CHAPTER FOUR

Mappings التطبيقات

Chapter One Contents:

1. Mappings التطبيقات

2. Types of mappings أنواع التطبيقات

3. Composition of mappings تركيب التطبيقات

4. Direct image and inverse image الصورة المباشرة والصورة العكسية

Mapping (Function) التطبيق او الدالة

Let A and B be two nonempty sets. A **relation** $\boldsymbol{f}$ from A to B ($f \subseteq A \times B$) is called a **mapping** or **function** if each element in A is related to a unique element in B. This relation is denoted by $f: A \to B$.

الدالة او التطبيق fهي **علاقة خاصة** من A الى B تربط كل عنصر في المجموعة A بعنصر وحيد في المجموعة B ويرمز لهذه العلاقة بالرمز $f: A \to B$.

Mathematically,

$$f: A \to B \text{ is a mapping } \Leftrightarrow \forall x \in A\ \exists!\, y \in B\ s.t.\ f(x) = y$$

<u>Remark</u> 1.1: A mapping is (generally) denoted by $f, Fg, G, h, H, \ldots.$

<u>Example 1.2</u>: Let $A = \{1,2,3,4\}$ and $B = Z$. Which of the following relations is mapping?

$R_1 = \{(x, y) \in A \times B : y = 2x\}$

$R_2 = \{(1,1), (1,2), (2,0), (3,-1), (4,1)\}$

$R_3 = \{(1,1), (2,0), (3,3)\}$

<u>Remark</u> 1.3: Every function is a relation but not every relation is a function

Mapping can be defined in another way:

Definition 1.4: Let A and B be two nonempty sets. A **relation f** from A to B ($f \subseteq A \times B$) is called a **mapping** or **function** if it satisfies two conditions:

1) Closure الانغلاق: $\forall x \in A \Longrightarrow f(x) \in B$ كل عنصر في المجال صورته تنتمي للمجال المقابل

2) Well-defined التعريف الجيد: كل عنصر في المجال له صورة وحيدة في المجال المقابل

If $x_1 = x_2$ then $f(x_1) = f(x_2) \quad \forall x_1, x_2 \in A$

Example 1.5: Determine whether $f: Z \to R$ is a function or not

a) $f(x) = \sqrt{x^2+1}$

1. closure: Let $\forall x, x \in Z \overset{?}{\Rightarrow} f(x) \in R$

$$\forall x, x \in Z \Longrightarrow x^2 + 1 \in Z \Longrightarrow \sqrt{x^2+1} \in R$$

$\therefore$ closure is held الانغلاق متحقق

2. well defined: $\forall x_1, x_2 \in Z, x_1 = x_2 \Longrightarrow x_1^2 = x_2^2 \Longrightarrow x_1^2 + 1 = x_2^2 + 1$

$$\Longrightarrow \sqrt{x_1^2+1} = \sqrt{x_2^2+1}$$

$$\Longrightarrow f(x_1) = f(x_2)$$

$\therefore f$ is mapping

b) $f(x) = \begin{cases} -x, & x \le 1 \\ x, & x \ge 1 \end{cases}$

f is not well defined because $x = 1 \in Z$

but $f(1) = 1$ and $f(1) = -1$ يوجد عنصر له صورتان

3) $f(x) = \frac{1}{x}$

Closure condition is not held شرط الانغلاق غير متحقق

Let $x = 0 \in Z$ but $f(0) = \frac{1}{0} \notin R$

$\therefore f$ is not a mapping

Example 1.6: (H.W.) Is f mapping?

1. Let $f: N \to N$ s.t. $f(x) = x/(|x| - 5)$

2. Let $f: R \to R$ s.t. $f(x) = \frac{\sqrt[3]{x}}{x-1}$

Graph of Mapping رسم الدالة

Let A and B be two non-empty sets and $f: A \to B$. The graph of f is denoted by $Graph\ f$ and is defined as

$$Graph\ f = \{(x, y): x \in A \text{ and } y = f(x)\}$$

Definition 1.7: Let $f: A \to B$ be a mapping. Then

1) The set A is called the **domain of f مجال الدالة** and is denoted by D_f

2) The set B is called the **Codomain of f المجال المقابل للدالة** and is denoted by Cod_f

3) If $f(x) = y$ then y is called the **image** of x and x is called the **preimage** of y

4) The set of all images of the elements of A is called the **range** of f and is denoted by R_f

$$R_f = f(A) = \{y = f(x): x \in A\} \subseteq B = Cod_f$$

ملاحظة 3: يمكن ايجاد المجال للدالة بالاعتماد على نوعها

١. الدالة الخطية مجالها جميع الاعداد الحقيقية $D_f = R$

٢. الدالة الكسرية مجالها جميع الاعداد الحقيقية ماعدا القيم التي تجعل المقام يساوي صفر

٣. الدالة الجذرية مجالها جميع الاعداد الحقيقية عدا القيم التي تجعل القيمة تحت الجذر سالبة

لإيجاد مجموعة الصور R_f هناك عدة طرق منها الاعتماد على منحني الدالة. اذا كان مجال الدالة مجموعة جزئية من مجموعة الاعداد الحقيقية فأن من الممكن ايجاد المدى عن طريق ايجاد قيم x بدلالة y

Example 1.8: Write the graph set, the domain and the range of the following functions:

1) Let $f:\{-2,-1,0,1,2\} \to Z$ s.t. $f(x) = x^3$

$$Graph\ f = \{(x, x^3): x \in \{-2,-1,0,1,2\} \text{ and } f(x) = x^3\}$$
$$= \{(-2,-8), (-1,-1), (0,0), (1,1), (2,8)\}$$

$D_f = \{-2,-1,0,1,2\}$

$R_f = \{-8,-1,0,1,8\} \subseteq Z = Cod_f$

2) Let $g: Z \to Z$ s.t. $g(x) = x^2$

$$Graph\ g = \{(x, x^2): x \in Z \text{ and } g(x) = x^2\}$$
$$= \{\dots, (-2,4), (-1,1), (0,0), (1,1), (2,4), \dots\}$$

$D_g = Z$

$$R_g = \{0,1,4,9,16, \dots\} \subseteq Z = Cod_g$$

Example 1.9: **(H. W.)** Find the domain and the range of the following functions:

1) $f(x) = \frac{x}{x+2}$ 2) $F(x) = \sqrt{1-2x}$ 3) $G(x) = \sqrt{\frac{x}{x+2}}$

