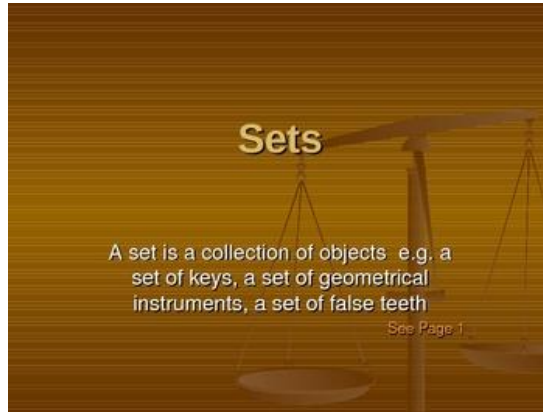


CHAPTER TWO

Set Theory

نظرية المجموعات



Chapter Two Contents:

1. Basic notion of sets مفهوم المجموعات
2. Subsets المجموعات الجزئية
3. Algebra of sets (union, intersection, difference, complement, symmetric difference) جبر المجموعات أو العمليات على المجموعات

Definition 2.1: Set

A set is an unordered collection of objects. The objects are called **the elements** or **members** of the set.

المجموعة هي تجمع من الأشياء المعرفة بدون ترتيب والتي تسمى بالعناصر أو أعضاء تنتمي للمجموعة.

Remark 2.2

1. The capital letters usually used to represent sets such as A, B, C, \dots etc.
2. The small letters such as a, b, c, d, \dots etc are used to represents the members or the elements of the set.
3. Membership in a set is denoted as follows: انتماء عنصر لمجموعة يعبر عنه بالشكل التالي

$a \in A$ denotes that a belongs to a set A

4. Non-membership to a set is denoted as follows: عدم انتماء عنصر لمجموعة يعبر عنه بالشكل التالي

$a \notin A$ denotes that a does not belong to a set A

Specifying a Set: طرق التعبير عن المجموعة**1. Listing members of a set:** الطريقة الجدولية

In this way, we list all non-repeated members of a set separated by commas and contained in braces $\{ \}$. The members are not in an order.

الطريقة الجدولية أو طريقة القائمة: في هذه الطريقة نضع جميع العناصر الغير المعادة بين قوسي مجموعة وبفواصل تفصل بينها. عناصر المجموعة لايشترط أن تكون مرتبة بطريقة معينة

Example2.3:

1. $A = \{1, 2, -5, 0, 9\}$, $B = \{x, y, \text{Ali, fish}\}$, $C = \{y_1, y_2, y_3\}$ are sets
2. The set of vowel letters in English: $V = \{a, e, i, o, u\}$
3. The set of even positive numbers less than 8 is: $W = \{0, 2, 4, 6\}$.
4. The set of positive numbers less than 50 is: $K = \{1, 2, \dots, 49\}$

2. Listing a set property: استخدام الصفة المميزة للمجموعة

In this way, we state the property that characterize the elements in a set in as follows: $\{x: p(x)\}$, where x is a variable and $p(x)$ is an open sentence.

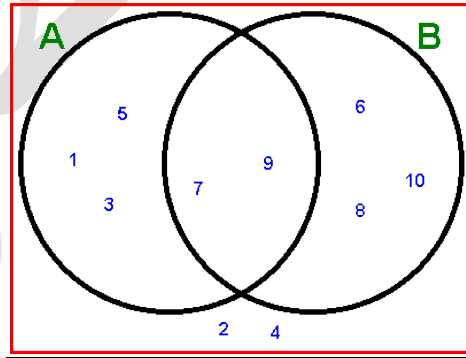
Example2.4: $A = \{x: x \in Q\}$

$$B = \{x: x \text{ is positive odd and } x < 10\} = \{1, 3, 5, 7, 9\}$$

$$C = \{x \in N: -3 \leq x \leq 5\} = \{1, 2, 3, 4, 5\}$$

3. Venn Diagrams: مخططات فن

في هذه الطريقة توضع عناصر المجموعة داخل منحنى مغلق يمثل المجموعة وتستخدم هذه الطريقة لأغراض توضيحية فقط.

**Definition2.5: Empty Set المجموعة الخالية**

The set that contains no elements is called an empty set and is denoted by $\{ \}$ or \emptyset .

تسمى المجموعة التي لاتحتوي أي عنصر بالمجموعة الخالية.

Example 2.6: $A = \{x \in N: 2 < x < 3\} = \emptyset$

$$B = \{x \in E: \sqrt{x}=1\} = \emptyset$$

$$C = \{x \in N: x < 0\} = \{\}$$

Subsets: المجموعات الجزئية

The set A is a subset of a set B ($A \subseteq B$) if and only if every element of A is an element of B . In other words,

$$A \subseteq B \text{ iff } \forall x, x \in A \Rightarrow x \in B$$

Remark 2.7: A is **not a subset** of B is denoted by $A \not\subseteq B$.

$$A \not\subseteq B \text{ if and only if } \sim[\forall x, x \in A \Rightarrow x \in B]$$

$$\text{if and only if } \exists x; x \in A \wedge x \notin B$$

Example 2.8: Consider the sets $A = \{2\}$, $B = \{1, 2, 3\}$ and $C = \{4, 5\}$ and $D = \{-2, 1, 2, 3, 4, 5\}$. Then $A \subseteq B$, $A \subseteq D$, $B \subseteq D$ and $C \subseteq D$.

It is true that $A \subseteq A$, $B \subseteq B$, $C \subseteq C$ and $D \subseteq D$.

Example 2.9: Let $A = \{4, 9\}$ and $B = \{x \in N: 1 < x < 10\}$. Determine whether $A \subseteq B$ or $B \subseteq A$.

