Ordinary differential equations

Chapter -1-

Basic concepts of differential equations

Definition of differential equations

Is an equation content of un known function and derivative or an equation containing the derivatives of one or more dependent variables w.r.t. one or more independent variables.

Types of differential equations

There are two types of differential equations:

1- Ordinary differential equations(O.D.E)

Is an equation contains only ordinary derivatives of one or more dependent variables w.r.t. a single independent variable.

Remark:-

Ordinary derivatives will be written by using either Leibniz notation

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3} \dots\dots\dots\dots.., \frac{d^ny}{dx^n}$$

Or the prime notation $y', y'', y''', y^{(4)} \dots\dots\dots.., y^{(n)}$

Ex:-

1- $2y' + 3y = 0$

Or $2\frac{dy}{dx} + 3y = 0$

2- $y'' - 4y' + 2y = e^x$

Or $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 2y = e^x$

3- $y^{(4)} + 2y''' + y'' = sinx$

Or $\frac{d^4y}{dx^4} + 2\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} = sinx$

2- Partial differential equations (P.D.E)

Is an equation contains partial derivatives of one or more dependent variables of two or more independent variables.

Remark:-

Partial derivatives will be written

$$\frac{\partial z}{\partial x}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^3 z}{\partial x^3} \dots \dots \dots \dots ..$$

$$\frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^3 z}{\partial y^3} \dots \dots \dots \dots \dots$$

Ex.:-

1- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

2- $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial \mathrm{u}}{\partial \mathrm{t}}$

3- $\frac{\partial \mathrm{u}}{\partial \mathrm{y}} = \frac{\partial \mathrm{v}}{\partial \mathrm{x}}$

Order of differential equation

Is an order of the highest derivative in equation.

Ex.:-

1- $\frac{d^2 y}{dx^2} + 4(\frac{dy}{dx})^3 - 2y = e^x \quad (second\ order)$

2- $y''' - y'' - 3y = cosx \quad (Third\ order)$

Degree of differential equation

Is a degree of the highest derivative in equation.

Ex. -1-

1- $(y^{(4)})^2 + y'' = 0 \quad (furth\ order\ and\ second\ degree)$

2- $(\frac{dy}{dx})^3 + 2ytanx = sinx \quad (first\ order\ and\ third\ degree)$

Ex.-2-

Consider the following equations:

1- $\cos\left(\frac{dy}{dx}\right) = \frac{dy}{dx} + x$

2- $lny' = x^3 + 2$

3- $siny'' - y = 0$

Does not have a degree

Remark:-

O.D.E. has the general form:

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots\dots\dots\dots \frac{d^ny}{dx^n}\right) = 0 \;\; or \;\; F\left(x, y, y', y'', \dots\dots\dots y^{(n)}\right) = 0$$

n-th order O.D.E. where F is a real –valued function of (n+2) variables $x, y, y', y'', \dots\dots\dots y^{(n)}$

$\frac{dy}{dx} = f(x, y)$

.$\frac{d^2y}{dx^2} = f(x, y, y')$

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$$\frac{d^ny}{dx^n} = f(x, y, y', y'', \dots\dots.., y^{(n-1)})$$

Linear O.D.E.

is an equation that is linear in the un known function and its derivative.

The general form of linear O.D.E. is:-

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots \dots\dots\dots\dots. + a_1(x)y' + a_o y = f(x) \dots\dots\dots.(*)$$

Where $a_o, a_1, \dots \dots \dots ., a_n, f$ are functions to x , $x \in [a, b]$

- if $a_n \neq 0$

The equation (*) is called of the n-th order.

- if $a_o, a_1, \dots \dots, a_n$ are constants then the equation (*) it is called of the constant coefficients.