Types of Mappings أنواع التطبيقات

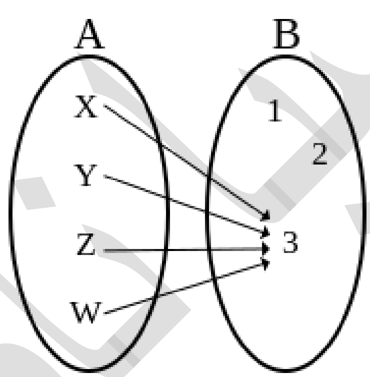
1. **Constant Mapping التطبيق الثابت**

A mapping $f: A \to B$ is called

constant map دالة ثابتة $\Leftrightarrow \exists! c \in B \ s.t. \ f(x) = c \quad \forall x \in A$

or

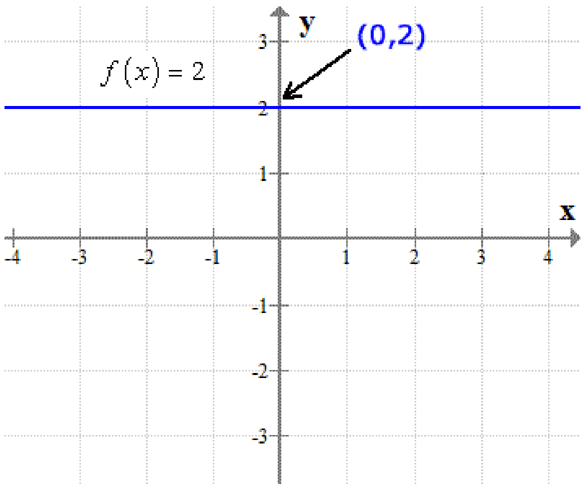
f is called constant $\Leftrightarrow R_f = \{c\}$



Constant Mapping

Example 1.10: Let $f: R \to R$ s.t. $f(x) = 2 \quad \forall x \in R$

f is constant function



<u>Example 1.11</u>: Give two examples of non-constant functions

1. let $f: R \to R$ s.t. $f(x) = \begin{cases} 2, & x \geq 1 \\ -3, & x < 1 \end{cases}$

f is not a constant mapping because $f(1) = 2$ and $f(0) = -3$

Give another example (H. W.)

2. <u>Identity Mapping الدالة الذاتية</u>

A mapping $f: A \to A$ is called **identity map دالة ذاتية** denoted by i_A
$\Leftrightarrow \quad f(x) = x \quad \forall x \in A$

<u>Example 1.12</u>:

1. let $f = \{(1,1), (3,3), (0,0), (-6,-6)\}$

f is identity function defined on $A = \{0,1,3,-6\}$

$\therefore f = i_A$

2. let $f: Z \to Z$ s.t. $f(x) = x \quad \forall x \in Z$

$f = i_Z$

3. let $f: Z \to Z$ s.t. $f(x) = |x| \quad \forall x \in Z$

$f(x) = f(-x) = x \quad \forall x \in Z$

$\therefore f$ is not identity function

4. let $f: N \to N$ s.t. $f(x) = |x| \quad \forall x \in N$

$f(x) = x \quad \forall x \in N$

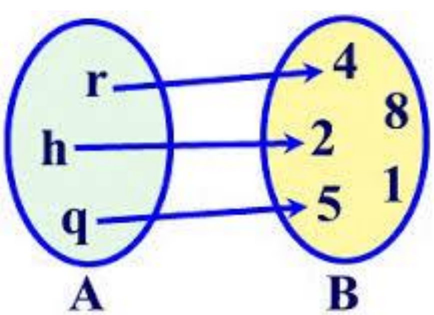
$\therefore f$ is identity function (i.e., $f = i_N$)

3. <u>Injective Mapping التطبيق المتباين</u>

A function $f: A \to B$ is called **one to one** (1-1) or **injective** if different elements in the domain A have different images in B

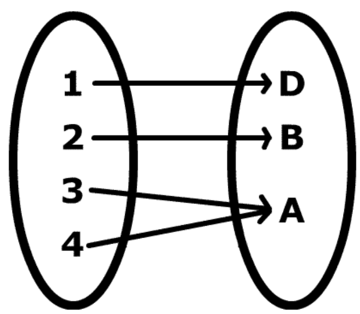
تسمى الدالة متباينة اذا كانت للعناصر المختلفة في المجال صورا مختلفة في المجال المقابل

$f: A \to B$ is called **1-1** $\Leftrightarrow \forall\, x_1, x_2 \in A;\ if\ x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$



Or

$f: A \to B$ is called **1-1** $\Leftrightarrow \forall\, x_1, x_2 \in A;\ if\ f(x_1) = f(x_2)$ then $x_1 = x_2$



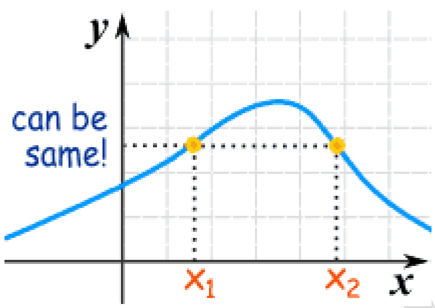
$f: A \to B$ is **not 1-1** $\Leftrightarrow \exists\, x_1, x_2 \in A;\ x_1 \neq x_2 \wedge f(x_1) = f(x_2)$

One to one function not one to one function

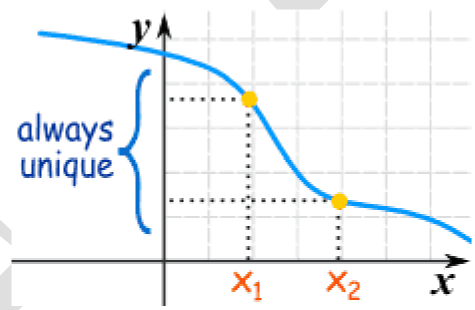
$f(4) = f(3) = A$

f is many to one

<u>Remark 1.13</u>: In the **graph of Injective** map, a horizontal line should never intersect the curve at 2 or more points**.**



not one to one (many to one)



Injective (one to one) function

4. <u>Surjective Mapping التطبيق الشامل</u> A function $f: A \to B$ is called (onto) or **surjective if** every element in "B" has **at least one** relating element in "A" (maybe more than one).

الدالة f تكون شاملة اذا كان كل عنصر في المجال المقابل هو صورة لعنصر واحد او اكثر في المجال

Mathematically,

A function $f: A \to B$ is called (onto) or **surjective** $\Leftrightarrow$ $R_f = Cod_f$

Or

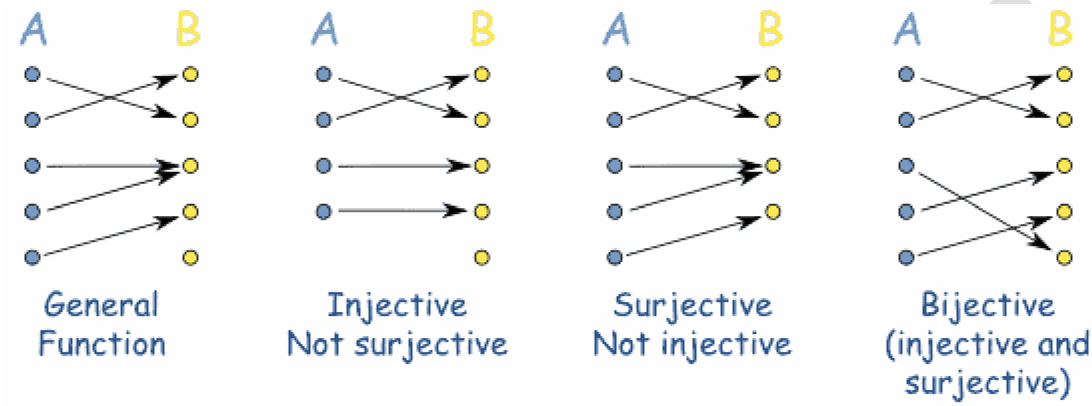
A function $f: A \to B$ is called (onto) or **surjective** $\Leftrightarrow$ $\forall y \in B\ \exists x \in A\ s.t.\ f(x) = y$