Solution: The set B can be written as $B = \{2, \dots, 9\}$. Then

$$\forall x, x \in A \Rightarrow x \in B$$

Hence, $A \subseteq B$.

But $B \not\subseteq A$ because, for example, $\exists x = 5 \in B \wedge x \notin A$.

Example 2.10: Let $A = \{x \in N: x > 3\}$ and $B = \{x \in N: x^2 > 4\}$. Is $A \subseteq B$?

Is $B \subseteq A$?

Solution: Let $x \in A \Rightarrow x \in N$ and $x > 3$
 $\Rightarrow x^2 > 9$
 $\Rightarrow x^2 > 4 \Rightarrow x \in B.$
 $\Rightarrow A \subseteq B.$

Is $B \subseteq A$?

Example2.11: (H. W.) Let $A = \{-2, 3\}, B = \{x \in Z: x^3 - x^2 - 6x = 0\}$. Determine whether $A \subseteq B$ or $B \subseteq A$?

Example2.12: (H. W.) Let $A = \{x \in N: x \geq 4\}$ and $B = \{x \in N: x < 9\}$. Determine whether $A \subseteq B$ or $B \subseteq A$?

Theorem2.13: Let A, B and C be any sets, then

1. $\emptyset \subseteq A$
2. $A \subseteq A$
3. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

Proof 1: T. P $\emptyset \subseteq A$, i.e., T. P $\forall x \in \emptyset \Rightarrow x \in A$
 $F \Rightarrow (T \text{ or } F) = T$
 $\Rightarrow \emptyset \subseteq A.$

Proof 2: T. P $A \subseteq A$, i.e., T. P $\forall x \in A \Rightarrow x \in A$
 $T \Rightarrow T = T$
 $\Rightarrow A \subseteq A.$

Proof 3: T. P If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

T. P $\forall x, x \in A \Rightarrow x \in C$
 $\therefore A \subseteq B \Rightarrow \forall x, x \in A \Rightarrow x \in B$
 $\therefore B \subseteq C \Rightarrow \forall x, x \in B \Rightarrow x \in C$
 $\therefore \forall x, x \in A \Rightarrow x \in B \Rightarrow x \in C$
 $\therefore \forall x, x \in A \Rightarrow x \in C$
 $\therefore A \subseteq C.$

Definition 2.14: Proper Subset المجموعة الجزئية الفعلية

A set A is called a **proper** subset of B and denoted by $(A \subset B)$ if and only if $A \subseteq B$ and there exist an element $x \in B$ that is $x \notin A$.

$$\text{i.e., } A \subset B \text{ iff } \{\forall x, x \in A \Rightarrow x \in B\} \wedge \{\exists y, y \in B \wedge y \notin A\}$$

Example 2.15: Let $A = \{x \in \mathbb{N} \vee x^2 - 16 = 0\}$

$$B = \{x \in \mathbb{N} : x^2 - 16 = 0\}$$

Determine if $A \subset B$ or $B \subset A$.

Solution: $A = \{1, 2, 3, \dots\} \cup \{4, -4\}$ and $B = \{4\}$

It is clear that $B \subset A$ because $B \subseteq A$ and

$$\exists y = \{1, 2, 3, 5, \dots\} \in A \wedge y \notin B.$$

Example 2.16: (H. W.) Let $A = \{\text{fish, dog, bird}\}$, $B = \{x, y, z, w\}$.

Determine if $A \subset B$ or $B \subset A$.

Example 2.17: (H. W.) Let $A = \{x \in \mathbb{Z} : -2 \leq x \leq 10\}$

$$B = \{x \in \mathbb{Z} \vee x^2 + 9 = 0\}$$

Determine if $A \subset B$ or $B \subset A$.

Solution: $A = \{-2, -1, 0, 1, \dots, 10\}$ and $B = \{0, \pm 1, \pm 2, \dots\} \cup \{3i, -3i\}$

Complete the solution!

Definition 2.18: Equal Sets المجموعات المتساوية

Two sets A and B are equal if they both have the same elements or, equivalently, if each is contained in the other.

يقال أن المجموعة A تساوي المجموعة B إذا كانت لهما نفس العناصر أو إذا كانت كل مجموعة جزئية من الأخرى.

$$\begin{aligned}
A = B & \text{ iff } A \subseteq B \wedge B \subseteq A \\
& \leftrightarrow \{ \forall x, x \in A \rightarrow x \in B \} \wedge \{ \forall x, x \in B \rightarrow x \in A \} \\
& \leftrightarrow \{ \forall x, x \in A \leftrightarrow x \in B \}
\end{aligned}$$

Example2.19: Let $A = \{x \in Z \wedge 5x^2 + 2 = 0\}$
 $B = \{x \in N: 2x + 3 = 0\}$

Is $A = B$?

Solution: $A = Z \cap \{\mp \sqrt{\frac{2}{5}} i\} = \emptyset$

$$B = \emptyset$$

$$\Rightarrow A = B$$

Lemma2.20: (H. W.) Prove that: $A = A$, for any set A .

Definition2.21: Universal Set المجموعة الشاملة

Universal set U is the set that contains all the elements or the sets we have under discussion.

المجموعة الشاملة: هي المجموعة التي تحوي جميع العناصر أو المجموعات قيد المناقشة ويرمز لها بالرمز U .

Example2.22: Let $A = \{x, y, 3\}$, $B = \{2, -5, 100\}$, $C = \{2, 3, 1\}$
 Find a universal set U .

Example2.23: Let $A = \{x \in R: 2 \leq x \leq 5\}$ and $B = \{x \in R: -1 \leq x \leq 2\}$

Find a universal set U .

Definition2.24: Family of Sets عائلة المجموعات

Family of sets is a set that have other sets as members.

