CHAPTER FIVE

Cardinality of Sets قدرة المجموعات

Chapter Two Contents:

1. Equivalent of Two Sets التكافؤ بين مجموعتين

2. Finite and infinite sets المجموعات المنتهية وغير المنتهية

3. Countably infinite sets المجموعات الغير النتهية القابلة للعد

Equivalent of Two Sets التكافؤ بين مجموعتين

Two sets A and B are called **equivalent ($\boldsymbol{A \approx B}$)** if and only if there exist a bijective map between them.

المجموعتان A و B تسمى مجموعتين **متقابلتين** او **متكافئتين** اذا وجدت دالة متقابلة تربط بينهما

Mathematically,

$$\boldsymbol{A \approx B} \Leftrightarrow \exists f \ s.t. f: A \to B \text{ is } \textbf{bijective}$$

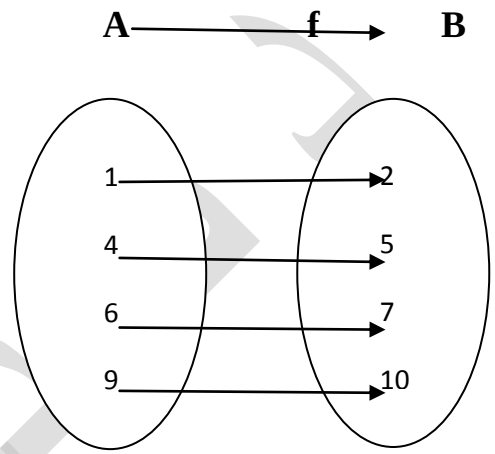
$$\boldsymbol{A \approx B} \Leftrightarrow \forall f \ s.t. f: A \to B \text{ is } \textbf{not bijective}$$

Remark 2.1: If $\boldsymbol{A \approx B}$**,** we say that there is a **"one to one correspondence"** between A and B.

<u>Example</u> 2.2: Let $A = \{1,4,6,9\}$, $B = \{2,5,7,10\}$. Show that $A \approx B$. Is $B \approx A$?

<u>Solution</u>: Define $f: A \to B\ s.t. f(1) = 2,\ f(4) = 5, f(6) = 7, f(9) = 10$

In general, $f(x) = x + 1\ \ \forall x \in A$



1-1? $\forall x_1, x_2 \in A, x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2) \Longrightarrow f$ is 1-1

Onto? $R_f = \{2,5,7,10\} = cod_f$

$\therefore f$ is bijective $\Longrightarrow\ A \approx B$ (there is a one to one correspondence between A and B)

Give another bijective function that makes $A \approx B$?

Define $h: A \to B\ s.t. h(1) =\ \ ,\ h(4) =\ \ , h(6) =\ \ , h(9) =$

Is $B \approx A$?

Define $g: B \to A\ s.t\ \ g(2) = 1, g(5) = 4, g(7) = 6, g(10) = 9$

Or $g(x) = x - 1\ \ \ \forall x \in B$

g is 1-1 and onto (check)

$\therefore g$ is bijective $\Longrightarrow\ B \approx A$

طريقة ثانية لبرهان ان $B \approx A$

Since $A \approx B \Longrightarrow \exists f^{-1}: B \to A \; s.t. f^{-1}(y) = x$

لإيجاد قيمة x

Let $y = x + 1 \Longrightarrow \mathrm{x} = y - 1$

Substitute the value of x in $f^{-1}(y)$

$\therefore \; f^{-1}: B \to A \; s.t. \; f^{-1}(y) = y - 1$

Show that f^{-1} is bijective (H.W.)

$\therefore \; B \approx A$

Example 2.3: Let $A = \{x, y, z\}, \;\; B = \{0, -1\}$

Is $A \approx B$? Is $B \approx A$?

Solution:

$A \approx B \Leftrightarrow \exists f \; s.t. f: A \to B$ is bijective

Let $f(x) = 0, f(y) = -1, f(z) = -1$

This relation is not 1-1 mapping because two elements $y, z \in A$ have the same image.