A function $f: A \to B$ is not (onto) or not **surjective** $\Leftrightarrow$ $R_f \neq Cod_f$

5. bijective Mapping التطبيق المتقابل

A function $f: A \to B$ is called **bijective** $\Leftrightarrow$ f is 1-1 and onto

A function $f: A \to B$ is called **not bijective** $\Leftrightarrow$ f is not 1-1 or f is not onto



Example 1.14: Which of the following functions are injective? Surjective? Bijective?

1. $f: \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = x$ (**H. W.)**

2. $f: \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = 3$ (**H. W.)**

3. $f: \mathbb{R}^+ \to \mathbb{R}$ s.t. $f(x) = x^2$ (**H. W.)**

4. $f: \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = \sqrt{x^2 + 9}$

5. $f: \mathbb{R} \setminus \{\frac{5}{2}\} \to \mathbb{R}$ s.t. $f(x) = \frac{x+4}{2x-5}$ (**H. W.)**

6. $f: \mathbb{R} \to [1, \infty)$ s.t. $f(x) = |x - 4| + 1$

7. $f: \mathbb{R} \to [-4, \infty)$ s.t. $f(x) = -4 + (x - 4)^2$ (**H. W.)**

8. $f:\mathbb{R} \to \mathbb{R}$ s.t. $f(x) = \begin{cases} x^3, & x < 0 \\ x^2, & x \geq 0 \end{cases}$

<u>Solution4</u>: $f:\mathbb{R} \to \mathbb{R}$ s.t. $f(x) = \sqrt{x^2 + 9}$

$D_f = \mathbb{R}$ and $cod_f = \mathbb{R}$

Surjective? We need to find R_f

When $x \in \mathbb{R} \Longrightarrow f(x) = y \geq 3$

$$R_f = \{y : y \geq 3\} = [3, \infty) \neq \mathbb{R} = cod_f$$

$\therefore f$ is not surjective (not onto)

Injective? Let $f(x_1) = f(x_2) \overset{?}{\to} x_1 = x_2$

$$f(x_1) = f(x_2) \Longrightarrow \sqrt{x_1^2 + 9} = \sqrt{x_2^2 + 9}$$

$$\Longrightarrow x_1^2 + 9 = x_2^2 + 9$$

$$\Longrightarrow x_1^2 = x_2^2$$

$$\Longrightarrow \quad x_1 = \mp x_2$$

$\Longrightarrow x_1 = x_2$ **....(1)**

$Or, \quad x_1 = -x_2 \Longrightarrow x_1 \neq x_2$ **.....(2)**

From (2), f is not injective

$\therefore f$ is not bijective

<u>Solution6</u>: $f:R \to [1, \infty)$ s.t. $f(x) = |x - 4| + 1 = \begin{cases} x - 3, & x \geq 4 \\ -x + 5, & x < 4 \end{cases}$

$D_f = \mathbb{R}$ and $cod_f = [1, \infty)$

Injective? Let $f(x_1) = f(x_2) \overset{?}{\rightarrow} x_1 = x_2$

$$f(x_1) = f(x_2) \Longrightarrow |\mathrm{x}_1 - 4| + 1 = |\mathrm{x}_2 - 4| + 1$$

$$\Longrightarrow |\mathrm{x}_1 - 4| = |\mathrm{x}_2 - 4|$$

Either $x_1 - 4 = x_2 - 4$

$\Longrightarrow x_1 = x_2$ **....(1)**

$Or, x_1 - 4 = -x_2 + 4$

$\Longrightarrow x_1 = 8 - x_2 \Longrightarrow \mathrm{x}_1 \neq \mathrm{x}_2$ **.....(2)**

From (2), f is not injective (1-1)

Surjective? We need to find R_f

$$f(x) = \begin{cases} x - 3, & x \geq 4 \\ -x + 5, & x < 4 \end{cases}$$

If $x \geq 4 \Longrightarrow x - 3 \geq 1 \Longrightarrow y \geq 1$ **……(1)**

If $x < 4 \Longrightarrow -x > -4 \Longrightarrow 5 - x > 1 \Longrightarrow y > 1$ **…....(2)**

∴ From (1) and (2), $R_f = \{y : y \geq 1 \text{ or } y > 1\} = [1, \infty) = cod_f$

∴ f is surjective (onto)

طريقة اخرى لاختبار ان الدالة شاملة وهي ايجاد x بدلالة y

If $x \geq 4 \Longrightarrow \mathrm{y} = x - 3 \Longrightarrow x = y + 3$

لكي تكون $x \geq 4$ يجب ان تكون قيم y اكبر او تساوي ال 1

$$\Longrightarrow y \geq 1 \;\ldots\ldots(\mathbf{1})$$

If $x < 4,\; y = -x + 5 \Longrightarrow x = 5 - y$

لكي تكون $x < 4$ يجب انتكون قيم y اكبر من ال 1

$\Longrightarrow\ y > 1$ **…….(2)**

$\therefore$ From (1) and (2), $R_f = \{y : y \geq 1$ or $y > 1\} = [1, \infty) = cod_f$

$\therefore\ f$ is onto

$\therefore\ f$ is not bijective

<u>Solution8</u>: $f: \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = \begin{cases} x^3, & x < 0 \\ x^2, & x \geq 0 \end{cases}$

$D_f = \mathbb{R}$ and $cod_f = \mathbb{R}$

Surjective (onto)? We need to find R_f

When $x < 0 \Longrightarrow \mathrm{y} = f(x) = x^3 < 0 \Longrightarrow y < 0$

When $x \geq 0 \Longrightarrow \mathrm{y} = f(x) = x^2 \geq 0 \Longrightarrow \mathrm{y} \geq 0$

$$R_f = \{y : y < 0 \ \ or \ y \geq 0\} = \mathbb{R} = cod_f$$

$\therefore\ f$ is onto

Injective (1-1)? Let $f(x_1) = f(x_2) \overset{?}{\to} x_1 = x_2$

$$f(x_1) = f(x_2) \Longrightarrow \begin{cases} x_1^3 = x_2^3, & x_1 < 0, x_2 < 0 \\ x_1^2 = x_2^2, & x_1 \geq 0,\ x_2 \geq 0 \\ x_1^3 = x_2^2\,, & x_1 < 0,\ x_2 \ \geq 0 \\ x_1^2 = x_2^3, & x_1 \geq 0, x_2 < 0 \end{cases}$$

$$\Longrightarrow \begin{cases} x_1 = x_2\ , x_1 < 0, x_2 < 0 \\ x_1 = x_2,\ x_1 \geq 0, x_2 \geq 0 \\ \end{cases}$$

$\therefore\ x_1 = x_2 \Longrightarrow\ f$ is injective

∴ f is bijective

<u>Inverse Mapping</u>: الدالة العكسية

Let f be a bijective mapping from A to B then f^{-1} is a mapping from B to A such that $f^{-1}(y) = x$.

