"Integrating factors"

Def.:-When an O.D.E. is not exact then multiplying by a factor; obtain an exact O.D.E. this factor is called an integrating factor of this equation.

i.e. if M(x,y)dx+N(x,y)dy=0 inexact O.D.E.

if there exist a factor u(x,y) s.t. uM(x,y)dx+UN(x,y)dy=0 exact O.D.E.

$$\therefore \frac{\partial(\mathrm{uM})}{\partial \mathrm{y}} = \frac{\partial(\mathrm{uN})}{\partial \mathrm{x}}$$

U(x,y) is called integrating factor.

The combination	Integrating factor	Exact differential
xdy-ydx	$\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{xy}, \frac{1}{x^2+y^2}, \frac{1}{x^2-y^2} \dots\dots\dots, \frac{1}{ax^2+bxy+cy^2}$	$\frac{y}{x}, \frac{-x}{y}, ln\frac{y}{x}, \dots.$
Pydx+qxdy ydx+xdy	$x^{p-1}y^{q-1}$ 1	$x^p y^q$ xdy − xy
dy+p(x)ydx	$e^{\int p(x)dx}$	$ye^{\int p(x)dx}$

Ex.:- **Solve the following equations:**

(1) $\boldsymbol{xdy - ydx = x^2y^3dx}$

Solution

$$\text{multiply } \textit{the equation} \text{ by the integrating factor (I.F.) } \frac{1}{x^2} \text{ or } \frac{1}{y^2} \text{ or } \frac{1}{xy}$$

$$\frac{xdy - ydx}{x^2} = y^3dx$$

$$d\left(\frac{y}{x}\right) = y^3 dx$$

Let $z = \frac{y}{x} \rightarrow y = xz$

$$dz = x^3 z^3 dx$$

$$\frac{dz}{z^3} = x^3 dx$$

By integrating

$$\frac{-1}{2z^2} = \frac{x^4}{4} + c$$

$$\frac{-1}{2\left(\frac{y}{x}\right)^2} = \frac{x^4}{4} + c \textit{ the general solution}$$

(2) $\boldsymbol{xdy - \left(3y + \frac{x^4}{y}\right) dx = 0}$

Solution

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \text{ is inexact}$$

$$\mathrm{xdy} - 3\mathrm{ydx} - \frac{\mathrm{x}^4}{\mathrm{y}}\mathrm{dx} = 0$$

$$\mathrm{xdy} - 3\mathrm{ydx} = \frac{\mathrm{x}^4}{\mathrm{y}}\mathrm{dx}$$

p=-3 , q=1

The integrating factor (I.F.) is $x^{p-1}y^{q-1} = x^{-4} = \frac{1}{x^4}$

$$\therefore d(x^{-3}y) = -3x^{-4}ydx + x^{-3}dy$$

$$= x^{-4}(-3ydx + xdy)$$

$$\therefore \frac{xdy - 3ydx}{x^4} = \frac{dx}{y}$$

$$d\left(\frac{y}{x^3}\right) = \frac{1}{y}dx$$

$$z = \frac{y}{x^3} \rightarrow y = x^3 z$$

$$dz = \frac{1}{x^3 z}dx$$

$$zdz = x^{-3}dx$$

By integrating

$$\frac{z^2}{2} = \frac{x^{-2}}{-2} + c$$

$$\frac{1}{2}\left(\frac{y}{x^3}\right)^2 = \frac{-1}{2x^2} + c \textit{ the general solution}$$

(3) $\boldsymbol{\frac{dy}{dx} + \frac{y}{x} = 3x + 2}$

<u>Solution</u>

$$dy + \frac{y}{x}dx = (3x + 2)dx$$

The integrating factor (I.F.) is $e^{\int \frac{1}{x}dx} = e^{lnx} = x$

$$\therefore x\left(dy + \frac{y}{x}dx\right) = (3x^2 + 2x)dx$$

$$d(xy) = (3x^2 + 2x)dx$$

By integrating

$$\therefore xy = x^3 + x^2 + c \;\; \textit{the general solution}$$

Exercises

Solve the following equations by using integrating factor:-

(1) $4ydx + xdy = xy^2dx$

(2) $2ydx + 3xdy = 3x^{-1}dx$

(3) $xdy - 2ydx = x^3y^4dx$

(4) $dy + \frac{3}{x}ydx = x^{-3}e^xdx$

(5) $\frac{dy}{dx} + 2xy = xe^{-x^2}$

3-Linear O.D.E.

The general form of non-homo. linear O.D.E. of the first order is:-

$$\frac{dy}{dx} + p(x)y = \varphi(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (*)$$

Where p, φ *are functions of* x

We can solve this equation (*) by using the integrating factor, as follows:-

$$dy + p(x)ydx = \varphi(x)dx$$

The integrating factor is $e^{\int p(x)dx}$

$$\therefore\ e^{\int p(x)dx}\,(dy + p(x)ydx) = e^{\int p(x)dx}\varphi(x)dx$$

$$\therefore\ d(ye^{\int p(x)dx}) = e^{\int p(x)dx}\varphi(x)dx$$

by integrating the general solution of (*) is:-

$$\therefore\ ye^{\int p(x)dx} = \int e^{\int p(x)dx}\ \varphi(x)dx + c$$

Or $y = e^{-\int p(x)dx}\int e^{\int p(x)dx}\ \varphi(x)dx + ce^{-\int p(x)dx}$

Ex.:- Solve the following equations:

(1) $\frac{dy}{dx} + \frac{y}{x} = x^2 - 3$

Solution

$$\mathrm{dy} + \frac{1}{\mathrm{x}}\mathrm{ydx} = (\mathrm{x}^2 - 3)\mathrm{dx}$$