In fact, each $f: A \to B$ is not 1-1$\Longrightarrow f$ is not bijective $\Longrightarrow$ A is not equivalent to B.

$B \approx A \Leftrightarrow \exists g \; s.t. g: B \to A$ is bijective

Let $g(0) = x, g(-1) = y$

This relation is not onto mapping because $R_g \neq cod_g = A$

In fact, each $g: B \to A$ is not onto$\Longrightarrow$ g is not bijective $\Longrightarrow$ B is not equivalent to A.

<u>Example</u> 2.4: Show that $N \approx B = \{5,10,15,20, \dots\}$

<u>Solution</u>: Define $f: N \to B \; s.t. f(x) = 5x \;\; \forall x \in N$

1-1? Let $x_1, x_2 \in N \; s.t. f(x_1) = f(x_2) \Longrightarrow 5x_1 = 5x_2$

$\Longrightarrow x_1 = x_2$

$\Longrightarrow$ f is injective

Onto? $R_f = \{y \in B: y = 5x, x \in N\} = \{5,10,15,20, \dots\dots\} = B$

$\therefore f$ is onto

$\therefore \; N \approx B$

<u>Example</u> 2.5: (**H.W.**) Show that $N \approx \{2,4,6, \dots\dots\}$

$N \approx \{-2, -4, -6, \dots\dots\}$

$Z^- = \{-1, -2, -3, \dots\dots\} \approx \; N$

<u>Theorem</u> 2.6: The equivalent relation ($\approx$) on sets is an equivalence relation

<u>Proof</u>:

1. $\approx$ reflexive? T.P. $A \approx A$ for any set A

$\exists i_A: A \to A \;\; s.t. \; i(x) = x \;\; \forall x \in A$

i_A is bijective $\Longrightarrow A \approx A \;\; \forall A$

$\therefore \; \approx$ reflexive

2. $\approx$ symmetric? Let A and B are two sets such that $A \approx B$ **T.P.** $B \approx A$

$A \approx B \Longrightarrow \exists f \; s.t. f: A \to B$ is bijective

Since f is bij. $\Longrightarrow \exists f^{-1}: B \to A$ such that f^{-1} is bij.

$$\Longrightarrow \; B \approx A$$

$\approx$ is symmetric

3. $\approx$ transitive? Let A, B and C are sets such that $A \approx B$ and $B \approx C$ **T.P.** $A \approx C$

$A \approx B \Longrightarrow \exists f \; s.t. f: A \to B$ is bijective

$B \approx C \Longrightarrow \exists g \; s.t. g: B \to C$ is bijective

$\therefore \; \exists gof: A \to C$ is bij. (by theorem 4.25 (chapter4))

$\therefore \; A \approx C$

$\therefore \approx$ is transitive

$\therefore \approx$ is an equivalence relation

Finite and Infinite Sets المجموعات المنتهية وغير المنتهية

A set A is said to be **finite** if A is empty or if A contains exactly m elements where
m is a positive integer; otherwise A is **infinite.**

المجموعة A تسمى **منتهية** اذا كانت مجموعة خالية او اذا كانت تحوي على عدد منتهي m من العناصر

وفيما عدا ذلك تكون المجموعة A **غير منتهية**.

<u>Remark</u> 2.7: The number of the elements in a finite set A is called the **size of A** and is denoted by $n(A)$ or $\#(A)$ or $|A|$.