يقال للمجموعة التي يكون كل عنصر من عناصرها مجموعة أنها عائلة مجموعات

Example 2.25:

1. $A = \{\{1\}, \{2\}\}$ is a family of sets
2. $B = \{\emptyset\}$ is a family of a set
3. $X = \{X\}$ is a family of a set
4. $A = \{x, \{y, z\}, \{1, \dots, 5\}\}$
5. $H = \{A: A \text{ is a subset of } \{1, 2, 3\}\}$
6. $K = \{A_i: A_i = \{2, \frac{2}{i}\}, i = 1, 2, 3\}$

Definition 2.26: Power Set مجموعة القوى أو مجموعة الأجزاء

Given a set X , the **power set of X** is the set of all subsets of X . The power set of X is denoted by $P(X)$.

لتكن X مجموعة يقال لمجموعة كل المجموعات الجزئية من X أنها مجموعة القوى ل X ويرمز لها بالرمز $P(X)$.

$$P(X) = \{A: A \subseteq X\}, \quad A \in P(X) \Leftrightarrow A \subseteq X$$

Example 2.27: Find $P(X)$ for the following sets X :

1. $X = \{1, 2, a\}$, $P(X) = \{\emptyset, X, \{1\}, \{2\}, \{a\}, \{1, 2\}, \{1, a\}, \{2, a\}\}$
2. $X = \{\emptyset\}$, $P(X) = \{\emptyset, X\}$
3. $X = \{-2, 3\}$, $P(X) = \{\emptyset, X, \{-2\}, \{3\}\}$

Remark 2.28: 1. Since $X \subseteq X$, then $P(X) \neq \emptyset$.

2. If X is finite and has n elements, then $P(X)$ has 2^n elements.

Theorem 2.29: Let X, Y be any sets, then $X \subseteq Y \Leftrightarrow P(X) \subseteq P(Y)$.

Proof: (\Rightarrow) Let $X \subseteq Y$ T.P $P(X) \subseteq P(Y)$

Let $A \in P(X) \Rightarrow A \subseteq X$ (By def. of $P(X)$)

$\Rightarrow A \subseteq Y$ ($X \subseteq Y$)

$\Rightarrow A \in P(Y)$

$\therefore P(X) \subseteq P(Y)$

(\Leftarrow) Let $P(X) \subseteq P(Y)$ To Prove $X \subseteq Y$

Let $x \in X \Rightarrow \{x\} \subseteq X$

$\Rightarrow \{x\} \in P(X)$

$\Rightarrow \{x\} \in P(Y)$ { $P(X) \subseteq P(Y)$ }

$\Rightarrow \{x\} \subseteq Y$

$\Rightarrow x \in Y$

$\therefore X \subseteq Y$

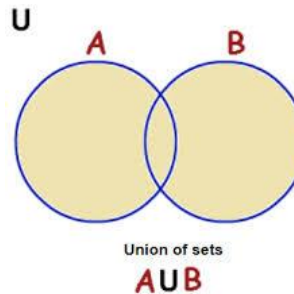
Algebra of sets:

1. Union الاتحاد

The **union** of the sets A and B , denoted by $A \cup B$, is the set of elements which belong to A or to B .

إتحاد مجموعتين A و B هي مجموعة العناصر التي تنتمي للمجموعة A أو المجموعة B .

$$\begin{aligned} A \cup B &= \{x, x \in A \vee x \in B\} \\ x \in A \cup B &\Leftrightarrow x \in A \vee x \in B \\ x \notin A \cup B &\Leftrightarrow x \notin A \wedge x \notin B \end{aligned}$$



Example 2.30: Let $A = \{x \in \mathbb{N} : 1 \leq x \leq 5\} = \{1, 2, 3, 4, 5\}$

$B = \{x \in \mathbb{N} : 8 \leq x \leq 12\} = \{8, 9, 10, 11, 12\}$

Find $A \cup B$, $B \cup A$, $A \cup A$, and $B \cup \emptyset$

Solution: $A \cup B = B \cup A = \{1, \dots, 5, 8, \dots, 12\}$

$$A \cup A = A$$

$$B \cup \emptyset = B$$

Example 2.31: Let $A = \{x \in \mathbb{R} : -2 \leq x \leq 5\} = [-2, 5]$,
 $B = \{x \in \mathbb{E} : x^2 - 16 = 0\} = \{4, -4\}$
 $C = \{1, 4\}$

Find $A \cup (B \cup C)$, $(A \cup B) \cup C$, $P(B)$, $P(C)$, $P(B \cup C)$

Definition 2.32: Generalization of the union تعميم الاتحاد

Let $A_1, A_2, \dots, A_n, A_{n+1}, \dots$ be any sets. Then:

$$\bigcup_{i=1}^n A_i = A_1 \cup A_2 \cup \dots \cup A_n$$

اتحاد عدد منتهى من المجموعات

In general,

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup \dots \cup A_n \cup A_{n+1} \cup \dots$$

اتحاد عدد غير منتهى من المجموعات

Example 2.33: Let $H = \{A_i; A_i = \{2i + 3\}, i \in \mathbb{Z}\}$

Find

$$\bigcup_{i=1}^4 A_i, \text{ and } \bigcup_{i=-1}^3 A_i$$

Solution: $\bigcup_{i=1}^4 A_i = A_1 \cup A_2 \cup A_3 \cup A_4 = \{5\} \cup \{7\} \cup \{9\} \cup \{11\}$
 $= \{5, 7, 9, 11\}$

$$\bigcup_{i=-1}^3 A_i = A_{-1} \cup A_0 \cup A_1 \cup A_2 \cup A_3 = \{1\} \cup \{3\} \cup \{5\} \cup \{7\} \cup \{9\}$$

$$= \{1, 3, 5, 7, 9\}$$

Example 2.34: Let $K = \{A_n; A_n = (-n, n), n \in \mathbb{N}\}$

Find $A_1 \cup \bigcup_{n=2}^{\infty} A_n$

Solution: $A_1 \cup \bigcup_{n=2}^{\infty} A_n = (-1, 1) \cup (-2, 2) \cup \dots \cup (-k, k) \cup \dots$