- if f(x)=0 then the equation(*) it is called homogenous

- if $f(x) \neq 0$ then the equation(*) it is called non homogenous

Ex.:-

1- $4xy' + 2e^x y = sinx$ (linear ,non homo.,non with constant coefficients)

2- $y''' + cosy = 0$ (non-linear)

3- $y^{(4)} + y^3 = 0$ (non-linrear)

4- $(1 - y)y'' + 4y = lnx$ (non-linear)

5- $y'' + y' = 0$ (linear,homo., with constant coefficients)

Exercises

Classify the following equations:

1- $y'' + 4y = x^3$

2- $(x + y)y'' + 4y = 2$

3- $(2 + 3(y')^3)^{1/3} = 2y''$

5- $\sqrt{y'} = x + y$

6- $y'' + y'tanx = lnx$

7- $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$

8- $cosy'' + y'' = x + e^{y'}$

9- $y^2 + 3xyy'' + (yy')^3 = x$

10- $\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = 0$

Solution of differential equations

any function φ defined on an interval I and possessing at least n derivatives that are continuous on I ,which when substitute into an n th-order O.D.E. reduces the equation to an identity, is said to be a **solution** of the equation on the interval I .

if $\varphi(x)$ is a solution of $F\left(x, y, y', y'', \dots \dots \dots, y^{(n)}\right) = 0$

Then F(x ,$\varphi(x), \varphi'(x), \varphi''(x), \dots \dots . . \varphi^{(n)}(x)$)= 0 for all x in I.

Ex.1 prove that $y = e^{3x}$ is a solution of $y' - 3y = 0$

Solution:-

$$y' = 3e^{3x}$$

Substituted in given equation obtain:

$$3e^{3x} - 3e^{3x} = 0$$

EX.2 is $y = x^2$a Solution of $x^2y'' - 3xy' + 4y = 0$?

Solution:-

$$y' = 2x \, , y'' = 2$$

Substituted in given equation obtain:

$$2x^2 - 6x^2 + 4x^2 = 0$$

$$\therefore y = x^2 \; is \; a \; solution \; of \; given \; equation$$

Ex.3 prove that $y = xlnx - x$ $is \; a \; solution \; of \; xy' = x + y \, ; x > 0$

Solution:-

$$y' = 1 + lnx - 1 = lnx$$

$$\therefore xlnx = x + xlnx - x$$

$$\therefore xlnx = xlnx \quad \rightarrow \therefore y = xlnx - x \; is \; a \; solution \; of \; given \; equation$$

EX.4 is $\boldsymbol{x = ce^{-kt}}$ **a Solution of** $\boldsymbol{\frac{dx}{dt} = -kx}$?

Solution:-

$$x' = -cke^{-kt}$$

$$\therefore -cke^{-kt} = -kce^{-kt}$$

is a Solution of $\frac{dx}{dt} = -\mathrm{k}x \therefore x = ce^{-kt}$

Exercises

1- prove that :

a- $\boldsymbol{y = \sqrt{1 - x^2}\ is\ a\ solution\ of\ yy' + x = 0\ on(-1, 1).}$

b- $\boldsymbol{y = \frac{1}{16}x^4\ is\ a\ solution\ of\ y' = xy^{1/2}}$

2- Is $\boldsymbol{y = e^{5x}\ a\ solution\ of\ y'' - y' + y = 0}$?

3- Is $\boldsymbol{y = xe^{x}\ a\ solution\ of\ y'' - 2y' + y = 0}$?

Types of the solution for O.D.E.

1-The general solution

Is a solution of O.D.E. has arbitrary constants.

Ex.

1- $y = sinx + c$ *is a general solution of* $y' = cosx$

2- $y = x^4 + c_1x + c_2$ *is a general solution of* $y'' = 12x^2$

3- $y = ce^x$ *is a general solution of* $y' = y$

2- The particular solution

Is a solution of O.D.E. we get it from the general solution.

Ex.1

$y = sinx \qquad (c = 0)$

$y = sinx + \frac{1}{2} \qquad (c = \frac{1}{2})$

$y = sinx - 1 \qquad (c = -1)$

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Are particular solutions of $y' = cosx$

Ex.2

$y = x^4 \qquad (c_1 = c_2 = 0)$

$y = x^4 + x - 1 \qquad (c_1 = 1, c_2 = -1)$

$y = x^4 + \sqrt{2}x - 3 \qquad (c_1 = \sqrt{2}, c_2 = -3$

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Are particular of $y'' = 12x^2$

Ex.3

$$y = e^x \qquad (c = 1)$$

$$y = -e^x \qquad (c = -1)$$

$$y = \sqrt{2}e^x \qquad (c = \sqrt{2})$$

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Are particular solutions 0f $y' = y$

3- Implicit solution of an O.D.E.

