# Polynomials

A polynomial is a mathematical expression constructed with constants and one variable x involving only non-negative integer powers of x, using the four operations:

That is a polynomial is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \ldots \ldots \ldots (1)$$

Where $a_n, a_{n-1}, \ldots \ldots \ldots a_2, a_1, a_0$ are the coefficients of the polynomial (real numbers or complex numbers).

**Definition:**

The **degree of a polynomial** is the highest power of x in its expression.

**Remark**

1- If $a_n \neq 0$ then the polynomial(1)of degree n.

2- The leading coefficient of the polynomial(1) is $a_n \neq 0$

3- Constant (non-zero) polynomials, linear polynomials, quadratics, cubic's and quartics are polynomials of degree 0, 1, 2, 3 and 4 respectively.

| Polynomial | Example | Degree |
|---|---|---|
| Constant | 1 | 0 |
| Linear | $2x+1$ | 1 |
| Quadratic | $3x^2+2x+1$ | 2 |
| Cubic | $4x^3+3x^2+2x+1$ | 3 |
| Quartic | $5x^4+4x^3+3x^2+2x+1$ | 4 |

**Example:**

Is the following function polynomial write in standard form, state its degree and write the leading coefficient?

**1-** $\boldsymbol{f(x) = \frac{1}{2}x^2 - 3x^4 - 7}$

The function is a polynomial function.

Its standard form is $f(\boldsymbol{x}) = -3\boldsymbol{x}^4 + \frac{1}{2}\boldsymbol{x}^2 - 7.$

It has degree 4, so it is a quartic function.

The leading coefficient is $-3$.

H.W.

**2-** $\boldsymbol{f(x) = 3^x + x^3}$

3- $f(x) = 4x^3 + \sqrt{x} - 1$

4- $f(x) = 5x^4 - 2x^2 + 3/x$

5- $f(x) = 0.5x + \pi x^2 - \sqrt{2}$

## (3.1) Properties of Polynomials

Let

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 = \sum_{i=0}^{n} a_i x^i$$

$$q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_2x^2 + b_1x + b_0 = \sum_{i=0}^{m} b_i x^i$$

Be two polynomials. Then

1- $p\ (x)=q(x) \Leftrightarrow n=m\ and\ a_n = b_m\ , a_{n-1} = b_{m-1}\ , \ldots\ a_0 = b_0$ .
2- If $m \le n$ then $p(x) + q(x)$ is a polynomial of degree $\le n$.
3- $p(x).q(x)$ is a polynomial of degree $n$ + m
4- If $p\ (x) \neq 0$ and $p\ (x) \cdot (x) = 0$ then $q\ (x) = 0$
5- If $q\ (x) \neq 0$ and $f\ (x) \cdot q\ (x) = (x) \cdot q(x)$ then $f(x) = p(x)$ .

**Example:**

Let $f(x) = 1 + 2x^2$ and $g(x) = 3 + 4x - 2x^2$ find f+g , f.g and their degree

**Solution** $f(x) + g(x) = 1 + 2x^2 + 3 + 4x - 2x^2 = 4x + 4$ (first degree)

$f(x).g(x) = (1 + 2x^2)\quad(3 + 4x - 2x^2) = 3 + 4x + 4x^2 + 8x^3 - 4x^4$

**Definition**

A number $r$ is said to be a zero or root of a function f($x$) if f($r$) = 0, and this occurs when (x−r) is a factor of f(x).

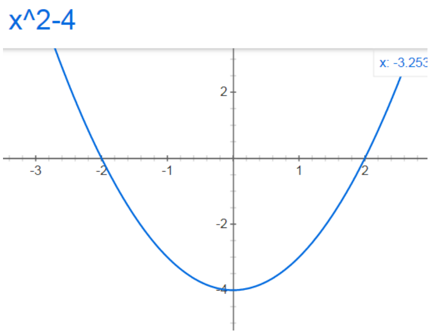
The zeros of a polynomial are the solutions of the equation. So if the coefficients of a polynomial are real numbers, then the real zeros of a polynomial are just the $x$ intercepts of the graph of a polynomial.

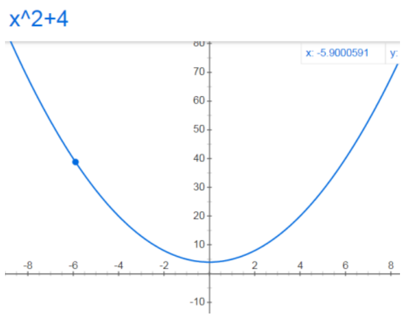
**Example:** the real zeros of the polynomial $\boldsymbol{P(x) = x^2 - 4}$ are 2 and $-2$, the $x$- intercepts of the graph of the polynomial

However, a polynomial may have zeros that are not $x$ - intercepts.