Example 2.35: (H. W.) Let $K = \{A_n; A_n = [n, n + 10), n \in \mathbb{Z}\}$

Find $\bigcup_{n=-2}^2 A_n$

Example 2.36: (H. W.) Let $K = \{A_j; A_j = \{j + 1, j + 2\}, j \in N\}$
 Find $\bigcup_{j=3}^{10} A_j$

Theorem 2.37: Let A, B and C any three sets. Then:

1. $A \cup \emptyset = A$ (Identity law)
2. $A \cup A = A$ (Idempotent law)
3. $A \cup U = U$ (Domination law)
4. $(A \cup B) \cup C = A \cup (B \cup C)$ (H.W.)
5. $A \cup C = C \cup A$
6. $A \subseteq B \Leftrightarrow A \cup B = B$
 $B \subseteq A \Leftrightarrow A \cup B = A$
7. $A \subseteq A \cup B$ (H.W.)
 $B \subseteq A \cup B$
8. $P(A) \cup P(B) \subseteq P(A \cup B)$

Proof 1: To prove $A \cup \emptyset \subseteq A \wedge A \subseteq A \cup \emptyset$

T. P $A \cup \emptyset \subseteq A$ (T. P $\forall x \in A \cup \emptyset \Rightarrow x \in A$)

Let $x \in A \cup \emptyset \Rightarrow x \in A \vee x \in \emptyset$ (def. of \cup)
 $\Rightarrow x \in A \vee F$
 $\Rightarrow x \in A$ ($p \vee F = p$)
 $\therefore A \cup \emptyset \subseteq A$ (1)

Let $x \in A \Rightarrow x \in A \vee F$ ($p \vee F = p$)
 $\Rightarrow x \in A \vee x \in \emptyset$
 $\Rightarrow x \in A \cup \emptyset$ (def. of \cup)
 $\therefore A \subseteq A \cup \emptyset$ (2)

From (1) & (2), $A \cup \emptyset = A$

Proof 2: To prove $A \cup A \subseteq A \wedge A \subseteq A \cup A$

T. P $A \cup A \subseteq A$ (T. P $\forall x \in A \cup A \Rightarrow x \in A$)

Let $x \in A \cup A \Rightarrow x \in A \vee x \in A$ (def. of \cup)
 $\Rightarrow x \in A$ ($p \vee p = p$)
 $\therefore A \cup A \subseteq A$ (1)

Let $x \in A \Rightarrow x \in A \vee x \in A$ ($p \vee p = p$)

$$\begin{aligned} &\Rightarrow x \in A \cup A \quad (\text{def. of } \cup) \\ &\therefore A \subseteq A \cup A \dots\dots\dots(2) \end{aligned}$$

From (1) & (2), $A \cup A = A$

Proof 3: To prove $A \cup U \subseteq U \wedge U \subseteq A \cup U$

$$\begin{aligned} &\text{T. P } A \cup U \subseteq U \quad (\text{T. P } \forall x \in A \cup U \Rightarrow x \in U) \\ &\text{Let } x \in A \cup U \Rightarrow x \in A \vee x \in U \quad (\text{def. of } \cup) \\ &\quad \Rightarrow x \in U \vee x \in U \quad (A \subseteq U) \\ &\quad \Rightarrow x \in U \\ &\therefore A \cup U \subseteq U \dots\dots\dots(1) \end{aligned}$$

$$\begin{aligned} &\text{Let } x \in U \Rightarrow x \in U \vee x \in A \quad (\text{T } \vee p = \text{T}) \\ &\quad \Rightarrow x \in A \cup U \quad (\text{def. of } \cup) \\ &\therefore U \subseteq A \cup U \dots\dots\dots(2) \end{aligned}$$

From (1) & (2), $A \cup U = U$

Proof 5: To prove $A \cup C \subseteq C \cup A \wedge C \cup A \subseteq A \cup C$

$$\begin{aligned} &\text{T. P } A \cup C \subseteq C \cup A \\ &\text{Let } x \in A \cup C \Rightarrow x \in A \vee x \in C \quad (\text{def. of } \cup) \\ &\quad \Rightarrow x \in C \vee x \in A \quad (\vee \text{ is commutative}) \\ &\quad x \in C \cup A \quad (\text{def. of } \cup) \\ &\therefore A \cup C \subseteq C \cup A \dots\dots\dots(1) \end{aligned}$$

Similarly, show that $C \cup A \subseteq A \cup C$ (H. W.)(2)