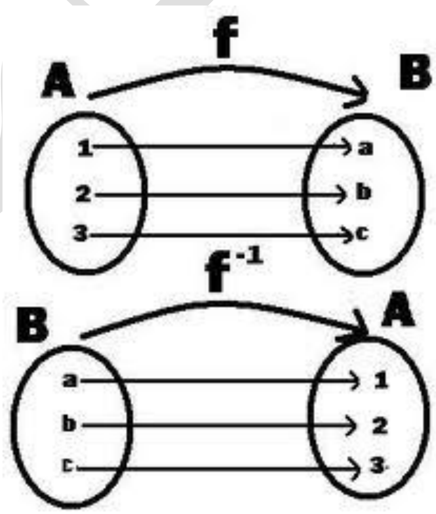
Mathematically,

$f: A \to B$ s.t. $f(x) = y$ is a bijective mapping $\Leftrightarrow$ $f^{-1}: B \to A$ is a map. s.t. $f^{-1}(y) = x$

<u>Example 1.15:</u> Let $f: \{1,2,3\} \to \{a, b, c\}$ s.t.

$f(1) = a,\ \ f(2) = b,\ f(3) = c$

Or $f = \{(1, a), (2, b), (3, c)\}$



Since f is 1-1 and onto (bijective) $\Longrightarrow$ $f^{-1}: \{a, b, c\} \to \{1,2,3\}$ s.t.

$f^{-1}(a) = 1,\ f^{-1}(b) = 2,\ \ f^{-1}(c) = 3$

or $f^{-1} = \{(a,1),(b,2),(c,3)\}$

Example 1.16: Let $f: R^{+} \to R^{+}$ s.t. $f(x) = \frac{x}{2}$. Find f^{-1} (if exist)?

Solution: f^{-1}exist $\Leftrightarrow$ f is bijective

1-1? Let $f(x_1) = f(x_2) \Rightarrow \frac{x_1}{2} = \frac{x_2}{2} \Rightarrow x_1 = x_2 \Rightarrow$ f is 1-1

Onto? $y = \frac{x}{2} \Rightarrow x = 2y$

$$\Rightarrow R_f = R^{+} = cod_f$$

$\therefore$ f is bijective $\Rightarrow f^{-1}$ is a mapping $\Rightarrow$ f^{-1} exist

لايجاد f^{-1} نجد x بدلالة y

$y = \frac{x}{2} \Rightarrow x = 2y$

$\therefore$ $f^{-1}: R^{+} \to R^{+}$ s.t. $f^{-1}(y) = 2y$

Remark 1.17:

1. If a function f is **not injective** then f^{-1} is **not a mapping** (f^{-1}does not exist)

2. If a function f is **not surjective** then f^{-1} is **not a mapping** (f^{-1}does not exist)

3. If a function f is **bijective** then f^{-1} is **a mapping** (f^{-1} exist)

Example 1.18: $f: \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = 4 + (x-4)^2$. Find f^{-1} (if exist)

Solution:

f^{-1} exist if and only if f is 1-1 and onto

1-1? Let $f(x_1) = f(x_2) \Rightarrow 4 + (x_1 - 4)^2 = 4 + (x_2 - 4)^2$

$$\Rightarrow (x_1 - 4)^2 = (x_2 - 4)^2$$

بجذر الطرفين

$$\Longrightarrow x_1 - 4 = \mp(x_2 - 4)$$

Either, $x_1 - 4 = x_2 - 4 \Longrightarrow x_1 = x_2$

Or, $x_1 - 4 = -(x_2 - 4) \Longrightarrow x_1 \neq x_2$

$\therefore$ f is not 1-1

$\therefore$ f^{-1} is not defined

Example 1.19: $f: R \rightarrow R$ s.t. $f(x) = \begin{cases} x^3, & x < 0 \\ x^2, & x \geq 0 \end{cases}$

Find f^{-1} (if exist)?

Solution:

f is bijective (see Example 4.14(8))

$\therefore$ f^{-1} is defined

$\therefore$ $f^{-1}: R \rightarrow R$ s.t. $f^{-1}(y) = \begin{cases} \sqrt[3]{y}, & y < 0 \\ \sqrt{y}, & y \geq 0 \end{cases}$

Example 1.20: $f: [3, \infty) \rightarrow [0, \infty)$ s.t. $f(x) = \sqrt{x - 3}$. Find f^{-1} (if exist)

Let $f(x_1) = f(x_2)$

$\Longrightarrow x_1 - 3 = x_2 - 3$

$\Longrightarrow x_1 = x_2 \Longrightarrow$ f is 1-1

Onto? Is $R_f = cod_f$?

نجد x بدلالة y

$y = \sqrt{x - 3} \Longrightarrow x = y^2 + 3 \Longrightarrow y \geq 0$ (because $x \geq 3$)

يمكن ايجاد المدى بطريقة اخرى

$x \geq 3 \Longrightarrow \mathrm{x} - 3 \geq 0$

$\Longrightarrow \sqrt{x-3} \geq 0 \Longrightarrow y \geq 0$

$\Longrightarrow \mathrm{R_f} = [0, \infty) = \mathrm{cod_f}$

f is bijective

$\therefore\ f^{-1}$ is defined

لايجاد f^{-1} نجد x بدلالة y

$y = \sqrt{x-3} \Longrightarrow x = y^2 + 3$

$$\therefore\ f^{-1}: [0, \infty) \to [3, \infty)\ \ s.t. f^{-1}(y) = y^2 + 3$$

Remark 1.21:

1. $f^{-1} \neq \frac{1}{f}$

2. $(f^{-1})^{-1} = f$

3. $f = f^{-1} \Leftrightarrow f = i_A$

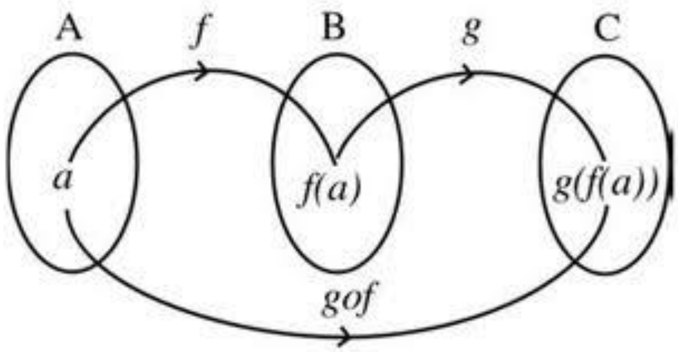
Composition of Mappings تركيب التطبيقات

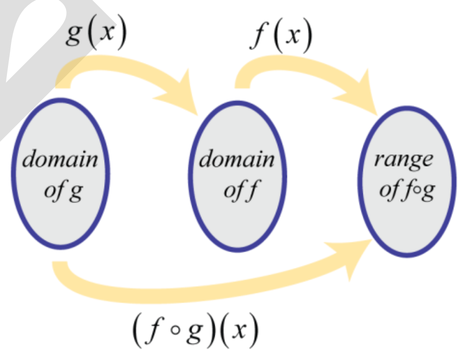
Let $f: A \to B$ be a mapping and $g: B \to C$ be a mapping. The **composition** of g and f is a mapping denoted by gof and is defined as

$$(gof)(a) = g\big(f(a)\big) \quad \forall a \in A$$

Mathematically,

If $f: A \rightarrow B$ and $g: B \rightarrow C$ then $gof: A \rightarrow C$ is a map. $\Leftrightarrow \forall a \in A$,
$\exists!\, g(f(a)) \in C$ s.t. $(gof)(a) = g(f(a))$





<u>Remark</u> 1.22: Let $f: A \rightarrow B$ and $g: B \rightarrow C$. Then

1. gof is defined (exist) if and only if $R_f \subseteq D_g$

2. fog is defined (exist) if and only if $R_g \subseteq D_f$

3. $fog \neq gof$ (in general)

Example 1.23: Let $f: \mathbb{R} \rightarrow [2, \infty)\ s.t. f(x) = x^4 + x^2 + 2$

$$g: \mathbb{R}^+ \rightarrow [-4, \infty)\ s.t. g(x) = \sqrt{x} - 4$$

Find fog and gof (if exist)

Solution: fog exist when $R_g \subseteq D_f$

Rg= $[-4, \infty)$ and $D_f = \mathbb{R} \Longrightarrow R_g \subseteq D_f$

$\therefore fog$ is defined

$$(fog)(x) = f(g(x)) = f(\sqrt{x} - 4) = (\sqrt{x} - 4)^4 + (\sqrt{x} - 4)^2 + 2$$

gof is defined when $R_f \subseteq D_g$

$R_f = [2, \infty)$ and $D_g = \mathbb{R}^+ \Longrightarrow R_f \subseteq D_g$

$\therefore gof$ is defined

$$(gof)(x) = g(f(x)) = g(x^4 + x^2 + 2) = \sqrt{x^4 + x^2 + 2} - 4$$

Example 1.24: Let $f: \mathbb{R} \rightarrow \mathbb{R}\ s.t. f(x) = \sin(x)$

$$g: [0, \infty) \rightarrow (-\infty, 0]\ s.t. g(x) = -\sqrt{x}$$

Find fog and gof (if exist)

Solution: fog exist when $R_g \subseteq D_f$

Find R_g? $g(x) = -\sqrt{x}$

$\sqrt{x} \geq 0 \Longrightarrow g(x) = -\sqrt{x} \leq 0$

$\therefore R_g = (-\infty, 0] \subseteq D_f = \mathbb{R}$

$\therefore fog$ is defined

$$(fog)(x) = f(g(x)) = f(-\sqrt{x}) = \sin(-\sqrt{x})$$

To find gof, we need to check if $R_f \subseteq D_g$

$$f(x) = \sin(x) \in [-1,1]$$

$$\therefore R_f = [-1,1] \not\subseteq D_g = [0, \infty)$$

$\therefore gof$ is not a map (does not exist)

<u>Theorem</u> 1.25: Let $f: A \to B$ and $g: B \to C$ be two mappings, then

1. If f and g are 1-1 then gof is 1-1

2. If gof is 1-1 then f is 1-1

3. . If f and g are onto then gof is onto

4. If gof is onto then g is onto

<u>Proof</u> 1: T.P. $gof: A \to C$ is 1-1 (injective)

$\forall x_1, x_2 \in A, (gof)(x_1) = (gof)(x_2)$ T.P. $x_1 = x_2$

$(gof)(x_1) = (gof)(x_2) \Longrightarrow g\big(f(x_1)\big) = g(f(x_2))$ (def. of gof)

$$\Longrightarrow f(x_1) = f(x_2) \qquad (g \text{ is 1-1})$$

$$\Longrightarrow x_1 = x_2 \qquad (f \text{ is 1-1})$$

$\therefore gof$ is 1-1

<u>Proof</u> 2: Let gof is 1-1 **T.P.** f is 1-1

$\forall x_1, x_2 \in A, f(x_1) = f(x_2)$ T.P. $x_1 = x_2$

$f(x_1), f(x_2) \in D_g$ and $f(x_1) = f(x_2)$

$\Longrightarrow g\big(f(x_1)\big) = g(f(x_2))$ (g is well defined)

$\Longrightarrow (gof)(x_1) = (gof)(x_2)$ (def. of gof)

$\Longrightarrow x_1 = x_2$ (gof is 1-1)

$\Longrightarrow f$ is 1-1

Proof 3: Assume that f and g are onto **T.P.** gof is onto

T.P. $\forall z \in C, \exists x \in A\ s.t. (gof)(x) = z$

Since f is onto $\Longrightarrow\ \forall y \in B, \exists x \in A\ s.t. f(x) = y$**……(1)**

Since g is onto $\Longrightarrow\ \forall z \in C, \exists y \in B\ s.t. g(y) = z$**……(2)**

Substitute (1) in (2)

$\Longrightarrow \forall z \in C, \exists y = f(x) \in B\ s.t. g(f(x)) = z$

$\Longrightarrow \forall z \in C, \exists x \in A\ s.t. (gof)(x) = z$

$\Longrightarrow gof$ is onto

Proof 4: Let gof is onto **T.P.** g is onto

T.P. $\forall z \in C, \exists y \in B\ s.t. g(y) = z$

Since gof is onto $\Longrightarrow\ \forall z \in C, \exists x \in A\ s.t. (gof)(x) = z$

$\Longrightarrow \forall z \in C, \exists y = f(x) \in B\ s.t.\ g(f(x)) = z$

$\Longrightarrow \forall z \in C, \exists y \in B\ s.t.\ g(y) = z$

$\Longrightarrow g$ is onto

Equal Mappings تساوي التطبيقات

Let f and g are two mappings, then

$$f = g \iff \mathrm{D}_f = \mathrm{D}_g \wedge f(x) = g(x) \quad \forall x \in D_f = D_g$$

Theorem 1.25: Let $f: A \to A$ be a mapping and $i_A: A \to A$ be the identity mapping, then $i_A of = f$ and $f o i_A = f$

Proof: T.P. $\underbrace{i_A o f = f}_{(1)} \ \wedge \ \underbrace{f o i_A = f}_{(2)}$

$i_A o f : A \to A$ is defined

T.P. $i_A o f = f$ لكي نبرهن المساواة يجب تحقق شرطين

1) $D_{i_A o f} = D_f$?

