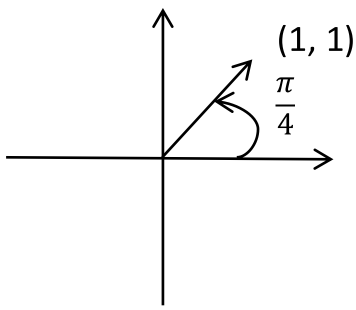
Example: Find the principal argument Arg z when

1. $z = 1 + i$



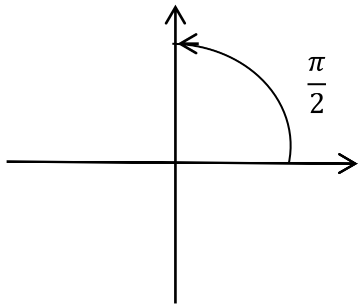
Solution: $\arg z = \text{Arg } z + 2n\pi$

$$= \frac{\pi}{4} + 2n\pi$$

$\therefore \text{Arg } z = \frac{\pi}{4}$

2. $z = i$

Solution: $r = 1,\ \theta = \frac{\pi}{2} + 2n\pi = arg\, i$



$\arg z = \text{Arg } z + 2n\pi$

$= \frac{\pi}{2} + 2n\pi$

$\therefore \text{Arg } z = \frac{\pi}{2}$

$\therefore i = z = 1.\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{2}\right)$

Example: Let $z = -1 - i$, write z in polar form and find Arg z.

Solution: $r = \sqrt{1+1} = \sqrt{2}$

$x = r\cos\theta \rightarrow -1 = \sqrt{2}\cos\theta \rightarrow \cos\theta = \frac{-1}{\sqrt{2}}$

$y = r\sin\theta \rightarrow -1 = \sqrt{2}\sin\theta \rightarrow \sin\theta = \frac{-1}{\sqrt{2}}$

$\theta = \tan^{-1}(1) = \frac{\pi}{4}$

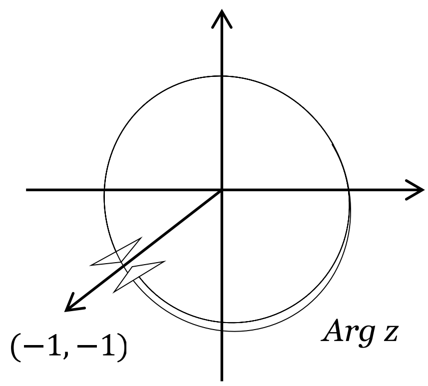
(Since θ is located in the third quarter) $\theta = \frac{\pi}{4} + \pi = \frac{5\pi}{4} + 2n\pi$

$= argz$

$\therefore \text{Arg } z = \arg z - 2\pi$

$= \frac{5\pi}{4} - 2\pi = \frac{-3\pi}{2} \in [-\pi, \pi]$

$z = -1 - i = \sqrt{2}\left(\cos\frac{-3\pi}{2} + i \sin\frac{-3\pi}{2}\right)$



Example: Let $z_1 = 1 + \sqrt{3}\, i$, $z_2 = -1 - \sqrt{3}\, i$, write z_1, z_2 in polar form and find Arg z_1, Arg z_2.

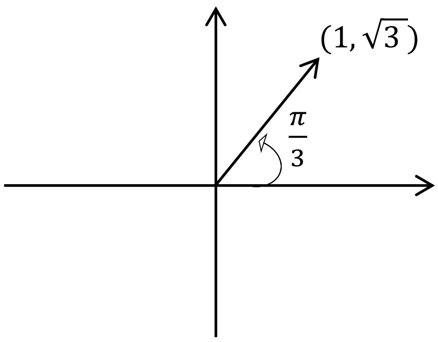
Solution: $z_1 = r_1 = \sqrt{x^2 + y^2} = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = \sqrt{1+3} = 2$