<u>Example</u>:

Let $A = \{1,2\}$ finite set $\Longrightarrow \mathrm{n}(A) = 2$ عدد عناصر المجموعة

Let $A = \{\quad\}$ finite set $\Longrightarrow \#(A) = 0$

Let $A = \{\emptyset\}$ finite set $\Longrightarrow \#(A) = 1$

Let $A = \{1, \emptyset, \{1,3\}, [0,1], N\}$ finite set $\Longrightarrow |A| = 5$

<u>Remark</u> 2.8: If A is an infinite set $\Longrightarrow |A|$ or $\#(A)$ is not defined (does not exist)

<u>Example</u> 2.9:

Let $A = Z$ infinite set $\Longrightarrow |A|$ is not defined

Let $A = (9,40]$ infinite set $\Longrightarrow \#(\mathrm{A})$ does not exist

Cardinality of Finite Sets قدرة المجموعات المنتهية

Let $J_m = \{1,2,\dots,m\}$ be a set. A set A is called finite of size m $(n(A) = m)$ if and only if $A \approx J_m = \{1,2,\dots,m\}$. The positive number m is called **the cardinality of A**.

اذا كانت المجموعة منتهية فان عدد عناصرها يسمى **قدرة المجموعة**

Mathematically,

$$n(A) = m \Leftrightarrow A \approx \{1,2,\dots,m\} = J_m$$

$$n(A) = m \Leftrightarrow \exists f: A \to \{1,2,\dots,m\} \text{ s.t. } f \text{ is bijective}$$

<u>Example 2.10</u>: Let $A = \{\alpha, \beta, \sin(x)\}$. Find the cardinality of A.

Solution: Let $J_3 = \{1,2,3\}$ T.P. $A \approx J_3$

Define $f: A \to J_3\ s.t. f(\alpha) = 1, f(\beta) = 2,\ f(\sin(x)) = 3$

It is clear that f is 1-1 and onto $\Longrightarrow n(A) = 3$

<u>Theorem</u> 2.11: Any two finite sets have the same cardinal number if and only if there is a bijective map. between them

i.e., Let A and B be two finite sets. Then $n(A) = n(B) \Leftrightarrow A \approx B$

<u>Proof</u>: $\Longrightarrow$) let $n(A) = n(B) = m$ $\boldsymbol{T.P.}$ $A \approx B$

$n(A) = m \Longrightarrow A \approx \{1,2,\dots,m\}$….(1) (def. of $n(A)$)

$n(B) = m \Longrightarrow B \approx \{1,2,\dots,m\}$….(2) (def. of $n(B)$)

From (1) &(2)

$A \approx \{1,2,\dots,m\} \wedge B \approx \{1,2,\dots,m\}$

$\Longrightarrow A \approx \{1,2,\dots,m\} \wedge \{1,2,\dots,m\} \approx B$ [$\approx$ symmetric]

$\Longrightarrow A \approx B$ [$\approx$ transitive]

$\Longleftarrow$) Suppose $A \approx B$ **T.P.** $n(A) = n(B)$

Let $n(A) = m \Longrightarrow A \approx \{1,2,\dots,m\}$ (def. of $n(A)$)

$\Longrightarrow A \approx B \wedge A \approx \{1,2,\dots,m\}$

$\Longrightarrow B \approx A \wedge A \approx \{1,2,\dots,m\}$ [$\approx$ symmetric]

$\Longrightarrow B \approx \{1,2,\dots,m\}$ [$\approx$ transitive]

$\Longrightarrow n(B) = m$

$\Longrightarrow \mathrm{n}(A) = n(B)$

<u>Theorem</u> 2.12: Let A be a finite set and $A_1, A_2, \dots, A_n$ be a partition of A. Then

$$n(A) = n(A_1) + \cdots + n(A_n) \quad \text{(بدون برهان)}$$

<u>Theorem</u> 2.13: Let A and B be finite sets. Then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

<u>Proof</u>: Take $A\backslash B, A \cap B, B\backslash A$ as partition for $A \cup B$

$\Longrightarrow$ n$(A \cup B) = n(A\backslash B)+n(A \cap B) + n(B\backslash A)$ ....(1) (by theorem 2.12)

