring

2. If the ring $(R,+,.)$ has unity 1 and the ring $(R',+',.')$ has unity $\bar{1}$, then the ring $R \times R'$ has

unity $(1,1')$

3. If rings $(R,+,\cdot)$, $(R',+',\cdot')$ have unities $1, 1'$ and $a \in R \wedge b \in R'$ such that $a$ is an invertible in $R \times R'$ and $(a^{-1}, b^{-1}) = (a,b)^{-1}$.

Example:

let $Z_2 \times Z_3 = \{(\bar{0},\bar{0}), (\bar{0},\bar{1}), (\bar{0},\bar{2}), (\bar{1},\bar{0}), (\bar{1},\bar{1}), (\bar{1},\bar{2})\}$

and
$$(\bar{a},\bar{b}) \oplus (\bar{c},\bar{d}) = (\bar{a} +_2 \bar{c}, \bar{b} +_3 \bar{d})$$
$$(\bar{a},\bar{b}) \odot (\bar{c},\bar{d}) = (\bar{a} \cdot_2 \bar{c}, \bar{b} \cdot_3 \bar{d})$$

Then $(Z_2 \times Z_3, \oplus, \odot)$ is a ring.

| $\oplus$ | $(\bar{0},\bar{0})$ | $(\bar{0},\bar{1})$ | $(\bar{0},\bar{2})$ | $(\bar{1},\bar{0})$ | $(\bar{1},\bar{1})$ | $(\bar{1},\bar{2})$ |
|---|---|---|---|---|---|---|
| $(\bar{0},\bar{0})$ | $(\bar{0},\bar{0})$ | $(\bar{0},\bar{1})$ | $(\bar{0},\bar{2})$ | $(\bar{1},\bar{0})$ | $(\bar{1},\bar{1})$ | $(\bar{1},\bar{2})$ |
| $(\bar{0},\bar{1})$ | $(\bar{0},\bar{1})$ | $(\bar{0},\bar{2})$ | $(\bar{0},\bar{0})$ | $(\bar{1},\bar{1})$ | $(\bar{1},\bar{2})$ | $(\bar{1},\bar{0})$ |
| $(\bar{0},\bar{2})$ | $(\bar{0},\bar{2})$ | $(\bar{0},\bar{0})$ | $(\bar{0},\bar{1})$ | $(\bar{1},\bar{2})$ | $(\bar{1},\bar{0})$ | $(\bar{1},\bar{1})$ |
| $(\bar{1},\bar{0})$ | $(\bar{1},\bar{0})$ | $(\bar{1},\bar{1})$ | $(\bar{1},\bar{2})$ | $(\bar{0},\bar{0})$ | $(\bar{0},\bar{1})$ | $(\bar{0},\bar{2})$ |
| $(\bar{1},\bar{1})$ | $(\bar{1},\bar{1})$ | $(\bar{1},\bar{2})$ | $(\bar{1},\bar{0})$ | $(\bar{0},\bar{1})$ | $(\bar{0},\bar{2})$ | $(\bar{0},\bar{0})$ |
| $(\bar{1},\bar{2})$ | $(\bar{1},\bar{2})$ | $(\bar{1},\bar{0})$ | $(\bar{1},\bar{1})$ | $(\bar{0},\bar{2})$ | $(\bar{0},\bar{0})$ | $(\bar{0},\bar{1})$ |

النظائر الجمعية

$-(\bar{0},\bar{0}) = (\bar{0},\bar{0})$

$-(\bar{0},\bar{1}) = (\bar{0},\bar{2})$ $\qquad -(\bar{1},\bar{1}) = (\bar{1},\bar{2})$

$-(\bar{0},\bar{2}) = (\bar{0},\bar{1})$ $\qquad -(\bar{1},\bar{2}) = (\bar{1},\bar{1})$

$-(\bar{1},\bar{0}) = (\bar{1},\bar{0})$

Definition : let $(R,+,\cdot)$ be a ring and $a,b \in R$ , $a \neq 0$ , $b \neq 0$ . If $a.b = 0$ , then a is called left zero divisor and b is called right zero divisor.

In Comm. ring , every left zero divisor is right zero divisor and conversely.

Examples :

1. In $(Z_3, +_3, \cdot_3)$ , $\bar{0}, \bar{1}, \bar{2}$ are not zero divisors

2. In $(Z_4, +_4, \cdot_4)$ , $\bar{0}, \bar{1}, \bar{3}$ are not zero divisors, but $\bar{2}$ is zero divisor , (since $\bar{2} \cdot_4 \bar{2} = \bar{0}$) .

3. In $(Z_6, +_6, \cdot_6)$ , $\bar{0}, \bar{1}, \bar{5}$ are not zero divisors. But $\bar{2}, \bar{3}, \bar{4}$ are zero divisors.

Remark :- let $(R,+,\cdot)$ be a Comm. ring with unity 1 . If $a \in R$, $a \neq 0$ , a is an invertible element , then a is not zero divisor .