$i_A o f : A \to A$

$\therefore \ D_{i_A o f} = A = D_f$

2) $(i_A o f)(x) = f(x)$

$(i_A o f)(x) = i_A(f(x))$ (def. of o)

$= f(x)$ (def. of i_A)

$\therefore \ (i_A o f)(x) = f(x) \quad \forall x \in A$

From (1) and (2) $\Longrightarrow$ $i_A o f = f$

Similarly, prove that $f o i_A = f$ (**H.W.**)

Theorem 1.26: Let A, B and C are nonempty sets. Then:

1. If $f: A \to B$ is bijective and $f^{-1}: B \to A$ then $f^{-1} o f = i_A$, $f o f^{-1} = i_B$
2. If $f: A \to A$ is bijective and $f^{-1}: A \to A$ then $f^{-1} o f = f o f^{-1} = i_A$ (**H.W.**)
3. If $f: A \to B$ be a bijective mapping then $f^{-1}: B \to A$ is a bijective mapping.
4. If $f: A \to B$, $g: B \to C, h: C \to D$ then $(h o g) o f = h o (g o f)$
5. If $f: A \to B$ is bijective and $g: B \to C$ is bij. then $(g o f)^{-1} = f^{-1} o g^{-1}$

Proof 1: Let $f: A \to B$ is bijective and $f^{-1}: B \to A$

T.P. $f^{-1}of = i_A$

$f^{-1}of: A \to A$ is defined

1) T.P. $D_{f^{-1}of} = \mathrm{D}_{\mathrm{i_A}}$

$f^{-1}of: A \to A \Longrightarrow D_{f^{-1}of} = A$

$i_A: A \to A \Longrightarrow \mathrm{D}_{\mathrm{i_A}} = A$

$\therefore \quad D_{f^{-1}of} = \mathrm{D}_{\mathrm{i_A}}$

2) T.P. $(f^{-1}of)(x) = i_A(x) \quad \forall x \in A$

$(f^{-1}of)(x) = f^{-1}\big(f(x)\big) \ldots\ldots(\mathbf{1})$

Since f is a map. $\Longrightarrow \forall x \in A, \exists y \in B \; s.t. \; f(x) = y$ **…..(2)**

Substitute (2) in (1) $\Longrightarrow (f^{-1}of)(x) = f^{-1}(y) = x$

$\therefore (f^{-1}of)(x) = x = i_A(x)$

From (1) and (2) $\Longrightarrow f^{-1}of = i_A$

Next, we prove $fof^{-1} = i_B$

$fof^{-1}: B \to B$ is defined

1) T.P. $D_{fof^{-1}} = \mathrm{D}_{\mathrm{i_B}}$

$fof^{-1}: B \to B \Longrightarrow D_{fof^{-1}} = B$

$i_B: B \to B \Longrightarrow \mathrm{D}_{\mathrm{i_B}} = B$

$\therefore \quad D_{fof^{-1}} = \mathrm{D}_{\mathrm{i_B}}$

2) T.P. $(fof^{-1})(y) = i_B(y) \quad \forall y \in B$

$(fof^{-1})(y) = f\big(f^{-1}(y)\big) \ldots\ldots(\mathbf{1})$

Since f^{-1} is a map. $\Longrightarrow \forall y \in B, \exists x \in A \; s.t. \; f^{-1}(y) = x$ **…….(2)**

Substitute (2) in (1) $\Longrightarrow$ $(fof^{-1})(y) = f(x) = y$

$\therefore$ $(fof^{-1})(y) = y = i_B\ (y)$

From (1) and (2) $\Longrightarrow$ $f^{-1}of = i_B$

<u>Proof</u> 3: Let $f: A \to B$ be a bijective mapping (1-1 and onto)

T.P. $f^{-1}: B \to A$ is a bijective mapping.

f^{-1} is 1-1? Let $y_1, y_2 \in B$ s.t. $f^{-1}(y_1) = f^{-1}(y_2)$ T.P. $y_1 = y_2$

since $y_1, y_2 \in B = R_f \Longrightarrow \exists x_1, x_2 \in A$ s.t. $f(x_1) = y_1, f(x_2) = y_2$ [f is onto

$\Longrightarrow \exists x_1, x_2 \in A$ s.t. $f^{-1}(f(x_1)) = f^{-1}(y_1)$ and $f^{-1}(f(x_2)) = f^{-1}(y_2)$ [$f^{-1}$is well defined]

$\Longrightarrow \exists x_1, x_2 \in A$ s.t. $f^{-1}\big(f(x_1)\big) = f^{-1}(f(x_2)$ [$f^{-1}(y_1) = f^{-1}(y_2)$]

$\Longrightarrow \exists x_1, x_2 \in A$ s.t. $(f^{-1}of)(x_1) = (f^{-1}of)(x_2)$ [def. of $f^{-1}of$]

$\Longrightarrow \exists x_1, x_2 \in A$ s.t. $i_A(x_1) = i_A(x_2)$ [from part (1)]

$\Longrightarrow x_1 = x_2$

$\Longrightarrow f(x_1) = f(x_2)$ [f is well defined]

$\Longrightarrow y_1 = y_2$

$\therefore f^{-1}$is 1-1

f^{-1} is onto? T.P. $R_{f^{-1}} = A$

$R_{f^{-1}} = \{x \in A: x = f^{-1}(y)\} = \{x \in A: f(x) = y\} = A$

$\therefore f^{-1}$ is onto

f^{-1} is 1-1 and onto $\Longrightarrow$ f^{-1} is bijective

<u>Proof</u> 4: let $f: A \to B,\ g: B \to C, h: C \to D$ **T.P.** $(hog)of = ho(gof)$

1) $D_{(hog)of} = D_{ho(gof)} = A$

2) T.P. $\forall x \in A,\ [(hog)of](x) = [ho(gof)](x)$

$$[(hog)of](x) = (hog)(f(x) \quad \text{(def. of o)}$$

$$= h(g(f(x)) \quad \text{(def. of o)}$$

$$= h\big(gof(x)\big)$$

$$= [ho(gof)](x)$$

$\therefore\ (hog)of = ho(gof)$

Proof 5: Let $f: A \to B$ is bijective and $g: B \to C$ is bijective

T.P. $(gof)^{-1} = f^{-1}og^{-1}$

Let $h = gof: A \to C$ T.P. $h^{-1} = f^{-1}og^{-1}$

T.P. $hoh^{-1} = i_C$

$$hoh^{-1} = (gof)o(f^{-1}og^{-1})$$

$$= go(fof^{-1})og^{-1} \quad \text{[o is associative]}$$

$$= goi_Bog^{-1} \quad [fof^{-1} = i_B]$$

$$= gog^{-1} \quad [goi_B = g]$$

$$= i_C \quad [gog^{-1} = i_C]$$

$\therefore\ (gof)^{-1} = f^{-1}og^{-1}$

Direct Image الصورة المباشرة

Let $f: A \to B$ be a mapping and $C \subseteq A$. Then the **direct image** of C under f is defined as

$$f(C) = \{y \in B;\ \exists x \in C\ s.t.\ y = f(x)\}$$

الصورة المباشرة $f(C)$ هي مجموعة جزئية من المجال المقابل B والتي كل عنصر فيها هو صورة لعنصر او اكثر من عناصر المجموعة C الجزئية من المجال. وتسمى $f(C)$ الصورة المباشرة ل Cبفعل التطبيق f.

$$y \in f(C) \Leftrightarrow \exists x \in C \; s.t. \; y = f(x)$$

$$y \notin f(C) \Leftrightarrow \forall x \in C \; s.t. \; y \neq f(x)$$

Example 1.27: Let $f: N \backslash \{1\} \to N \; s.t. f(n) = n^2 - 1$

Let $C = \{2,3,4\}$. Find $f(C)$.