Take $A\backslash B, A \cap B,$ as partition for A $\Longrightarrow n(A) = n(A\backslash B)+n(A \cap B)$ (by Theorem 2.12)

$$\Longrightarrow n(A\backslash B)= n(A) - n(A \cap B)\ldots(2)$$

Take $B\backslash A,$ $A \cap B,$ as partition for B $\Longrightarrow n(B) = n(B\backslash A)+n(A \cap B)$ (by Theorem 2.12)

$$\Longrightarrow n(B\backslash A)= n(B) - n(A \cap B)\ldots(3)$$

Substitute (2)&(3) into (1)

$$\Longrightarrow \text{n}(A \cup B)$$
$$= n(A) - n(A \cap B) + n(A \cap B) + n(B) - n(A \cap B)$$
$$\Longrightarrow \text{n}(A \cup B) = n(A) + n(B) - n(A \cap B)$$

<u>Theorem</u> 2.14: Let A and B be finite sets. Then $n(A \times B) = n(A).n(B)$

<u>Proof</u>:

A is finite set $\Longrightarrow n(A) = m$ s.t. $A = \{a_1, a_2, \ldots, a_m\}$

B is finite set $\Longrightarrow n(B) = n$ s.t. $B = \{b_1, b_2, \ldots, b_n\}$

$A \times B = \{(a, b): a \in A, b \in B\}$

$A \times \{b_1\} = \{(a_1, b_1), (a_2, b_1), \ldots, (a_m, b_1)\}$

$A \times \{b_2\} = \{(a_1, b_2), (a_2, b_2), \ldots, (a_m, b_2)\}$

$A \times \{b_n\} = \{(a_1, b_n), (a_2, b_n), \ldots, (a_m, b_n)\}$

$\Longrightarrow A \times B = A \times \{b_1\} \cup A \times \{b_2\} \cup \ldots \cup A \times \{b_n\}$

Such that $A \times \{b_i\} \cap A \times \{b_j\} = \emptyset \quad \forall i \neq j$

$\therefore \quad A \times \{b_1\}, A \times \{b_2\}, \ldots, A \times \{b_n\}$ is a partition for $A \times B$

$$n(A \times B) = n(A \times \{b_1\}) + n(A \times \{b_2\}) + \cdots + n(A \times \{b_n\})$$

$$= \underbrace{m + m + \cdots + m}_{n-times} = mn = n(A).n(B)$$

Cardinality of infinite Sets قدرة المجموعات غير المنتهية

Let A be an infinite set. Then the cardinality of A is not a finite positive number. The cardinality of A is denoted by $\aleph_0$

المجموعة غير المنتهية تكون لها قدرة ولكن القدرة هنا لاتكون عدد معرف موجب

ترمز لقدرة المجموعة الغير منتهية الى حجم تلك المجموعة

<u>Example</u> 2.15: The cardinal number of N is denoted by $\aleph_0$

i.e., $n(N) = \#(N) = \aleph_0$

Countably Infinite Set المجموعة غير المنتهية القابلة للعد

An infinite set A is called **countable** if it is equivalent to the set of natural number. Thus, the cardinality of an infinite countable set is $\aleph_0$

المجموعة غير المنتهية تكون قابلة للعد اذا كافئت مجموعة الاعداد الطبيعية

Mathematically,

A is countable infinite set $\Leftrightarrow A \approx N \Leftrightarrow n(A) = n(N) = \aleph_0$
A is not countable infinite set $\Leftrightarrow A \not\approx N$

<u>Example</u> 2.16: Show that N is countably infinite

Solution: T. P. $N \approx N$

Define $f: N \to N \; s.t. \; f(x) = i_N(x) = x \quad \forall x \in N$

f is bijective (**prove**!)