That is, a is invertible element $\Rightarrow$ a is not zero divisor

Proof :- If $a \in R$ , $a \neq 0$ and a is an invertible element.

$\therefore \exists\ a^{-1} \in R$ s.t. $a^{-1}.a = a.a^{-1} = 1$

Suppose a is a zero divisor:

Then $\exists\ b \in R,\ b \neq 0$ s.t. $a.b = 0$

$a^{-1}.(a.b) = a^{-1}.0 \Rightarrow (a^{-1}.a).b = 0$

$\Rightarrow 1.b = 0 \Rightarrow b = 0$ C! (since $b \neq 0$).

Thus a is not zero divisor.

والعلاقة العكسية خاطئة

a is zero divisor $\Rightarrow$ a is not invertible.

Now, we have the following example:

Example:

Consider the direct sum of $(\mathbb{R}, +, .)$ with $(\mathbb{R}, +, .)$

$(a, 0) \odot (0, b) = (0, 0) \quad \forall a \neq 0, \forall b \neq 0$

Then all element of the form $(a, 0), (0, b)$ where $a \neq 0, b \neq 0$ are zero divisors.

Theorem:

let R be a comm. ring. Then R has no zero divisor $\Leftrightarrow \forall a \in R, a \neq 0$, $a.b = a.c$ which implies $b = c$.

The Cancellation law holds with respect to multiplication operation

proof: $\Rightarrow$ If R has no zero divisor

let $a.b = a.c$ and $a \neq 0$

$a.b + (-ac) = ac + (-ac)$

$a.b - ac = 0 \Rightarrow a(b - c) = 0$.

but $a \neq 0$ and $R$ has no Zero divisor.
Then $b - c = 0$ and hence $b = c$.

$\Leftarrow$ suppose $R$ has Zero divisor, so

$\exists\, a \neq 0, b \neq 0$ s.t. $a.b = 0$

let $a.0 = 0$ (by properties of ring)
Then $a.b = a.0$ and hence $b = 0$ C!

Therefore $R$ has no Zero divisor.

Theorem: let $R$ be a Comm. ring with unity 1

and $R$ has no zero divisor. Then the equation
$a^2 = a$ has only two slution $a = 0$ or $a = 1$.

"Integral domain" (ﺣﻠﻘﺔ)

Definition: let $(R, +, \cdot)$ be a Comm. ring with

unity 1. Then $R$ is Called an integral domain
$\Leftrightarrow$ $R$ has no Zero divisor.

Example: ① All rings $(\mathbb{Z}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$,

$(\mathbb{C}, +, \cdot)$ have no Zero divisors. Then all these rings
are integral domains.

② $(\mathbb{Z}_3, +_3, \cdot_3)$, $(\mathbb{Z}_5, +_5, \cdot_5)$ are integral domains
③ $(\mathbb{Z}_4, +_4, \cdot_4)$, $(\mathbb{Z}_6, +_6, \cdot_6)$ are not integral domains

Theorem: $(\mathbb{Z}_n, +_n, \cdot_n)$ is an integral domain $\Leftrightarrow$ n is prime number.

Proof: $\rightarrow$ If $(\mathbb{Z}_n, +_n, \cdot_n)$ is an integral domain

To prove n is a prime number
suppose n is not prime number
$\therefore \exists\, m, k \in \mathbb{Z}_+$ s.t $\quad n = m \cdot k \quad 1 < m < n$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 1 < k < n$

$\Rightarrow \bar{0} = \bar{m} \cdot_n \bar{k}$ and $\bar{m}, \bar{k}$ are no zero element.

Then $\bar{m}, \bar{k}$ are zero divisors in $\mathbb{Z}_n$.
which is contradiction since $\mathbb{Z}_n$ is an integral domain.
Thus n is prime number.

$\leftarrow$ $(\mathbb{Z}_n, +_n, \cdot_n)$ is comm. ring with unity 1
To prove $(\mathbb{Z}_n, +_n, \cdot_n)$ is an integral domain,
we must prove that $\mathbb{Z}_n$ has no zero divisor
suppose $\mathbb{Z}_n$ has zero divisor
$\exists\, \bar{m}, \bar{k} \in \mathbb{Z}_n$ s.t. $\bar{m} \neq \bar{0}$ and $\bar{k} \neq \bar{0}$, $\bar{m} \cdot \bar{k} = 0$

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$\Rightarrow \bar{m} \cdot \bar{k} = n \cdot r$ for some $r \in \mathbb{Z}_+$

$\Rightarrow n / mk$. Then $n/m$ or $n \backslash k$ (since n is prime)

If $n/m \Rightarrow m$ = multiply of $n \Rightarrow \bar{m} = \bar{0}$ C!
If $n/k \Rightarrow k$ = multiply of $n \Rightarrow \bar{k} = \bar{0}$ C!

Thus $\mathbb{Z}_n$ has no zero divisor.

Definition:
let $(R, +, \cdot)$ be a comm. ring with unity 1
. Then $(R, +, \cdot)$ is called field $\Leftrightarrow \forall a \in R, a \neq 0$, a