**Example:** $\boldsymbol{Q(x) = x^2 + 4}$ has zeros $2i$ and $-2i$, but its graph has no

$x$ – intercepts.




## (3.2) Polynomial Division (Long Division)

The process is very like the long division of numbers. Example below will illustrate the process.

**Example**

Divide $P(x) = 3x^3 - 5 + 2x^4 - x$ by $(2 + x)$.

**Solution:** First, rewrite the dividend the polynomial in descending powers of $x$, inserting 0 as the coefficient for any missing terms of degree less than 4:

$P(x) = 2x^4 + 3x^3 + 0x^2 - x - 5.$

Similarly, rewrite the divisor $(2 + x)$ in the form $(x + 2)$. Then divide the first term $x$ of the divisor into the first term $2x^4$ of the dividend. The result will be

$2x^3$, Multiply it by the divisor, obtaining $2x^4 + 4x^3$. Line up like terms, subtract as in arithmetic, and bring down $0x^2$. Repeat the process until the degree of the remainder is less than the degree of the divisor.

$$
\begin{array}{lrl}
 & \quad 2x^3 - x^2 + 2x - 5 & \text{Quotient} \\
\text{Divisor} \quad x + 2 & \overline{)\,2x^4 + 3x^3 + 0x^2 - x - 5} & \text{Dividend} \\
 & \underline{2x^4 + 4x^3} \qquad\qquad\qquad & \text{Subtract} \\
 & -x^3 + 0x^2 \qquad\qquad & \\
 & \underline{-x^3 - 2x^2} \qquad\qquad & \text{Subtract} \\
 & 2x^2 - x \qquad & \\
 & \underline{2x^2 + 4x} \qquad & \text{Subtract} \\
 & -5x - 5 & \\
 & \underline{-5x - 10} & \text{Subtract} \\
 & 5 & \text{Remainder}
\end{array}
$$

Then

$$\frac{2x^4 + 3x^3 - x - 5}{x + 2} = 2x^3 - x^2 + 2x - 5 + \frac{5}{x + 2}$$

H.W: check your answer using multiplication.

## (3.3) Division Algorithm Theorem for Polynomials

**(**Dividend = Quotient × Divisor + Remainder)

Suppose f(x) and g(x) are the two polynomials, where g(x) ≠0, there are unique polynomials q(x) and r(x) such that:

f(x) = q(x) g(x) + r(x)

Where r(x) is the remainder polynomial and degree r(x) < degree g(x).

**That is** this leads to : For each polynomial $P(x)$ of degree greater than 0 and each number $r$, there exists a unique polynomial $Q(x)$ of degree less than $P(x)$ and a unique number $R$ such that

$$P(x) = (x - r)Q(x) + R.$$

The polynomial P$(x)$ is called the **dividend**, Q$(x)$ is the **quotient,** $(x - r)$ is the **divisor,** and $R$ is the **remainder.**

Note that $R$ may be 0.

**(3.3.1)Verification of Division Algorithm**

Take the above example and verify it:

$$P(x) = 2x^4 + 3x^3 + 0x^2 - x - 5 = (x+2)(2x^4 + 4x^3)+5$$

## (3.3.2)Horner's Method (Synthetic Division)

To divide the polynomial $P(x)$ by $x - r$:

**Step 1.** Arrange the coefficients of $P(x)$ in order of descending powers of $x$. Write 0 as the coefficient for each missing power.

**Step 2.** After writing the divisor in the form $x - r$, use $r$ to generate the second and third rows of numbers as follows. Bring down the first coefficient of the dividend and multiply it by $r$; then add the product to the second coefficient of the dividend. Multiply this sum by $r$, and add the product to the third coefficient of the dividend. Repeat the process until a product is added to the constant term of $P(x)$.

**Step 3.** The last number to the right in the third row of numbers is the remainder. The other numbers in the third row are the coefficients of the quotient, which is of degree less than $P(x)$.

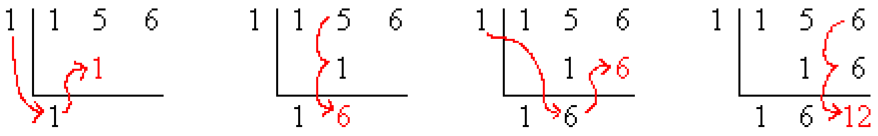
**Example:**

Divide $x^2 + 5x + 6$ by $x - 1$ by using synthetic division?