Solution:

$f(C) = \{f(2), f(3), f(4)\} = \{3,8,15\}$

Theorem 1.28: Let $f: A \to B$ be a mapping, let C, D are subsets of A. Then:

1. If $C \subseteq D$ then $f(C) \subseteq f(D)$
2. $f(C \cap D) \subseteq f(C) \cap f(D)$ **(H.W.)**
3. If f is injective (1-1) then $f(C \cap D) = f(C) \cap f(D)$
4. $f(C \cup D) = f(C) \cup f(D)$
5. $f(C) \backslash f(D) \subseteq f(C \backslash D)$
6. $f(C \backslash D) \subseteq f(C)$ **(H.W.)**

Proof1: Let $C \subseteq D$ T.P. $f(C) \subseteq f(D)$

Let $y \in f(C) \Longrightarrow y \in B; \; \exists x \in C \; s.t. \; y = f(x)$ (def. of $f(C)$)

$\Longrightarrow y \in B; \; \exists x \in D \; s.t. \; y = f(x) \qquad (C \subseteq D)$

$\Longrightarrow$ y ∈ f(D) (def. of $f(D)$)

$\therefore\ f(C) \subseteq f(D)$

Proof3: Suppose f is 1-1 T.P. $\underbrace{f(C\cap D) \subseteq f(C)\cap f(D)}_{(1)}$ $\wedge$

$$\underbrace{f(C)\cap f(D) \subseteq f(C\cap D)}_{(2)}$$

From (2), $f(C\cap D) \subseteq f(C)\cap f(D)$ ….(1) (**H.W.**)

T.P. $f(C)\cap f(D) \subseteq f(C\cap D)$

Let $y \in f(C)\cap f(D)$ T.P. $y \in f(C\cap D)$

$y \in f(C)\cap f(D) \Longrightarrow y \in f(C) \wedge y \in f(D)$ (def. of ∩)

$\Longrightarrow \exists\ x_1 \in C, y = f(x_1) \wedge \exists\ x_2 \in D, y = f(x_2)$ (def. of direct image)

$\Longrightarrow y = f(x_1) = f(x_2)$

$\Longrightarrow x_1 = x_2$ (f is 1-1)

$\Longrightarrow \exists\ x = x_1 = x_2 \in C\cap D$ s.t. $y = f(x) \in f(C\cap D)$ (def. of direct image)

$\therefore\ f(C)\cap f(D) \subseteq f(C\cap D)$ ….(2)

From (1) and (2), $f(C)\cap f(D) = f(C\cap D)$

Proof 4: T.P. $\underbrace{f(C\cup D) \subseteq f(C)\cup f(D)}_{(1)}$ $\wedge$ $\underbrace{f(C)\cup f(D) \subseteq f(C\cup D)}_{(2)}$

Let $y \in f(C\cup D) \Leftrightarrow y \in B;\ \exists x \in C\cup D\ s.t.\ y = f(x)$ (def. of $f(C\cup D)$)

$\Leftrightarrow y \in B;\ \exists x \in C$ v $x \in D\ s.t.\ y = f(x)$ (def. of ∪)

$\Leftrightarrow\ y \in B; [\exists x \in C\ \ s.t.\ \ y = f(x)] \text{v} [x \in D\ \ s.t.\ \ y = f(x)]$

$\Leftrightarrow \text{y} \in f(C) \text{ v } y \in f(D)$ (def. of direct image)

$\Leftrightarrow \text{y} \in f(C) \cup f(D)$ (def. of ∪)

$\therefore\ f(C \cup D) = f(C) \cup f(D)$

<u>Proof</u> 5: T.P. $f(C)\backslash f(D) \subseteq f(C\backslash D)$

Let $y \in\ f(C)\backslash f(D) \Longrightarrow\ y \in\ f(C)\ \wedge y \notin f(D)$ (def. of \)

$\Longrightarrow \exists x \in C\ \ s.t.\ \ y = f(x)\ \wedge\ \ \forall x \in D;\ y \neq f(x)$ (def. of direct image)

$\Longrightarrow \exists x \in C\ \ s.t.\ \ y = f(x)\ \wedge\ \ \text{x} \notin \text{D};\ \ \text{y} = f(x)$

$\Longrightarrow \exists x \in C\ \ \wedge\ \ \text{x} \notin \text{D};\ \text{y} = f(x)$

$\Longrightarrow\ \ y \in\ f(C\backslash D)$

$\therefore\ f(C)\backslash f(D) \subseteq f(C\backslash D)$

Inverse Image الصورة العكسية

Let $f: A \to B$ be a mapping and $D \subseteq B$. Then the **inverse image** of D under f is defined as

$$f^{-1}(D) = \{x \in A: f(x) \in D\ \}$$

الصورة العكسية $f^{-1}(D)$ هي مجموعة جزئية من المجال A والتي تنتمي صورة كل عنصر فيها الى المجموعة D الجزئية من المجال. وتسمى $f^{-1}(D)$ الصورة العكسية ل D بفعل التطبيق f.

$$x \in f^{-1}(D) \Leftrightarrow\ x \in A\ s.t.\ f(x) \in D$$

$$x \notin f^{-1}(D) \Leftrightarrow\ x \in A\ s.t.\ f(x) \notin D$$

<u>Example 1.29</u>: Let $f: N\backslash\{1\} \to N\ s.t.\ f(n) = n^2 - 1$

Let $D = \{2,3,4,8\}$. Find $f^{-1}(D)$

Solution: $f^{-1}(D) = \{n \in N\backslash\{1\}: f(n) \in \{2,3,4,8\}\}$

$= \{n \in N\backslash\{1\}:\ n^2 - 1 = 2 \vee n^2 - 1 = 3 \vee n^2 - 1 = 4 \vee n^2 - 1 = 8\}$

$= \{n \in N\backslash\{1\}:\ n^2 = 3 \vee n^2 = 4 \vee n^2 = 5 \vee n^2 = 9\}$

$= \{n \in N\backslash\{1\}:\ n=\sqrt{3} \notin N \vee n=2 \in N \vee n = \sqrt{5} \notin N \vee n = 3 \in N\}$

