الفصل الاول

Definition (chapter one)

let R be a non-empty set and let $+$, $\cdot$ be two binary operations on R. Then $(R,+,\cdot)$ is a ring if:

1. $(R,+)$ is an abelain group, that is

a. $+$ is closed on R

b. $+$ is associative

c. $\forall a \in R$, $\exists$ an element $0 \in R$ s.t. $a+0=a$, 0 is Called the zero element.

d. $\forall a \in R$, $\exists\, -a \in R$ s.t. $a+(-a)=0$, $-a$ is Called the additive inverse of a.

e. $+$ is Commutative.

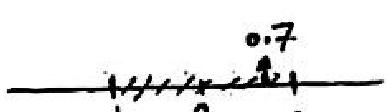
2. $(a\cdot b)\cdot c = a\cdot(b\cdot c)$ $\quad \forall a,b \in R$

3. $a\cdot(b+c) = a\cdot b + a\cdot c$, $(b+c)\cdot a = b\cdot a + c\cdot a$ $\quad \forall a,b,c \in R$.

Examples

1. $(\mathbb{R},+,\cdot)$, $(\mathbb{Q},+,\cdot)$, $(\mathbb{Z},+,\cdot)$, $(\mathbb{C},+,\cdot)$ are rings

2. $((-1,1),+,\cdot)$ is not ring, for example

$0.7 \in (-1,1)$ 

, but $0.7+0.7=1.4 \notin (-1,1)$

3. $(\mathbb{Z}_e,+,\cdot)$ be a ring, where $\mathbb{Z}_e = \{x : x = 2k \text{ for some } k \in \mathbb{Z}\}$

4. let n be a fixed positive integer and $n \neq 1$.

let $A=\{x \mid x=nk$ for some $k\in\mathbb{Z}\}$. Then $(A,+,\cdot)$ is a ring. show that (H.W.)

Definition:

Let R be a ring. R is said to be Commutative ring, if $a.b=b.a$ $\forall a,b\in R$.

Definition:

Let R be a ring. Then if $\exists 1\in R$ such that $a.1=1.a=a$ $\forall a\in R$. R is Called a ring with identity (or unitary ring). Also, let $a\in R$. Then a is Called invertible element, if $\exists a^{-1}\in R$ st $a.a^{-1}=a^{-1}.a=1$

Examples:

1. $(\mathbb{Z},+,\cdot)$, $(\mathbb{Q},+,\cdot)$, $(\mathbb{R},+,\cdot)$, $(\mathbb{C},+,\cdot)$ are a Commutative ring with identity.

2. let $R=M_{n\times n}(\mathbb{R})=\left\{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix} ; a_{ij}\in\mathbb{R} \ \forall i,j=1,\ldots,n\right\}$

be the set of all metrices $n\times n$ with usual addition and multiplicative for matrices. Then $(R,+,\cdot)$ is a non Commutative ring with identity

Since, let $A=(a_{ij})_{n\times n}$ and $B=(b_{ij})_{n\times n}$. Then

1. $A+B=(a_{ij}+b_{ij})_{n\times n}\in M_{n\times n}(\mathbb{R})$ is Comm. gp.

2. $A.B=\left(\sum_{k=1}^{n} a_{ik}b_{kj}\right)$, There associative and distributive over $+$

$$I_n=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1\end{pmatrix}_{n\times n}\in M_{n\times n}(\mathbb{R})$$

$A \cdot I_n = I_n \cdot A = A \qquad \forall A \in M_{n\times n}$

$\therefore (M_{n\times n}, +, \cdot)$ is a ring with unity, but $(M_{n\times n}, +, \cdot)$ is not Comm. ingeneral.

3. $(\mathbb{Z}_e, +, \cdot)$ is Comm. ring without identity

4. let X be a non-empty set. Then $(\mathbb{P}(X), \Delta, \cap)$ is a ring, where $A \Delta B = (A \cup B) - (A \cap B)$
$= (A - B) \cup (B - A)$.

This ring is Comm. with identity X. For example:

let $X = \{1, 2\}$. Then $\mathbb{P}(X) = \{\emptyset, \{1\}, \{2\}, X\}$
Δ is closed on $\mathbb{P}(X)$ and $\cap$ is Comm. and associa.