A relation G(x,y)=0 is said to be an implicit solution of an O.D.E.
$F\left(x, y, y', y'', \dots \dots, y^{(n)}\right) = 0$

On an interval I, provided that there exists at least one function φ that satisfies the relation as well as the O.D.E. on I.

EX.

The relation $x^2 + y^2 = 25$

Is an implicit solution of the O.D.E. $\frac{dy}{dx} = -\frac{x}{y}\ on\ (-5,5)$

$$\therefore y = \varphi_1(x) = \sqrt{25 - x^2}\ , \qquad y = \varphi_2(x) = -\sqrt{25 - x^2}$$

Are solutions of $\frac{dy}{dx} = -\frac{x}{y}$

Integral curves

Are the curves of the general solution.

Solution curve

Is the graph of a solution of an O.D.E.

Ex. $\mathbf{y' = 2x}$

Solution:-

$$\frac{dy}{dx} = 2x$$

dy=2xdx

By integrating; obtain the general solution:

$$\mathrm{y = x^2 + c\,, c \text{ is arbitrary constant}}$$

$$c = 0 \rightarrow y = x^2$$

$$c = 1 \rightarrow y = x^2 + 1$$

$$c = 2 \rightarrow y = x^2 + 2$$

$$c = -1 \rightarrow y = x^2 - 1$$

$$c = -2 \rightarrow y = x^2 - 2$$

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Are particular solution

Value problems

1- Initial value problem (I.V.P.)

O.D.E. with initial conditions it is called initial value problem (I.V.P.)

$$F(x, y, y') = 0 \;, \qquad y(x_o) = y_o$$

$$F(x, y, y', y'') = 0 \;, \qquad y(x_o) = y_o\,, y'(x_o) = y_1$$

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$$F\left(x, y, y', y'', \dots \dots .., y^{(n)}\right) = 0 \;, y(x_o) = y_o\,, y'(x_o) = y_1 \dots ., y^{(n-1)}(x_o) = y_{n-1}$$

Ex.1

If $\boldsymbol{y = x^2 + c \text{ is the solution of } I.V.P.\, y' = 2x\,, y(1) = 5 \text{ then find } c}$

Solution

5=1+c

C=4

$\therefore y = x^2 + 4$ is the solution of given I. V. P.

EX.2

$\boldsymbol{y = c_1 cos4x + c_2 sin4x \text{ is a solution of the equation}}$

$\boldsymbol{y'' + 16y = 0. \text{ find a solution of the } I.V.P.}$

$$\boldsymbol{y'' + 16y = 0\,, y\left(\frac{\pi}{2}\right) = -2\,, y'\left(\frac{\pi}{2}\right) = 1}$$

Solution

$$y\left(\frac{\pi}{2}\right) = -2$$

$$c_1 cos2\pi + c_2 sin2\pi = -2$$

$$c_1 = -2$$

$$y'(x) = -4c_1 sin4x + 4c_2 cos4x$$

$$y'\left(\frac{\pi}{2}\right) = 1$$

$$4c_2 = 1 \rightarrow c_2 = \frac{1}{4}$$

The solution of I.V.P. is $y = -2\cos4x + \frac{1}{4}\sin4x$

2- Boundary value problem(B.V.P.)

When an O.D.E. is to be solved under conditions involving dependent variable and its derivatives at two different values of independent variable then the problem under consideration is said to be a B.V.P.

Ex.