$\therefore \; N \approx N$

<u>Example</u> 2.17: Show that the set of even positive numbers is countably infinite

Solution: T. P. $E^+ \approx N$

يمكن ايجاد دالة متقابلة بين المجموعتين اما بالتخمين او من خلال ايجاد معادلة مستقيم اذا علمت منه نقطتان

$\boldsymbol{E^+}$	0	2	4	6	8	10…
$\boldsymbol{N}$	1	2	3	4	5	6…

Let $f(0) = 1$ and $f(2) = 2$

$(x_1, y_1) = (0,1)$ and $(x_2, y_2) = (2,2)$

نجد معادلة المستقيم المحدد بالنقطتين (x_1, y_1) و (x_2, y_2)

$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$

$\frac{y-1}{x} = \frac{1}{2} \Longrightarrow \mathrm{y} - 1 = \frac{x}{2}$

$\Longrightarrow \mathrm{y} = \frac{\mathrm{x}}{2} + 1$

$\therefore \; f: E^+ \to N$ s.t. $\mathrm{f(x)} = \frac{\mathrm{x}}{2} + 1$

f is bijective (**prove**!)

$\therefore\ E^{+} \approx N$

<u>Example</u> 2.18: (H.W.) Show that the set of odd positive numbers is countably infinite

Solution: T. P. $N \approx O^{+}$

$\boldsymbol{N}$	1	2	3	4	5	6…….
$\boldsymbol{O^{+}}$	1	3	5	7	9	11 …..

Let $f(1) = 1$ and $f(2) = 3$

$(x_1, y_1) = (1,1)$ and $(x_2, y_2) = (2,3)$

نجد معادلة المستقيم المحدد بالنقطتين (x_1, y_1) و (x_2, y_2)

$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$

$\frac{y-1}{x-1} = \frac{2}{1} \Longrightarrow \mathrm{y} - 1 = 2x - 2$

$\Longrightarrow \mathrm{y} = 2\mathrm{x} - 1$

$\therefore\ f: N \to O^{+}$ s.t. f(x)= $2\mathrm{x} - 1$

f is bijective (**prove**!)

$\therefore\ N \approx O^{+}$

<u>Example</u> 2.19: (H. W.) Prove that $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots.\}$ is countably infinite set.

<u>Example</u> 2.20: (H. W.) Prove that $A = \{1, \sqrt{2}, \sqrt{3}, \dots.\}$ is countably infinite set

<u>Example</u> 2.21: Prove that $A = Z$ is countable infinite set

Solution: T. P. $Z \approx N$

T. P. $\exists f: Z \to N \; s.t. f$ is bijective

Z	0	1	-1	2	-2	3	-3…
N	1	2	3	4	5	6…	

1) Let $x \in Z \; s.t. x \geq 0$

$f(0) = 1, \;\; f(1) = 3$

نجد معادلة المستقيم المحدد بالنقطتين (x_1, y_1) و (x_2, y_2)

$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$

$\frac{y-1}{x} = \frac{2}{1} \Longrightarrow \mathrm{y} - 1 = 2x$

$\Longrightarrow \mathrm{y} = 2\mathrm{x} + 1 \dots..(1)$

2) Let $x \in Z \; s.t. x < 0$

$f(-1) = 2, \;\; f(-2) = 4$

نجد معادلة المستقيم المحدد بالنقطتين (x_1, y_1) و (x_2, y_2)

$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$

$\frac{y-2}{x+1} = \frac{2}{-1} \Longrightarrow \mathrm{y} - 2 = -2(x+1)$

$\Longrightarrow y = -2x$…..(2)

From (1) &(2), define $f: Z \to N\ s.t.\ f(x) = \begin{cases} 2x+1, & x \geq 0 \\ -2x, & x < 0 \end{cases}$

Show that f is bijective

$\therefore\ Z \approx N$

<u>Example</u> 2.22: Prove that $A_k = \{0, \mp k, \mp 2k, \mp 3k, \dots\}$ is countably infinite set

Solution:

سنبرهن بان $A_k \approx Z$ وبما ان $Z \approx N$ فحسب خاصية التعدي $A_k \approx N$

If $k = 1 \Longrightarrow \mathrm{A_k} = \mathrm{A_1} = \{0, \mp 1, \mp 2, \mp 3, \dots\} = \mathrm{Z}$