From (1) & (2), $A \cup C = C \cup A$

Proof 6: To prove $A \subseteq B \Leftrightarrow A \cup B = B$

$$\begin{aligned} &(\Rightarrow) \text{ if } A \subseteq B \text{ T. P } A \cup B = B \\ &\quad \text{T. P } A \cup B \subseteq B \wedge B \subseteq A \cup B \\ &\text{Let } x \in A \cup B \Rightarrow x \in A \vee x \in B \quad (\text{def. of } \cup) \\ &\quad \Rightarrow x \in B \vee x \in B \quad (\text{by hypo. } A \subseteq B) \\ &\quad \Rightarrow x \in B \quad (p \vee p = p) \\ &\therefore A \cup B \subseteq B \dots\dots\dots(1) \end{aligned}$$

$$\begin{aligned} \text{Let } x \in B &\Rightarrow x \in B \vee x \in A \quad (\text{T} \vee p = \text{T}) \\ &\Rightarrow x \in A \cup B \quad (\text{def. of } \cup) \end{aligned}$$

$$\therefore B \subseteq A \cup B \dots\dots\dots(2)$$

From (1) & (2), $A \cup B = B$

(\Leftarrow) If $A \cup B = B$ To prove $A \subseteq B$

$$\begin{aligned} \text{Let } x \in A &\Rightarrow x \in A \vee x \in B \quad (\text{T} \vee p = \text{T}) \\ &\Rightarrow x \in A \cup B \quad (\text{def. of } \cup) \\ &\Rightarrow x \in B \quad (\text{by hypo. } A \cup B = B) \end{aligned}$$

$$\therefore A \subseteq B$$

Proof 8: T. P $P(A) \cup P(B) \subseteq P(A \cup B)$

Let $X \in P(A) \cup P(B)$ To prove $X \in P(A \cup B)$

$$\begin{aligned} X \in P(A) \cup P(B) &\Rightarrow X \in P(A) \vee X \in P(B) && (\text{def. of } \cup) \\ &\Rightarrow X \subseteq A \vee X \subseteq B && (\text{def. of } P(A)) \\ &\Rightarrow X \subseteq A \cup B && (\text{def. of } \cup) \\ &\Rightarrow X \in P(A \cup B) && (\text{def. of } P(A \cup B)) \end{aligned}$$

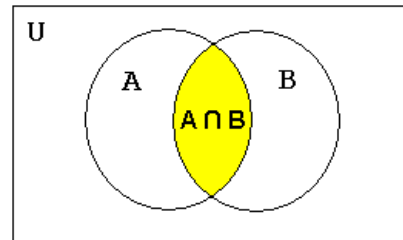
$$\therefore P(A) \cup P(B) \subseteq P(A \cup B)$$

2. Intersection التقاطع

The **intersection** of the sets A and B , denoted by $A \cap B$, is the set of elements which belong to both A **and** to B .

تقاطع مجموعتين A و B هي مجموعة العناصر التي تنتمي لكلا المجموعتين معاً.

$$\begin{aligned} A \cap B &= \{x; x \in A \wedge x \in B\} \\ x \in A \cap B &\Leftrightarrow x \in A \wedge x \in B \\ x \notin A \cap B &\Leftrightarrow x \notin A \vee x \notin B \end{aligned}$$



Example2.38: Let $A = [2,9], B = (5, 14], C = (8, 12)$
Find $A \cap B, A \cap \emptyset, B \cap B, B \cap C, A \cap C, (A \cap B) \cup (B \cap C)$

Definition2.39: Generalization of the intersection تعميم التقاطع

Let $A_1, A_2, \dots, A_n, A_{n+1}, \dots$ be any sets. Then:

$$A_1 \cap A_2 \cap \dots \cap A_n = \bigcap_{i=1}^n A_i = \{x, x \in A_i \forall i = 1, 2, \dots, n\}$$

تقاطع عدد منتهي من المجموعات

In general,

$$A_1 \cap A_2 \cap \dots \cap A_n \cap A_{n+1} \cap \dots = \bigcap_{i=1}^{\infty} A_i = \{x, x \in A_i, \forall i\}$$

تقاطع عدد غير منتهي من المجموعات

Example2.40: Let $X = \{A_i; A_i = \{1, 2, 3, \dots, i\}; i \in N\}$

Find

$$\left(\bigcup_{i=1}^5 A_i\right) \cap \left(\bigcap_{i=1}^5 A_i\right)$$

Solution: $A_1 = \{1\}, A_2 = \{1,2\}, \dots, A_5 = \{1, 2, 3, 4, 5\}$

$$\bigcup_{i=1}^5 A_i = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 = \{1,2,3,4,5\}$$

$$\bigcap_{i=1}^5 A_i = A_1 \cap \dots \cap A_5 = \{1\}$$

$$\left(\bigcup_{i=1}^5 A_i\right) \cap \left(\bigcap_{i=1}^5 A_i\right) = \{1\}$$

Example2.41: Let $X = \left\{A_j; A_j = \left(-\frac{1}{j}, \frac{1}{j}\right); j \in N\right\}$

Find

$$\bigcap_{j=2}^4 A_j \quad \text{and} \quad \bigcup_{j=2}^4 A_j$$

Solution:

$$\bigcap_{j=2}^4 A_j = A_2 \cap A_3 \cap A_4 = \left(-\frac{1}{2}, \frac{1}{2}\right) \cap \left(-\frac{1}{3}, \frac{1}{3}\right) \cap \left(-\frac{1}{4}, \frac{1}{4}\right) = \left(-\frac{1}{4}, \frac{1}{4}\right)$$

$$\bigcup_{j=2}^4 A_j = A_2 \cup A_3 \cup A_4 = \left(-\frac{1}{2}, \frac{1}{2}\right) \cup \left(-\frac{1}{3}, \frac{1}{3}\right) \cup \left(-\frac{1}{4}, \frac{1}{4}\right) = \left(-\frac{1}{2}, \frac{1}{2}\right)$$

Example2.42: (H. W.) Let $X = \{A_n; A_n = \{n^3 + 1\}; n \in Z\}$

Find

$$\bigcap_{n=-3}^0 A_n, \bigcap_{n=1}^{\infty} A_n, P(\bigcup_{n=1}^3 A_n)$$

Example2.43: (H. W.) Let $X = \{A_n; A_n = \{n - 2, n - 1, n\}; n \in N\}$

Find

$$\bigcap_{n=1}^{\infty} A_n, \bigcup_{n=1}^{\infty} A_n$$

Theorem2.44: Let A, B and C any three sets. Then:

1. $A \cap \emptyset = \emptyset$ (H. W.) (Domination law)
2. $A \cap A = A$ (H. W.) (Idempotent law)
3. $A \cap U = A$ (Identity law)
4. $(A \cap B) \cap C = A \cap (B \cap C)$
5. $A \cap C = C \cap A$ (H. W.)
6. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
7. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (H. W.)
8. $A \cap B \subseteq A$ (H. W.)
 $A \cap B \subseteq B$
9. $A \subseteq B \Leftrightarrow A \cap B = A$
10. $P(A) \cap P(B) = P(A \cap B)$ (H. W.)