I.F. $e^{\int \frac{1}{x}dx} = e^{lnx} = x$

$$\mathrm{x}(\mathrm{dy} + \frac{1}{\mathrm{x}}\mathrm{ydx}) = (\mathrm{x}^3 - 3\mathrm{x})\mathrm{dx}$$

$$\mathrm{d}(\mathrm{yx}) = (\mathrm{x}^3 - 3\mathrm{x})\mathrm{dx}$$

By integrating the general solution is

$$\mathrm{yx} = \frac{x^4}{4} - \frac{3}{2}x^2 + c \rightarrow \therefore \mathrm{y} = \frac{x^3}{4} - \frac{3}{2}x + \frac{c}{x}$$

$(\mathbf{2})\ \boldsymbol{x\frac{dy}{dx} + 2y = x^3}$

Solution

$$xdy + 2ydx = x^3dx$$

$$dy + \frac{2}{\mathrm{x}}ydx = x^2dx$$

I.F. $e^{\int \frac{2}{x}dx} = e^{2lnx} = e^{lnx^2} = x^2$

$\mathrm{d}(\mathrm{y}x^2) = x^4dx$

by integrating the general solution is:-

$$yx^2 = \frac{x^5}{5} + c$$

$$y = \frac{x^3}{5} + \frac{c}{x^2}$$

Exercises

Solve the following equations:-

(1) $\frac{dy}{dx} + 3xy = -2$

(2) $y' + x^2y^2 = x$

(3) $xy' - \frac{y}{x^2} = sinx$

(4) $y' - 2xy = e^{x^2}$

4- Bernollie's equation

This equation has the form:-

$$\frac{dy}{dx} + p(x)y = y^n\varphi(x) \dots\dots..(*)$$

If n=0 → $non - homo. Linear\ O.D.E.$

If n=1 → $homo. Linear\ O.D.E.$

The equation (*) is non-Linear transfer it to Linear and we can solve this equation by the following steps:-

(1) $[\frac{dy}{dx} + p(x)y = y^n\varphi(x)] \div y^n$

$$y^{-n}\frac{dy}{dx} + p(x)y^{1-n} = \varphi(x)$$

(2) let $z=y^{1-n}$

$$\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

$$\therefore y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dz}{dx}$$

(3) Substituted (2) in (1); obtain:-

$$\frac{1}{1-\mathrm{n}}\frac{\mathrm{dz}}{\mathrm{dx}}+p(x)z=\varphi(\mathrm{x})$$

$$(4)\ \frac{\mathrm{dz}}{\mathrm{dx}}+(1-\mathrm{n})p(x)z=(1-n)\varphi(\mathrm{x})\ \text{non}-\text{homo. Linear O.D.E.}$$

We can solve this equation by integrating factor

(5) I.F. $e^{\int(1-n)p(x)dx}$

(6) $\mathrm{d}(\mathrm{z}\, e^{\int(1-n)p(x)dx})=(1-n)\varphi(\mathrm{x})\ e^{\int(1-n)p(x)dx}$

(7) by integrating; obtain:-

$$\mathrm{z}\ e^{\int(1-n)p(x)dx}=\int(1-n)\ \varphi(\mathrm{x})\ e^{\int(1-n)p(x)dx}\mathrm{dx}+\mathrm{c}$$

(8) Substituted z=$\mathrm{y}^{1-\mathrm{n}}$ in (7)

<u>Ex.</u> Solve the following equation:-

$$\boldsymbol{xy-\frac{dy}{dx}=y^3e^{-x^2}}$$

<u>Solution</u>

Bernoulli's equation

$$\left[\frac{\mathrm{dy}}{\mathrm{dx}}-\mathrm{xy}=-\mathrm{y}^3\mathrm{e}^{-\mathrm{x}^2}\right]\div\mathrm{y}^3$$

(1) $\mathrm{y}^{-3}\ \frac{\mathrm{dy}}{\mathrm{dx}}-\mathrm{xy}^{-2}=-\mathrm{e}^{-\mathrm{x}^2}$

(2) z=y^{-2}

$$\frac{\mathrm{dz}}{\mathrm{dx}}=-2\mathrm{y}^{-3}\frac{\mathrm{dy}}{\mathrm{dx}}$$

$$\mathrm{y}^{-3}\frac{\mathrm{dy}}{\mathrm{dx}}=\frac{-1}{2}\frac{\mathrm{dz}}{\mathrm{dx}}$$

(3) $\frac{-1}{2}\frac{\mathrm{dz}}{\mathrm{dx}}-\mathrm{xz}=-\mathrm{e}^{-\mathrm{x}^2}$

$$\frac{dz}{dx} + 2xz = 2e^{-x^2}$$

(4) $dz + 2xzdx = 2e^{-x^2}dx$

I.F. is $e^{\int 2xdx} = e^{x^2}$

(5) $e^{x^2}(dx + 2xzdx) = 2dx$

(6) $d(ze^{x^2}) = 2dx$

By integrating

$$ze^{x^2} = 2x + c$$

The general solution is:-

$$y^{-2}e^{x^2} = 2x + c$$

Exercises

(1) $\boldsymbol{y' + xy = \frac{x}{y^3} , y \neq 0}$

(2) $\boldsymbol{xdy - ydx = y^2dx}$

(3) $\boldsymbol{xy' + y = y^2 lnx}$

(4) $\boldsymbol{2y' - \frac{y}{x} = 5x^3y^3 , x \neq 0}$