$A_1 \approx Z$ because $\exists f: A_1 \to Z\ s.t.\ f(x) = x$ is bijective

If $k = 2 \Longrightarrow \mathrm{A_k} = \mathrm{A_2} = \{0, \mp 2, \mp 4, \mp 6, \dots\}$

A_2	0	2	-2	4	-4	6 …
Z	0	1	-1	2	-2	3…

$A_2 \approx Z$ because $\exists f: A_2 \to Z\ s.t.\ f(x) = \frac{x}{2}$ is bijective

If $k = 3 \Longrightarrow \mathrm{A_k} = \mathrm{A_3} = \{0, \mp 3, \mp 6, \mp 9, \dots\}$

A_3	0	3	-3	6	-6	9 …
Z	0	1	-1	2	-2	3…

$A_3 \approx Z$ because $\exists f: A_3 \to Z\ s.t.\ f(x) = \frac{x}{3}$ is bijective

In general, $A_k \approx Z$ because $\exists f: A_k \to Z \; s.t. \; f(x) = \frac{x}{k}\ ,\ k \neq 0$ is bijective

Since, $A_k \approx Z \ \wedge Z \approx N \Longrightarrow A_k \approx N$ ($\approx$ is transitive)

$\therefore A_k$ is countable infinite set

<u>Theorem</u> 2.23: Any infinite subset of an infinite countable set is countable

i.e., If A is countable infinite set and $B \subseteq A$ then B is countable set

اي مجموعة جزئية من مجموعة قابلة للعد تكون قابلة للعد

<u>Theorem</u> 2.24: If A is countably infinite set then $A \cup \{a\}$ is also countably infinite set.

Proof: Let A be a countably infinite set $\Longrightarrow A \approx N$

$\exists f: A \to N$ bijective $s.t.\ f(a_1) = 1,,,, f(a_2) = 2,\ \ f(a_3) = 3, ...$

A	a_1	a_2	a_3…..
N	1	2	3…….

T.P. $A \cup \{a\} \approx N$

Define $g: A \cup \{a\} \to N$ s.t. $g(a) = 1,\ \ g(a_1) = 2,\ \ g(a_2) = 3,\ \ g(a_3) = 4, ...$

$A \cup \{a\}$	a	a_1	a_2	a_3…..
N	1	2	3	4…….

$g(a) = 1$

$g(a_1) = 2 = 1 + 1 = f(a_1) + 1$

$g(a_2) = 3 = 2 + 1 = f(a_2) + 1$

$g(a_3) = 4 = 3 + 1 = f(a_3) + 1$

In general,

$$g(x) = \begin{cases} 1, & x = a \\ f(x) + 1, & x \neq a \end{cases}$$

T.P. g is bijective

g is 1-1? Let $x_1, x_2 \in A \cup \{a\}\ s.t.\ g(x_1) = g(x_2)$ T.P. $x_1 = x_2$

$$g(x_1) = g(x_2) \implies \begin{cases} 1 = 1, & x_1 = a = x_2 \\ f(x_1) + 1 = f(x_2) + 1, & x_1 \neq a \text{ and } x_2 \neq a \end{cases}$$

$$\implies f(x_1) + 1 = f(x_2) + 1$$

$$\implies f(x_1) = f(x_2)$$

$$\implies \mathrm{x}_1 = x_2 \quad \text{(f is 1-1)}$$

$\therefore$ g is 1-1

g is onto? $R_g = \{y \in N: \exists x \in A \cup \{a\}, y = g(x)\}$

$=N$

g is onto

g is bijective

$\therefore\ A \cup \{a\} \approx N$

$\therefore\ A \cup \{a\}$ is countably infinite set

<u>Remark</u> 2.25:

1) Every finite set is countable

2) Q the set of rational numbers is countably infinite set

3) R the set of real numbers is uncountable infinite set

د. صبا ناصر