# Chapter -3-

## (1) Solve O.D.E. of the second order

### Reduction of order

The general form of non-homo. linear O.D.E. of second order is:

$$y'' + a_1(x)y' + a_2(x)y = f(x) \ldots\ldots\ldots\ldots\ldots (*)$$

Where $a_1, a_2, f$ $are\ function\ of\ x$

Let u(x) is a particular solution of homo. linear O.D.E.

$$y'' + a_1(x)y' + a_2(x)y = 0$$

We can reduce of the equation (*) by the following steps:

1- Suppose the solution has the form:-

y=uv

2- by derivative ,obtain:

y=uv

$$y' = uv' + vu'$$

$$y'' = uv'' + u'v' + u'v' + u''v$$

$$= uv'' + 2u'v' + u''v$$

Substituted y, $y', y''$ $in$ $(*)$

$$uv'' + 2u'v' + u''v + a_1(x)(uv' + vu') + a_2(x)uv = f(x)$$

$$uv'' + [2u' + a_1(x)u]v' + [u'' + a_1(x)u' + a_2(x)u]v = f(x)$$

$$u'' + a_1(x)u' + a_2(x)u = 0$$

$$\therefore uv'' + [2u' + a_1(x)u]v' = f(x) \ldots\ldots\ldots (**)$$

3- Let $p = v'\, then\, p' = v''$

Substituted in (**), obtain:

$$up' + (2u' + a_1(x)u)p = f(x)$$

It is non-homo. linear O.D.E. of the first order with dependent variable p independent x ;we can solve it and find the solution of given equation by original variables .

**<u>Ex:-1-</u> if $y = e^{-2x}$ is a solution of the homo. equation**

$$(1+x)y'' + (4x+5)y' + (4x+6)y = 0$$

**then find the solution of the equation**

$$(1+x)y'' + (4x+5)y' + (4x+6)y = e^{-2x}$$

**<u>Solution:-</u>**

Let $y = ve^{-2x}$

$$y' = -2ve^{-2x} + v'e^{-2x}$$

$$y'' = 4ve^{-2x} - 2v'e^{-2x} + v''e^{-2x} - 2v'e^{-2x}$$

Substituted in given equation

$$(1+x)(4ve^{-2x} - 2v'e^{-2x} + v''e^{-2x} - 2v'e^{-2x}) + (4x+5)(-2ve^{-2x} + v'e^{-2x}) + (4x+6)ve^{-2x} = e^{-2x}$$

$$(1+x)v''e^{-2x} + v'e^{-2x} = e^{-2x}] \div e^{-2x}$$

$$(1+x)v'' + v' = 1$$

$$p = v' \, then \, p' = v''$$

$$\therefore (1+x)p' + p = 1$$

( O.D.E. of the first order solving by integrating factor)$\therefore \frac{dp}{dx} + \frac{p}{1+x}dx = \frac{dx}{1+x}$

I.F. is $e^{\int \frac{1}{1+x}dx} = e^{\ln(1+x)} = 1 + x$

d(p(1+x))=dx

by integrating:

$$p(1+x) = x + c_1$$

$$\therefore p = \frac{x + c_1}{1 + x}$$

$$v' = \frac{x + c_1}{1 + x}$$

$$\therefore dv = \frac{x + c_1}{1 + x} dx$$

By integrating:

$v = \int \frac{x+1-1+c_1}{1+x} dx + c_2$

$\therefore v = \int dx + \int \frac{c_1-1}{1+x} dx + c_2$

$v = x + (c_1 - 1) \ln|1 + x| + c_2$

y=$e^{-2x}[x + (c_1 - 1) \ln|1 + x| + c_2]\therefore$

y=$xe^{-2x} + (c_1 - 1) \ln|1 + x| \, e^{-2x} + c_2 e^{-2x}\therefore$

**Ex:-2-** **if $y = x^3$ is a solution of the homo. equation then find the general solution of this equation and find $y'' + \frac{1}{x}y' - \frac{9}{x^2}y = 0$ the other particular solution of this equation.**

**Solution:-**

Let $y = x^3 v$

$$y' = 3vx^2 + v'x^3$$

$$y'' = 3v'x^2 + 6xv + 3v'x^2 + v''x^3$$

$$y'' = 6v'x^2 + 6xv + v''x^3$$

Substituted in given equation; obtain:

$$6v'x^2 + 6xv + v''x^3 + \frac{1}{x}(3vx^2 + v'x^3) - \frac{9}{x^2}x^3 v = 0$$

$(6x^2 + x^2)v' + (6x + 3x - 9x)v = 0v''x^3 +$

$$v''x^3 + 7x^2 v' = 0 \ldots \ldots (*)$$

$$let\ p = v'\ then\ p' = v''$$

Substituted in (*) ;obtain:

$$x^3 p' + 7x^2 p = 0$$

$$p' + \frac{7}{x}p = 0$$

( O.D.E. of the first order solving by integrating factor) $dp + \frac{7}{x}p\mathrm{dx} = 0$