$x = r\cos\theta \to 1 = 2\cos\theta \to \cos\theta = \frac{1}{2}$

$y = r\sin\theta \to \sqrt{3} = 2\sin\theta \to \sin\theta = \frac{\sqrt{3}}{2}$

$\therefore \theta = \tan^{-1}\frac{y}{x} = \frac{\pi}{3} + 2n\pi$

$z_1 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$



$\to z_2 = r_2 = \sqrt{(-1)^2 + \left(-\sqrt{3}\right)^2} = 2$

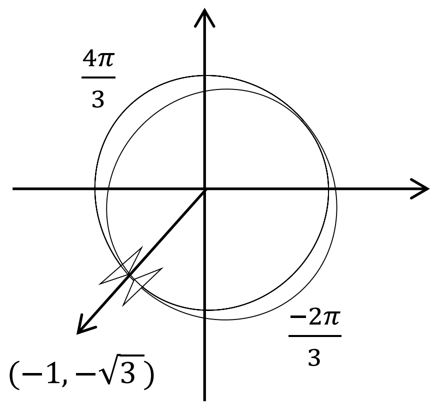
$x = r\cos\theta \to -1 = 2\cos\theta \to \cos\theta = \frac{-1}{2}$

$y = r\sin\theta \to -\sqrt{3} = 2\sin\theta \to \sin\theta = \frac{-\sqrt{3}}{2}$

$\therefore \theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{-\sqrt{3}}{-1} = \tan^{-1}\sqrt{3}$

$= \left(\pi + \frac{\pi}{3}\right) + 2n\pi$

$= \frac{4\pi}{3} + 2n\pi$



Arg $z_2 = \frac{4\pi}{3} - 2\pi$

$= \frac{-2\pi}{3}$

$z_2 = 2\left(\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right)$

Exercises: Find the principal argument Arg z when $z = -i, 1, -1$.

Note:

$$\left.\begin{matrix} 1 \mp i \\ -1 \mp i \end{matrix}\right\} \quad \text{Angle } 45°$$

$$\left.\begin{matrix} 1 \mp \sqrt{3}\, i \\ -1 \mp \sqrt{3}\, i \end{matrix}\right\} \text{Angle } 60°$$

$$\left.\begin{matrix} \sqrt{3} \mp i \\ -\sqrt{3} \mp i \end{matrix}\right\} \quad \text{Angle } 30°$$

Properties of $\arg z$:

1. $\arg(z_1.z_2) = \arg z_1 + \arg z_2$

2. $\arg\left(\frac{1}{z}\right) = -\arg z$

3. $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

4. $\arg \bar{z} = -\arg z$

Example: Let $z_1 = i$, $z_2 = -1 + \sqrt{3}\, i$ then find $\arg\left(i\left(1 + \sqrt{3}\, i\right)\right)$

Solution:

$\arg\left(i\left(1 + \sqrt{3}\, i\right)\right) = \arg i + \arg(1 + \sqrt{3}\, i)$

$= \left(\frac{\pi}{2} + 2n\pi\right) + \left(\frac{\pi}{3} + 2m\pi\right)$

$= \frac{5}{6}\,\pi + 2k\pi\,, \quad k = n + m$

Example: Let $z_1 = i$, $z_2 = -1 + \sqrt{3}\, i$

$\arg z_1 = \left(\frac{\pi}{2} + 2n\pi\right)$, $\arg z_2 = \left(\frac{\pi}{3} + 2n\pi\right)$

$\operatorname{Arg} z_1 = \frac{\pi}{2}$, $\operatorname{Arg} z_2 = \frac{\pi}{3}$

$z_1 z_2 = i\left(-1 + \sqrt{3}\, i\right) = -\sqrt{3} - i$

$\arg z_1 z_2 = \pi + \frac{\pi}{6} = \frac{7}{6}\,\pi + 2n\pi$

$\therefore \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) = \frac{7}{6}\pi \notin [-\pi, \pi]$