**Solution:**

First write the coefficients of the dividend and the *negative* of the constant term of the divisor in the format shown below at the left. Bring down the 1 as indicated next on the right, multiply by 1, and record the product 1. Add 5 and 1, bringing down their sum 6.

Repeat the process until the coefficients of the quotient and the remainder are obtained.



**Example:** Use synthetic division to divide

$$P(x) = 4x^5 - 30x^3 - 50x - 2 \text{ by } (x + 3).$$

Find the quotient and remainder. Write the conclusion in the form of Division Algorithm Theorem.

**Solution:** Because $x + 3 = x - (-3)$, we have $r = -3$ then the synthetic division is:

| | | | | | | |
|---|---|---|---|---|---|---|
| | 4 | 0 | −30 | 0 | −50 | −2 |
| -3 | | −12 | 36 | −18 | 54 | −12 |
| | 4 | −12 | 6 | −18 | 4 | −14 |

The quotient is $4x^4 - 12x^3 + 6x^2 - 18x + 4$

The remainder -14

The form of division algorithm is:

$4x^5 - 30x^3 - 50x - 2 = (x + 3)(4x^4 - 12x^3 + 6x^2 - 18x + 4) - 14$

**Theorem: (Remainder Theorem)**

If $R$ is the remainder after dividing the polynomial $P(x)$ by $(- r)$, then

$P(r) = R$.

**Example:**

If $P(x) = 4x^4 + 10x^3 + 19x + 5$, find $P(-3)$ by

(i) Using the remainder theorem and synthetic division.

(ii) Evaluating $(-3)$ directly.

**Solution:**

(i) Use synthetic division to divide $P(x)$ by $x-(-3)$.

| | | | | | |
|---|---|---|---|---|---|
| | 4 | 10 | 0 | 19 | 5 |
| | | −12 | 6 | −18 | −3 |
| −3 | 4 | −2 | 6 | 1 | 2 = R = P(−3) |

(ii) $P(-3) \; = \; 4(-3)^4 \; + 10(-3)^3 + \; 19(-3) \; + \; 5 = 2.$

**Theorem: (Factor Theorem)**

If $r$ is a zero of the polynomial $P(x)$, then $(x - r)$ is a factor of $(x)$. Conversely, if $(x-r)$ is a factor of $(x)$, then $r$ is a zero of $(x)$.

***Proof:***

The remainder theorem shows that the division algorithm equation,

$P(x) = (x-r)Q(x) + R$

Can be written in the form where $R$ is replaced by $(r)$:

$P(x) = (x-r)Q(x) + P(r)$

Therefore, $x$- $r$ is a factor of $P(x)$ if and only if $P(r) = 0$, that is, if and only if

$r$ is a zero of the polynomial $P(x)$ .

**Example:**

Use the factor theorem to show that $(x+1)$ is a factor of $P(x) = x^{25} + 1$ but is not a factor of $Q(x) = x^{25} - 1$

**Solution:**

Since x+1=x-(-1) , $P(-1) = (-1)^{25} + 1 = -1 + 1 = 0$

Then x+1 is a factor of $x^{25} + 1$.

$$Q(-1) = (-1)^{25} - 1 = -1 - 1 = -2$$

And x+1 is not a factor of $x^{25} - 1$

**Example:**

Find the remainder when $f(x) = 2x^4 - 3x^3 + 7x$ is divided by $(x + 2)$.

By using: (i) Remainder Theorem (ii) Long division (iii) synthetic division

**Solution:**

(i) Remainder Theorem. Since $(x - r) = (x + 2)$, it follows that $r = -2$. Thus,

$$R = f(-2) = 32 + 24 - 14 = 42.$$

(ii) Long Division.

$$\begin{array}{r|l} & 2x^3 - 7x^2 + 14x - 21 \\ \hline (x+2) & 2x^4 - 3x^3 + 0x^2 + 7x + 0 \\ & \mp 2x^4 \mp 4x^3 \\ \hline & -7x^3 + 0x^2 + 7x + 0 \\ & \pm 7x^3 \pm 14x^2 \\ \hline & 14x^2 + 7x + 0 \\ & \mp 14x^2 \mp 28x \\ \hline & -21x + 0 \\ & \pm 21x \pm 42 \\ \hline & +42 \end{array}$$

Then the remainder is 42 and the quotient $P(x)$ is

$$q(x) = 2x^3 - 7x^2 + 14x - 21.$$

(iii) Synthetic substitution.

$$\begin{array}{r|rrrrr} & 2 & -3 & +0 & +7 & +0 \\ & & -4 & +14 & -28 & +42 \\ \hline -2 & 2 & -7 & +14 & -21 & +42 = R = f(-2) \end{array}$$