I.F. is $e^{\int \frac{7}{x}dx} = e^{7\ln x} = e^{ln x^7} = x^7$

$$d(px^7) = 0$$

$$px^7 = c_1$$

$$p = c_1 x^{-7}$$

$$\therefore v' = c_1 x^{-7}$$

$$dv = c_1 x^{-7} dx$$

By integrating; obtain:

$$v = \frac{-c_1}{6}x^{-6} + c_2$$

$$y = x^3\left[\frac{-c_1}{6}x^{-6} + c_2\right]$$

$$y = \frac{-c_1}{6}x^{-3} + c_2 x^3$$

is particular solution of homo. equation$\therefore y_1 = x^3$

is particular solution of homo. equation$y_2 = \frac{-1}{6}x^3$

The linearly combination of $y_1, y_2$ is the general solution of given equation

## Exercise

**1- if $y_1 = e^x$ is a solution of the homo. equation $y'' - y = 0$ then use the reduction of order to find a second solution $y_2$ .**

**2- if y=x is a solution of the homo. equation $x^3y'' + xy' - y = 0$ then use the reduction of order to find the general solution.**

**Remark**

We can use the reduction of order to solve O.D.E. of the higher order.

**Ex. Solve the following equations:**

**1-** $\boldsymbol{xy''' - 2y'' = 0}$

**Solution**

Let $q = y''$

$$\therefore q' = y'''$$

$$\therefore xq' - 2q = 0$$

$$xdq - 2qdx = 0$$

$$\frac{dq}{q} - \frac{2}{x}dx = 0$$

by integrating

lnq-2lnx=lna

lnq-ln$x^2 = lna$

$$ln\frac{q}{x^2} = lna$$

$$\therefore q = ax^2$$

$$y'' = ax^2$$

By integrating

$$y' = a\frac{x^3}{3} + b$$

By integrating

$$y = a\frac{x^4}{12} + bx + c$$

**2-** $\boldsymbol{y''' - y'' = 1}$

**Solution**

Let $p = y''$

$$\therefore p' = y'''$$

$$\therefore p' - \mathrm{p} = 1$$

$$\frac{dp}{dx} - p = 1$$

$$dp - pdx = dx$$

$$dp = (p + 1)dx$$

$$\frac{dp}{p + 1} = dx$$

by integrating

$$ln|p + 1| = x + c_1$$

$$\therefore p + 1 = e^{x+c_1}$$

$$\therefore p = Ae^x - 1$$

$$y'' = Ae^x - 1$$

By integrating

$$y' = Ae^x - x + B$$

By integrating

$$y = Ae^x - \frac{x^2}{2} + Bx + c$$

## Exercise

**Solve the equation** $\mathbf{xy'' = y'}$

## (2) Linear O.D.E. with constant coefficients

**Def.**:- The functions $y_{1(x)}, y_{2(x)}, \dots \dots \dots \dots \dots, y_{n(x)}$ are linearly dependent on I if there is a set of the constants $c_1, c_2, \dots \dots \dots \dots \dots, c_n$

Not all zero s.t. $c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0$ In $a \le x \le b \dots \dots .. (**)$

And the functions $y_i(x), i = 1,2,3, \dots \dots . n$ are linearly independent on I if the constants $c_1, c_2, \dots \dots \dots \dots \dots, c_n$ all zero $c_1 = c_2 \dots \dots = c_n = 0$

[The left member of (**) is called a linear combination of the functions $y_1, y_2, \dots \dots . y_n$.]

**Ex.-1- prove that the functions $y_1 = e^x, y_2 = e^{2x}$**

**Are linear independent.**

**Solution:-**

$c_1 y_1 + c_2 y_2 = 0$

$c_1 e^x + c_2 e^{2x} = 0$

(derivative w.r.t x)$c_1 e^x + 2c_2 e^{2x} = 0$

---

$-c_2 e^{2x} = 0$

$e^{2x} \neq 0 \to c_2 = 0$

$\therefore c_1 = 0$

$\therefore c_1 = c_2 = 0$

The functions are linear independent for any x∴

**Theorem-1-**

Let $y_1, y_2, \dots \dots, y_n$

Be solutions of the homo. n-th order linear O.D.E. on an interval I Then the linear combination

Where $c_1, c_2, \dots \dots, c_n$ Are arbitrary constants

Is a solutio on I.

**Corollaries:-**

1- A constant multiple $y = c_1 y_1(x)$

Of a solution $y_1(x)$ Of a homo. linear O.D.E.

Is also solution.