$f^{-1}(D) = \{2,3\}$

<u>Example 1.30</u>: Let $f: \mathbb{R} \to \mathbb{R}\ s.t.\ f(x) = x^2 - 2$. Find $f^{-1}(\{2,7\})$

Solution: $f^{-1}(\{2,7\}) = \{x \in \mathbb{R}: f(x) \in \{2,7\}\}$

$= \{x \in R: x^2 - 2 = 2\ or\ x^2 - 2 = 7\} = \{2,-2,3,-3\}$

<u>Theorem</u> 1.31: Let $f: A \to B$ be a mapping, let $E \subseteq A$ and C, D are subsets of B. Then:

1. $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$
2. If $C \subseteq D$ then $f^{-1}(C) \subseteq f^{-1}(D)$ **(H.W.)**
3. $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$ **(H.W.)**
4. $f^{-1}(C\backslash D) = f^{-1}(C)\backslash f^{-1}(D)$
5. $f^{-1}(C\backslash D) \subseteq f^{-1}(C)$ **(H.W.)**
6. $E \subseteq f^{-1}(f(E))$ **(H.W.)**
7. If f is 1-1 if and only if $E = f^{-1}(f(E))$

8. $f\left(f^{-1}(C)\right) \subseteq C$ (**H.W.**)

9. If f is onto if and only if $f\left(f^{-1}(C)\right) = C$

Proof 1: Let $x \in f^{-1}(C \cap D) \Leftrightarrow f(x) \in C \cap D$ (def. of f^{-1})

$\Leftrightarrow f(x) \in C \wedge f(x) \in D$ (def. of $\cap$)

$\Leftrightarrow x \in f^{-1}(C) \wedge x \in f^{-1}(D)$ (def. of f^{-1})

$\Leftrightarrow x \in f^{-1}(C) \cap f^{-1}(D)$ (def. of $\cap$)

$\therefore f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

Proof 4: Let $x \in f^{-1}(C \backslash D) \Leftrightarrow f(x) \in C \backslash D$ (def. of f^{-1})

$\Leftrightarrow f(x) \in C \wedge f(x) \notin D$ (def. of $\backslash$)

$\Leftrightarrow x \in f^{-1}(C) \wedge x \notin f^{-1}(D)$ (def. of f^{-1})

$\Leftrightarrow x \in f^{-1}(C) \backslash f^{-1}(D)$ (def. of $\backslash$)

$\therefore f^{-1}(C \backslash D) = f^{-1}(C) \backslash f^{-1}(D)$

Proof 7: $\Longrightarrow$) Let f is 1-1 T.P. $E \subseteq f^{-1}\left(f(E)\right) \wedge f^{-1}\left(f(E)\right) \subseteq E$

From part (6), $E \subseteq f^{-1}\left(f(E)\right)$ …….(1)

T.P. $f^{-1}\left(f(E)\right) \subseteq E$

Let $x \in f^{-1}\left(f(E)\right) \Longrightarrow f(x) \in f(E)$ (def. of inverse image)

$\Longrightarrow \exists$ e $\in$ E s.t. $f(x) = f(e)$

$\Longrightarrow x = e \in E$ (f is 1-1)

$\Longrightarrow x \in E$

$\therefore f^{-1}\left(f(E)\right) \subseteq E$ ……(2)

From (1) and (2), $f^{-1}\left(f(E)\right) = E$

$\Leftarrow$) Assume that $f^{-1}\big(f(E)\big) = E$ T.P. f is 1-1

Suppose f is not 1-1 برهان غير مباشر

$\exists x_1, x_2 \in A$ s.t $f(x_1) = f(x_2)$ and $x_1 \neq x_2$

Let $E = \{x_1\} \Longrightarrow x_1 \in E \Longrightarrow f(x_1) \in f(E)$ (def. of direct image)

$\Longrightarrow f(x_2) \in f(E)$ $(f(x_1) = f(x_2))$

$\Longrightarrow f(x_1) \in f(E)$ and $f(x_2) \in f(E)$

$\Longrightarrow x_1 \in f^{-1}\big(f(E)\big) = E$ and $x_2 \in f^{-1}\big(f(E)\big) = E$

$\Longrightarrow \{x_1, x_2\} \in E = \{x_1\}$ تناقض

f is 1-1

<u>Proof</u> 9: $\Longrightarrow$) Let f is onto T.P. $f\big(f^{-1}(C)\big) \subseteq C \wedge \mathrm{C} \subseteq f\big(f^{-1}(C)\big)$

From part (8), $f\big(f^{-1}(C)\big) \subseteq C$ …….(1)

T.P. $\mathrm{C} \subseteq f\big(f^{-1}(C)\big)$

Let $y \in C \Longrightarrow \exists x \in A$ s.t. $y = f(x)$ (f is onto)

$\Longrightarrow x \in f^{-1}(y) \in f^{-1}(C)$ (def. of inverse image)

$\Longrightarrow f(x) = y \in f(f^{-1}(C))$

$\therefore \mathrm{C} \subseteq f\big(f^{-1}(C)\big)$ …..(2)

From (1) and (2), $\mathrm{C} = f\big(f^{-1}(C)\big)$

$\Leftarrow$) Assume that $f\big(f^{-1}(C)\big) = C$ T.P. f is onto

Assume f is not onto برهان بالتناقض

$\exists y \in B - f(A) \Longrightarrow y \neq f(x) \quad \forall x \in A$

$\Longrightarrow f^{-1}(y) \neq x \; \forall x \in A$

Let $C = \{y\} \Longrightarrow f^{-1}(C) = f^{-1}(\{y\}) = \emptyset$

$\Longrightarrow f\left(f^{-1}(C)\right) = f(\emptyset) = \emptyset$

$\Longrightarrow f\left(f^{-1}(C)\right) \neq C$ تناقض مع الفرض

$\therefore f$ is onto

Remark 1.32: Let $f: A \to B$ be a mapping, let $E \subseteq A$ and $C \subseteq B$. Then in general

1. $A \neq f^{-1}\left(f(A)\right)$

2. $B \neq f\left(f^{-1}(B)\right)$

For example,

Let $A = \{1,2,3\}, B = \{4,5,6,7\}$

$f: A \to B$ s.t. $f(1) = f(2) = 4$

$f(3) = 6$

Let $E = \{1,3\} \subseteq A$ and $C = \{4,5\} \subseteq B$

$f(E) = \{4,6\} \Longrightarrow f^{-1}\left(f(E)\right) = f^{-1}(\{4,6\}) = \{1,2,3\} \neq E$

$\Longrightarrow f^{-1}\left(f(E)\right) \neq E$

Also, $f^{-1}(C) = \{1,2\} \Longrightarrow f\left(f^{-1}(C)\right) = f(\{1,2\}) = \{4\} \neq C$

$\Longrightarrow f\left(f^{-1}(C)\right) \neq C$