$\emptyset$ = identity with respect to Δ and every element in $\mathbb{P}(X)$ has inverse, also Δ is associative, then $(\mathbb{P}(X), \Delta)$ is Comm. gp.

Now, $\cap$ is asso. and $\cap$ distributed over Δ.

Therefore $(\mathbb{P}(X), \Delta, \cap)$ is a Comm. ring with identity X. $\{$since, $X \cap \{1\} = \{1\}$, $X \cap \{2\} = \{2\}$, $X \cap \emptyset = \emptyset$ and $X \cap X = X \}$

5. let $\mathbb{Z}[\sqrt{3}] = \{a + b\sqrt{3} : a, b \in \mathbb{Z}\}$. Define $+, \cdot$ on $\mathbb{Z}[\sqrt{3}]$ by

$$(a + b\sqrt{3}) + (c + d\sqrt{3}) = (a + c) + (b + d)\sqrt{3}$$

$$(a + b\sqrt{3}) \cdot (c + d\sqrt{3}) = (ac + 3bd) + (bc + ad)\sqrt{3}$$

Then $(\mathbb{Z}[\sqrt{3}], +, \cdot)$ be a Comm. ring with identity $1 = 1 + 0\sqrt{3}$

6- let $x \oplus y = x + y \quad \forall x, y \in \mathbb{Z}$ and
$x \odot y = 0 \quad \forall x, y \in \mathbb{Z}$

defined on $\mathbb{Z}$. Then $(\mathbb{Z}, \oplus, \odot)$ is Comm. ring without unity.

Since, $(\mathbb{Z}, \oplus)$ is Comm. gp

① Now, To prove well-defined, if $x = x_1$ and $y = y_1$, implies $x \odot y = 0$ and $x_1 \odot y_1 = 0$

Therefore $x \odot y = x_1 \odot y_1$ and hence $\odot$ well defined

②. $\forall x, y \in \mathbb{Z}$, $x \odot y = 0 \in \mathbb{Z}$
$\Rightarrow \odot$ is closed on $\mathbb{Z}$

③ $x \odot y = y \odot x = 0$ which implies $\odot$ is Comm.

④. $\forall x, y, z \in \mathbb{Z}$, $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z) = 0 \oplus 0 = 0$
Then $\odot$ distributive over $\oplus$.

Thus $(\mathbb{Z}, \oplus, \odot)$ is Comm. ring and $\nexists\, b \in \mathbb{Z}$ s.t. $a \odot b = a \quad \forall a \in \mathbb{Z}$ and $a \neq 0$. Hence the ring has no unity.

7. let $C[0,1] = \{f : [0,1] \to \mathbb{R}\}$ which are Countinuas s.t $(f+g)_{(x)} = f(x) + g(x)$ and

$$(f \cdot g)_{(x)} = f(x) \cdot g(x)$$

Then $(C[0,1], +, \cdot)$ is a Comm. ring with unity $I(x)$ [identity function].

8. let $Z_n = \{\bar{0}, \bar{1}, \bar{2}, \ldots, \overline{n-1}\}$. Then $(Z_n, +_n, \cdot_n)$ is Comm. ring with identity 1

special Case : $(Z_4, +_4, \cdot_4)$ is Comm. ring with identity 1.

Properties of Rings مواصفات الحلقات

let $(R, +, \cdot)$ be a ring. Then for all $a, b, c \in R$.

1. $a.0 = 0 = 0.a$
2. $(-a).b = -(a.b) = a.(-b)$
3. $-(-a) = a$
4. $(-a).(-b) = a.b$
5. $-(a+b) = -a - b$
6. $a.(b-c) = a.b - b.c$
7. $(a-b).c = a.c - b.c$
8. If R has unity 1, then
   (i) $(-1)a = -a$
   (ii) $(-1)(-1) = 1$

Proofs

1. To prove $a.0 = 0$
   $a.0 = a.(0+0) = a.0 + a.0$
   $a.0 - a.0 = a.0$
   $0 = a.0$.