Proof 3: T.P $A \cap U \subseteq A \wedge A \subseteq A \cap U$

Assume that $x \in A \cap U \Rightarrow x \in A \wedge x \in U$
 $\Rightarrow x \in A$

$\therefore A \cap U \subseteq A \dots\dots\dots(1)$

Let $x \in A$ T.P $x \in A \cap U$
 $x \in A \Rightarrow x \in A \wedge x \in U$ [$A \subseteq U$]
 $\Rightarrow x \in A \cap U \dots\dots\dots(2)$

From (1) & (2), $A \cap U = A$

Proof 4:

Let $x \in (A \cap B) \cap C \Leftrightarrow x \in (A \cap B) \wedge x \in C$ (def. of \cap)
 $\Leftrightarrow (x \in A \wedge x \in B) \wedge x \in C$ (def. of \cap)
 $\Leftrightarrow x \in A \wedge (x \in B \wedge x \in C)$ (\wedge is assoc.)
 $\Leftrightarrow x \in A \wedge x \in B \cap C$ (def. of \cap)
 $\Leftrightarrow x \in A \cap (B \cap C)$ (def. of \cap)

$\therefore (A \cap B) \cap C = A \cap (B \cap C)$

Proof 6:

Let $x \in A \cap (B \cup C) \Leftrightarrow x \in A \wedge x \in B \cup C$ (def. of \cap)
 $\Leftrightarrow x \in A \wedge (x \in B \vee x \in C)$ (def. of \cup)
 $\Leftrightarrow (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$ (distribute \wedge on \vee)
 $\Leftrightarrow x \in A \cap B \vee x \in A \cap C$ (def. of \cap)
 $\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$ (def. of \cup)

Proof 9: T.P $A \subseteq B \Leftrightarrow A \cap B = A$

(\Rightarrow) Suppose $A \subseteq B$ T.P $A \cap B \subseteq A \wedge A \subseteq A \cap B$

Let $x \in A \cap B \Rightarrow x \in A \wedge x \in B$ (def. of \cap)
 $\Rightarrow x \in A$

$\therefore A \cap B \subseteq A \dots\dots\dots(1)$

Let $x \in A \Rightarrow x \in A \wedge x \in B$ [$A \subseteq B$]
 $\Rightarrow x \in A \cap B$
 $\Rightarrow A \subseteq A \cap B \dots\dots\dots(2)$

From (1) & (2), $A \cap B = A$

(\Leftarrow) Let $A \cap B = A$ **T. P** $A \subseteq B$ (**H. W.**)

3. The Complement المتممة أو المكملة

Let U be a universal set and A be any subset of U . The **complement** of a set A , denoted by A^c , is the set of elements which belong to U but do not belong to A .

متممة المجموعة A هي مجموعة العناصر التي تنتمي للمجموعة الشاملة ولا تنتمي للمجموعة A .

$$\begin{aligned} A^c &= \{x, x \in U \wedge x \notin A\} = U \setminus A \\ x \in A^c &\Leftrightarrow x \notin A \\ x \in A &\Leftrightarrow x \notin A^c \end{aligned}$$

Example 2.45: Let $U = Z, A = \{-1, 0, 1\}$. Find A^c .

Solution: $A^c = Z \setminus A$

Example 2.46: Let $U = [0, 8), A = \{1, 2\}$. Find A^c .

$$A^c = [0, 1) \cup (1, 2) \cup (2, 8)$$

Example 2.47: Let $U = \{1, 2, \dots, 10\}$,

$$A = \{x \in N: 1 \leq x \leq 3\} = \{1, 2, 3\}$$

$$B = \{x \in N: 8 \leq x \leq 10\} = \{8, 9, 10\}$$

$$C = \{x \in N: 1 \leq x \leq 2\} = \{1, 2\}$$

Find $A^c, B^c, C^c, (A \cup B)^c, (A \cap C)^c, (C \cup B)^c$

Solution: $A^c = \{4, 5, \dots, 10\}$

$$B^c = \{1, 2, \dots, 7\}$$

$$C^c =$$

$$(A \cup B)^c = \{1, 2, 3, 8, 9, 10\}^c = \{4, 5, 6, 7\}$$

$$(A \cap C)^c = \{1, 2\}^c = \{3, 4, \dots, 10\}$$

Theorem 2.48: Let A and B any two sets. Then:

1. $\emptyset^c = U, U^c = \emptyset$
2. $A^c \cap A = \emptyset$ (H. W.), $A^c \cup A = U, (A^c)^c = A$ (H. W.)
3. $(A \cup B)^c = A^c \cap B^c$
4. $(A \cap B)^c = A^c \cup B^c$ (H. W.)
5. $A \subseteq B \Leftrightarrow B^c \subseteq A^c$ (H. W.)
6. $A \subseteq B \Leftrightarrow A \cap B^c = \emptyset$