2- A homo. linear O.D.E. always has the trivial solution $y = 0$

**Def.**:-The function $y = c_1 y_1 + c_2 y_2 + \cdots \dots + c_n y_n$

Which is a linear combination for solutions is a general solution for homo. linear O.D.E. if $y_1, y_2, \dots \dots, y_n$ is linear independent.

**Ex.:-** The functions $y_1 = x^2, y_2 = x^2 lnx$

Are solution of

$0x^3 y''' - 2xy' + 4y =$

On $(0, \infty)$then

$\mathrm{y = c_1 x^2 + c_2 x^2 lnx}$

is a solution of this eq. on $(0, \infty)$

Is called the general solution.

**Def.**:- suppose each of the function $y_1, y_2, \dots \dots y_n$

Has at least (n-1) derivative; the determinate

$$w = w(y_1, y_2, \dots\dots, y_n) = \begin{vmatrix} y_1 & y_2 \dots\dots & y_n \\ y'_1 & y'_2 \dots\dots\dots . & y'_n \\ & . & \\ & . & \\ & . & \\ y^{(n-1)}{}_1 & y^{(n-1)}{}_2 \dots\dots\dots\dots & y^{(n-1)}{}_n \end{vmatrix}$$

Where the primes denoted derivative is called the **wronskian** of the functions.

**Theorem-2-:-**

The solutions $y_1, y_2, \dots \dots y_n$ for homo. linear O.D.E. are linear dependent iff $w(y_1, y_2, \dots \dots, y_n) = 0$

And $y_1, y_2, \dots \dots y_n$ are linear independent iff $w(y_1, y_2, \dots \dots, y_n) \neq 0$

**Ex.-1-prove that the functions $e^x, 4e^x, 3e^{-2x}$**

**Are linear dependent for** all **x.**

**Solution:**

$$w\left(e^x, 4e^x, 3e^{-2x}\right) = \begin{vmatrix} e^x & 4e^x & 3e^{-2x} \\ e^x & 4e^x & -6e^{-2x} \\ e^x & 4e^x & 12e^{-2x} \end{vmatrix}$$

$$=4\begin{vmatrix} e^x & e^x & 3e^{-2x} \\ e^x & e^x & -6e^{-2x} \\ e^x & e^x & 12e^{-2x} \end{vmatrix}$$

$$=0$$

The functions $e^x, 4e^x, 3e^{-2x}$ are linear dependent for all x.∴

**Ex.-2- prove that the functions $x^2, x^3, x^{-2}$**

**Are linear independent solutions for $x^3y''' - 6xy' + 12y = 0$**

**Solution:-**

$$w(x^2, x^3, x^{-2}) = \begin{vmatrix} x^2 & x^3 & x^{-2} \\ 2x & 3x^2 & -2x^{-3} \\ 2 & 6x & 6x^{-4} \end{vmatrix}$$

$$= 20$$

$$\neq 0$$

**Ex.-3- are the functions sinx ,cosx solution for $y'' + y = 0$?**

**and are these functions linear independent?**

$y = \sin x \to y' = \cos x \to y'' = -\sin x$

-sinx+sinx=0

sinx is a solution for this equation∴

$y = \cos x \rightarrow y' = -\sin x \rightarrow \boldsymbol{y''} = -\cos x$

-cosx+cosx=0

cosx is a solution for this equation∴

$$w(sinx, cosx) = \begin{vmatrix} sinx & cosx \\ cosx & -sinx \end{vmatrix}$$

$$= -sin^2x - cos^2x = -1 \neq 0$$

Sinx,cosx are linear independent

$y = c_1 sinx + c_2 cosx$

Is a general solution for this eq.

## "Solution of the homo. Linear diff.eq. with constant coefficients"

The general form of homo. linear diff. eq. with constant coefficients is:-

$\frac{\mathrm{dy}}{\mathrm{dx}} + a_n y = 0$ (n order) $\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots \ldots \ldots \ldots \ldots .. + a_{n-1}$

Where $a_1, a_2, \ldots \ldots \ldots \ldots \ldots .., a_{n-1}, a_n$ are constants.

We can write this eq. by using polynomial's operator:-

$F(D)y = (D^n + a_1 D^{n-1} + a_2 D^{n-2} + . \ldots \ldots \ldots \ldots + a_{n-1}\mathrm{D} + a_n)y = 0$

Let $F(m) = m^n + a_1 m^{n-1} + a_2 m^{n-2} + . \ldots \ldots \ldots \ldots + a_{n-1}\mathrm{m} + a_n$

Let $F(m) = 0$

$m^n + a_1 m^{n-1} + a_2 m^{n-2} + . \ldots \ldots \ldots \ldots + a_{n-1}\mathrm{m} + a_n = 0$∴

Characteristic eq. for the homo. linear diff. eq.