1- $a_2(x)y'' + a_1(x)y' + a_o(x)y = g(x), \quad y(a) = y_o\,, y(b) = y_1$

2- $y'' + y = 0\,, \quad y(a) = y_1\,, y(b) = y_2$

3- $y'' + 16y = 0\,, \quad y(0) = 0\,, y\left(\frac{\pi}{2}\right) = 1$

System of O.D.E

A system of O.D.E. is two or more equations involving the derivatives of two or more unknown functions of a single independent variable.

Ex.

System of O.D.E. of two first order where y,z are dependent variable and x is independent variable:

$$\frac{dy}{dx} = f(x, y, z)$$

$$\frac{dz}{dx} = g(x, y, z)$$

A solution of the system is a pair of functions $y = \varphi_1(x), z = \varphi_2(x)$

Defined on interval I that satisfy each equation of the system on this interval.

Finding O.D.E. from the general solution

We will find O.D.E. from general solution by show the relation between arbitrary constants and order of O.D.E.

Ex.

1- Find O.D.E. from general solution $y = ce^x$

$$y' = ce^x \rightarrow y' = y$$

Or $c = \frac{y'}{e^x}$

$$\mathrm{y} = \frac{\mathrm{y}'}{\mathrm{e}^{\mathrm{x}}}\mathrm{e}^{\mathrm{x}}$$

$\therefore y = y'$

2- Find O.D.E. from the general solution $y = c_1 x + c_2 x^3$

$$y' = c_1 + 3c_2 x^2$$

$$y'' = 6c_2 x \rightarrow c_2 = \frac{y''}{6x}$$

$$y' = c_1 + \frac{1}{2} x y''$$

$$c_1 = y' - \frac{1}{2} x y''$$

$$\therefore y = \left(y' - \tfrac{1}{2} x y''\right) x + \tfrac{1}{6} x^2 y''$$

$$= xy' - \frac{1}{3} x^2 y''$$

$$y - xy' + \frac{1}{3} x^2 y'' = 0 \quad (O.D.E.)$$

Remark

We can use the determent to finding O.D.E.

Ex.1

Find O.D.E. from the general solution $y = Ax + Bx^4$

Solution

$$\mathrm{y = Ax + Bx^4} \quad \rightarrow \quad \mathrm{Ax + Bx^4 - y = 0}$$

$$y' = A + 4Bx^3 \rightarrow A + 4Bx^3 - y' = 0$$

$$y'' = 12Bx^2 \quad \rightarrow 12Bx^2 - y'' = 0$$

$$\begin{bmatrix} x & x^4 & -y \\ 1 & 4x^3 & -y' \\ 0 & 12x^2 & -y'' \end{bmatrix} = 0$$

$$\mathrm{x} \begin{bmatrix} 4\mathrm{x}^3 & -\mathrm{y}' \\ 12\mathrm{x}^2 & -\mathrm{y}'' \end{bmatrix} - \begin{bmatrix} x^4 & -y \\ 12x^2 & -y'' \end{bmatrix} = 0$$

$$x(-4x^3 y'' + 12x^2 y') - (-x^4 y'' + 12x^2 y) = 0$$

(O.D.E.)$-3x^4y'' + 12x^3y' - 12x^2y = 0$

Ex.2

Find O.D.E. from the general solution $y = ce^x$ $by\ using\ determent$

Solution

$$y = ce^x \quad \rightarrow \; ce^x - y = 0$$

$$y' = ce^x \quad \rightarrow ce^x - y' = 0$$

$$\begin{bmatrix} e^x & -y \\ e^x & -y' \end{bmatrix} = 0$$

$$-e^x y' + e^x y = 0$$

$$\therefore -y' + y = 0$$

$$\therefore y' = y \quad (O.D.E.)$$

Exercises

1- Find c in $y = \frac{1}{3}x^3 + c$ $is\ a\ solution\ of\ I.V.P.\ y' = x^2\ , x = 1\ , y = 2.$

2- Find c in $y = \frac{1}{4}x^4 + c$ $is\ a\ solution\ of\ I.V.P.\ y' = x^3\ , y(1) = \frac{1}{2}.$

3- Find O.D.E from the general solution $y = c_1x + c_2x^2$ $by\ using\ determent.$