Proof 1: T. P $\emptyset^c = U$

Assume that $\emptyset^c \neq U$ برهان غير مباشر

$\exists x, x \in \emptyset^c \wedge x \notin U$

$x \in \emptyset^c \Rightarrow x \in U \wedge x \notin \emptyset$ (def. of \emptyset^c)

$\Rightarrow x \in U$ تناقض مع الفرض

$\therefore \emptyset^c = U$

Proof 1: T. P $U^c = \emptyset$

Assume that $U^c \neq \emptyset$ برهان غير مباشر

$\exists x, x \in U^c \wedge x \notin \emptyset$

$x \in U^c \Rightarrow x \in U \wedge x \notin U$ تناقض

$\therefore U^c = \emptyset$

Proof 2: T. P $A^c \cup A = U$

Assume that $A^c \cup A \neq U$ برهان غير مباشر

$\exists x, x \in A^c \cup A \wedge x \notin U \dots \dots (*)$

$x \in A^c \cup A \Rightarrow x \in A^c \vee x \in A$ (def. of \cup)

If $x \in A^c \Rightarrow x \in U \wedge x \notin A$ (contradiction with $x \notin U$ in $*$)

If $x \in A \Rightarrow x \in U \wedge x \notin A^c$ (contradiction with $x \notin U$ in $*$)

$\therefore A^c \cup A = U$

Proof 3: T. P $(A \cup B)^c = A^c \cap B^c$

$x \in (A \cup B)^c \Leftrightarrow x \notin A \cup B$ (def of complement)

$\Leftrightarrow x \notin A \wedge x \notin B$ (def of \cup)

$\Leftrightarrow x \in A^c \wedge x \in B^c$ (def. of A^c)

$\Leftrightarrow x \in A^c \cap B^c$ (def. of \cap)

Proof 6: (\Rightarrow) Let $A \subseteq B$ **T. P** $A \cap B^c = \emptyset$

Assume that $A \cap B^c \neq \emptyset$ برهان غير مباشر بالتناقض

$\exists x, x \in A \cap B^c \Rightarrow x \in A \wedge x \in B^c$

$$\begin{aligned} &\Rightarrow x \in B \wedge x \in B^c \text{ (by hypo. } A \subseteq B) \\ &\Rightarrow \text{تناقض} \\ &\therefore A \cap B^c = \emptyset \end{aligned}$$

(\Leftarrow) Let $A \cap B^c = \emptyset$ T. P $A \subseteq B$

$$\begin{aligned} \text{Let } x \in A &\Rightarrow x \notin B^c \text{ (by hypo. } A \cap B^c = \emptyset) \\ &\Rightarrow x \in B \text{ (def. of } B^c) \\ \therefore &A \subseteq B \end{aligned}$$

4. Difference or relative complement : **الفضلة أو الفرق**

Let A and B are two sets. The **difference** between A and B , denoted as $A-B$ or $A \setminus B$, is the set of elements which belong to A but do not belong to B .

يقال لمجموعة العناصر المنتمية إلى A وغير منتمية إلى B بأنها فضلة A على B .

$$\begin{aligned} A - B &= \{x, x \in A \wedge x \notin B\} = A \setminus B \\ x \in A - B &\Leftrightarrow x \in A \wedge x \notin B \\ x \notin A - B &\Leftrightarrow x \notin A \vee x \in B \end{aligned}$$

Example2.49: Let $A = \{x \in \mathbb{Z}: x \geq 1\} = N$

$$B = \{x \in \mathbb{R}: x^2 + 3 = 0\} = \emptyset$$

$$C = \{x \in \mathbb{Z}: -3 < x \leq 4\} = \{-2, -1, \dots, 4\}$$

$$D = \{x \in \mathbb{O}: x^2 - 9 = 0\} = \{3, -3\}$$

Find $A \setminus D$, $B - B$, $C \setminus (A \cap D)$, $(B \cup A) \setminus C$, $(C \cup A) \setminus (B \cap D)$

Solution: $A \setminus D = N \setminus \{3, -3\}$

$$(C \cup A) \setminus (B \cap D) = (N \cup \{-2, -1, 0\}) \setminus \emptyset = N \cup \{-2, -1, 0\}$$

$$B - B =$$

$$C \setminus (A \cap D) =$$

$$(B \cup A) \setminus C =$$

Theorem 2.50: Let A, B and C any three sets. Then:

1. $A \setminus A = \emptyset, A \setminus U = \emptyset$ (H. W.)
2. $A \setminus \emptyset = A, \emptyset \setminus A = \emptyset$ (H. W.)
3. $A \setminus B \subseteq A, B \setminus A \subseteq B$ (H. W.)
4. $A \subseteq B \Leftrightarrow A \setminus B = \emptyset$ (H. W.)
5. $A \setminus B \cap B = \emptyset$
6. $A \cap B = \emptyset \Leftrightarrow A \setminus B = A \wedge B \setminus A = B$
7. $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ (H. W.)
8. $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$
9. $A \setminus A^c = A, A^c \setminus A = A^c$ (H. W.)