$(m - m_1)(m - m_2)(m - m_3) \ldots \ldots \ldots \ldots .(m - m_n) = 0$∴

are roots for Characteristic eq.$m_1, m_2, \ldots \ldots \ldots . m_n$

**Remark-1-**

The general form the homo. linear diff. eq. with constant coefficients of the second order is:-

$\frac{dy}{dx} + a_2 y = 0 \dots\dots\dots\dots\dots\dots\dots . (*) \frac{d^2y}{dx^2} + a_1$

$D^2 y + a_1 \mathrm{D}y + a_2 y = 0$

$(D^2 + a_1 \mathrm{D} + a_2)y = 0$

$\mathrm{F(D)y} = 0$ where $\mathrm{F(D)} = \mathrm{D}^2 + a_1 \mathrm{D} + \mathrm{a}_2 \therefore$

$F(m) = \mathrm{m}^2 + a_1 \mathrm{m} + \mathrm{a}_2$

Let F(m)=0

$\therefore \mathrm{m}^2 + a_1 \mathrm{m} + \mathrm{a}_2 = 0$

$(m - m_1)(m - m_2) = 0$

Where $m_1, m_2$ are roots

**We can solve the homo. linear O.D.E. with constant coefficients by the following:-**

## Case-1-

**If the roots are real and $m_1 \neq m_2$** ;then the general solution for(*) is:-

$$\boldsymbol{y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$

## Case-2-

**If the roots are real and $m_1 = m_2 = m$** ;then the general solution for(*) is:-

$$\boldsymbol{y(x) = (c_1 x + c_2) e^{mx}}$$

## Case-3-

**If the roots are complex $m_1 = a + ib$ , $m_2 = a - ib$** ; then the general solution for(*) is:-

$$\boldsymbol{y(x) = e^{ax}(A\cos bx + B\sin bx)}$$

**Ex.:- solve the following eq.'s:-**

**1-**$\boldsymbol{y'' + y' - 2y = 0}$

**Solution:-**

$D^2 y + \mathrm{Dy} - 2y = 0$

$(D^2 + \mathrm{D} - 2)y = 0$

$\mathrm{F(D)y} = (\,\mathrm{D}^2 + \mathrm{D} - 2)\mathrm{y} = 0$

$F(m) = \mathrm{m}^2 + \mathrm{m} - 2$

Let F(m)=0

$\therefore\ \mathrm{m}^2 + \mathrm{m} - 2 = 0$

$(m-1)(m+2) = 0$

are roots $m_1 = 1\,, m_2 = -2$

$m_1 \neq m_2$

$\therefore\ \boldsymbol{y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$

$\therefore\ \boldsymbol{y(x) = c_1 e^{x} + c_2 e^{-2x}}$

**2-**$\boldsymbol{y'' - 4y' + 4y = 0}$

**Solution:-**

$D^2 y - 4\mathrm{Dy} + 4y = 0$

$(D^2 - 4\mathrm{D} + 4)y = 0$

$\mathrm{F(D)y} = (\,\mathrm{D}^2 - 4\mathrm{D} + 4)\mathrm{y} = 0$

$F(m) = \mathrm{m}^2 - 4\mathrm{m} + 4$

Let F(m)=0

$\therefore\ \mathrm{m}^2 - 4\mathrm{m} + 4 = 0$

$(m-2)(m-2) = 0$

are roots $m_1 = 2\,, m_2 = 2$

$m_1 = m_2$

$\therefore \boldsymbol{y(x) = e^{mx}(c_1 x + c_2)}$

$\therefore \boldsymbol{y(x) = e^{2x}(c_1 x + c_2)}$

**3-** $\boldsymbol{y'' + 2y' + 2y = 0}$

**Solution:-**

$D^2 y + 2\text{Dy} + 2y = 0$

$(D^2 + 2\text{D} + 2)y = 0$

$\text{F(D)} = \text{D}^2 + 2\text{D} + 2\text{F(D)y} = (\text{D}^2 + 2\text{D} + 2)\text{y} = 0 \rightarrow$

$F(m) = \text{m}^2 + 2\text{m} + 2$

Let F(m)=0

$\therefore \text{m}^2 + 2\text{m} + 2 = 0$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2\text{i}}{2} = \frac{2(-1 \pm \text{i})}{2} = -1 \pm i$$

are roots (a=-1, b=1)$m_1 = -1 + i\,, m_2 = -1 - i$

$\therefore \boldsymbol{y(x) = e^{ax}(Acosbx + Bsinbx)}$

$\therefore \boldsymbol{y(x) = e^{-x}(Acosx + Bsinx)}$

**Exercises:-** **solve the following eq.'s:-**

**1-** $\boldsymbol{y'' - 4y' + 3y = 0}$

**2-** $\boldsymbol{y'' + y' = 0}$

**3-** $\boldsymbol{y'' - 2y' + y = 0}$

**4-** $\boldsymbol{y'' + y' + y = 0}$