Proof 5: Assume that $A \setminus B \cap B \neq \emptyset$ برهان غير مباشر

$$\begin{aligned} \exists x, x \in A \setminus B \cap B &\Rightarrow x \in A \setminus B \wedge x \in B \\ &\Rightarrow (x \in A \wedge x \notin B) \wedge x \in B \quad (\text{def. of difference}) \\ &\Rightarrow x \in A \wedge (x \notin B \wedge x \in B) \quad (\wedge \text{ is asso.}) \\ &\Rightarrow x \in A \wedge F \\ &\Rightarrow x \in F \quad (\text{تناقض } (p \wedge F = F)) \\ \therefore A \setminus B \cap B &= \emptyset \end{aligned}$$

Proof 6: (\Rightarrow) Let $A \cap B = \emptyset$ T. P $A \setminus B = A \wedge B \setminus A = B$

$$\begin{aligned} \text{Let } x \in A \setminus B &\Leftrightarrow x \in A \wedge x \notin B \quad (\text{def. of } \setminus) \\ &\Leftrightarrow x \in A \wedge T \quad (\text{by hypo.}) \\ &\Leftrightarrow x \in A \quad (p \wedge T = p) \\ \therefore A \setminus B &= A \end{aligned}$$

Similarly, one can prove that $B \setminus A = B$

(\Leftarrow) Let $A \setminus B = A \wedge B \setminus A = B$ T. P $A \cap B = \emptyset$

Assume that $A \cap B \neq \emptyset$ برهان بالتناقض

$$\begin{aligned}
& \exists x, x \in A \cap B \Rightarrow x \in A \wedge x \in B \text{ (def. of } \cap) \\
& \Rightarrow \exists x, x \in A \setminus B \wedge x \in B \text{ (} A \setminus B = A) \\
& \Rightarrow \exists x, x \in A \setminus B \wedge x \in B \\
& \Rightarrow \exists x, (x \in A \wedge x \notin B) \wedge x \in B \\
& \Rightarrow \exists x, x \in A \wedge (x \notin B \wedge x \in B) \text{ (} \wedge \text{ is asso.)} \\
& \Rightarrow \exists x, x \in A \wedge F = F \text{ تناقض (} p \wedge F = F)
\end{aligned}$$

$$\therefore A \cap B = \emptyset$$

Proof 8: T. P $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

$$\begin{aligned}
& \text{let } x \in A \setminus (B \cap C) \Leftrightarrow x \in A \wedge x \notin B \cap C \text{ (def. of } \setminus) \\
& \Leftrightarrow x \in A \wedge (x \notin B \vee x \notin C) \text{ (def. of } \cap) \\
& \Leftrightarrow (x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C) \text{ (dist. } \wedge \text{ over } \vee) \\
& \Leftrightarrow x \in A \setminus B \vee x \in A \setminus C \text{ (def. of } \setminus) \\
& \Leftrightarrow x \in A \setminus B \cup A \setminus C \\
& \therefore A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)
\end{aligned}$$

5. Symmetric Difference الفرق التناظري

The **symmetric difference** between two sets A and B is denoted by $A \Delta B$ and is defined as:

$$\begin{aligned}
A \Delta B &= (A \setminus B) \cup (B \setminus A) \\
&= (A \cup B) \setminus (A \cap B)
\end{aligned}$$

Example 2.51: Let $A = \{x \in E: -8 \leq x < 9\} = \{-8, -6, \dots, 0, 2, \dots, 8\}$

$B = \{1, 2, 4, 6\}$. Find $A \Delta B$

Solution: $A \Delta B = (A \setminus B) \cup (B \setminus A)$
 $= \{-8, -6, \dots, 0, 8\} \cup \{1\}$

Theorem 2.52: Let A, B and C any three sets. Then:

1. $A \Delta A = \emptyset$ (H.W.), $A \Delta \emptyset = A$
2. $A \Delta B = B \Delta A$
3. $A \Delta B = \emptyset \Leftrightarrow A = B$
4. $A \Delta (B \Delta C) = (A \Delta B) \Delta C$ (بدون برهان)
5. $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$ (بدون برهان)

Proof 1: T. P $A \Delta \emptyset = A$

$$\begin{aligned} \text{Let } x \in A \Delta \emptyset &\Leftrightarrow x \in (A \setminus \emptyset \cup \emptyset \setminus A) \quad (\text{def. of } \Delta) \\ &\Leftrightarrow x \in A \setminus \emptyset \vee x \in \emptyset \setminus A \\ &\Leftrightarrow x \in A \cup \emptyset \quad (A \setminus \emptyset = A, \emptyset \setminus A = \emptyset) \\ &\Leftrightarrow x \in A \end{aligned}$$

$$\therefore A \Delta \emptyset = A$$

Proof 2: T. P $A \Delta B = B \Delta A$

$$\begin{aligned} \text{Let } x \in A \Delta B &\Leftrightarrow x \in (A \setminus B \cup B \setminus A) \quad (\text{def. of } \Delta) \\ &\Leftrightarrow x \in (B \setminus A \cup A \setminus B) \quad (\cup \text{ is comm.}) \\ &\Leftrightarrow x \in B \Delta A \end{aligned}$$

$$\therefore A \Delta B = B \Delta A$$

Proof 3: $A \Delta B = \emptyset \Leftrightarrow A = B$

(\Rightarrow) let $A \Delta B = \emptyset$ T. P $A = B$

$$\begin{aligned} \text{Suppose } A \neq B &\Rightarrow \exists x, x \in A \wedge x \notin B \\ &\Rightarrow \exists x, x \in A \setminus B \quad (\text{def. of } \setminus) \\ &\Rightarrow \exists x, x \in A \setminus B \vee x \in B \setminus A \quad (\text{T } \vee \text{ p=T}) \\ &\Rightarrow \exists x, x \in A \setminus B \cup B \setminus A \quad (\text{def. of } \cup) \\ &\Rightarrow \exists x, x \in A \Delta B = \emptyset \\ &\Rightarrow \exists x, x \in \emptyset \quad \text{contradiction} \\ &\therefore A = B \end{aligned}$$

(\Leftarrow) suppose $A = B$ T. P $A \Delta B = \emptyset$

$$\begin{aligned} A \Delta B &= A \setminus B \cup B \setminus A \quad (\text{def. of } \Delta) \\ &= A \setminus A \cup A \setminus A \quad (A = B) \\ &= \emptyset \cup \emptyset = \emptyset \end{aligned}$$

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