[7] Powers and Roots

Let $z = re^{i\theta}$ be a nonzero complex number, let n be an integer number then

$$z^n = r^n e^{in\theta}$$

Example: Find $(1+i)^{25}$

Solution: $r = \sqrt{x^2 + y^2} = \sqrt{2}$, $\theta = \frac{\pi}{4}$

$z^{25} = \left(re^{i\theta}\right)^{25}$

$= \left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)^{25}$

$= \left(\sqrt{2}\right)^{25} e^{i\,25.\frac{\pi}{4}}$

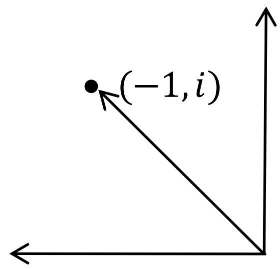
$= 12\sqrt{2}\left(cos\frac{\pi}{4} + i\ sin\frac{\pi}{4}\right)$

$= 12\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$

$= 12(1+i)$

Example: Find $(-1+i)^4$

Solution: $r = \sqrt{2}$, $\theta = \pi - \frac{\pi}{4} = \frac{3}{4}\pi$



$z^n = r^n e^{in\theta} = \left(\sqrt{2}\right)^4 e^{i\,4.\frac{3\pi}{4}}$

$= 4e^{i3\pi}$

$= 4(\cos 3\pi + i \sin 3\pi)$

$= 4(-1+0) = -4$

[8] De Moivre's Theorem

$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

Note: If $z^n = z_0$ then $z = z_0^{\frac{1}{n}}$ and $z = re^{i\theta} = \sqrt[n]{r_0}\, e^{i\left(\frac{\theta_0 + 2k\pi}{n}\right)} = z^{1/n}$ is called nth $-$ root of z.

Example: Calculate root of $z^3 = i$

Solution: $z^3 = i \rightarrow z = (i)^{1/3}$

$$\rightarrow re^{i\theta} = \left(1.\, e^{i\left(\frac{\pi}{2} + 2k\pi\right)}\right)^{1/3}$$

$$\text{s.t } \theta_0 = \frac{\pi}{2} + 2k\pi\,,\ k = 0, \mp 1, \mp 2, \ldots$$

$$\rightarrow re^{i\theta} = e^{i\,\frac{\pi}{6} + \frac{2}{3}k\pi}$$

$$\therefore r = 1\,,\ \theta = \frac{\pi}{6} + 2k\pi,\ k = 0, \mp 1, \mp 2, \ldots$$

To find the roots:

If $k = 0 \rightarrow \ \theta_1 = \frac{\pi}{6}$ (in the first quarter)

$$\rightarrow\ z_1 = 1.\, e^{i\,\frac{\pi}{6}}$$

If $k = 1 \rightarrow\ z_2 = 1.\, e^{i\,\frac{\pi}{6} + \frac{2\pi}{3}}$ (in the second quarter)

$$= \cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi$$

$$= \frac{-\sqrt{3}}{6} + \frac{i}{2}$$

If $k = 2 \rightarrow\ z_3 = 1.\, e^{i\,\frac{\pi}{6} + \frac{4\pi}{3}}$

$\frac{\pi 9}{6}$ nis $i + \frac{\pi 9}{6}$ soc $=$

$= -i$

Note:

1. If the complex number was raised to a fraction whether it was $\frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}$ then

the number of roots is $3, 4, \dots, n$. In the above example the number of roots is 3.

2. $z^n = z_0$ has n different roots only and they are located on the vertices of a

regular polygon centered at the origin.

Example: $z^2 = 1 + i$ has two different roots

Solution:

$$z^2 = 1 + i \rightarrow z = (1 + i)^{1/2}$$

$$r_0 = \sqrt{2}\,,\ \theta_0 = \frac{\pi}{4} + 2n\pi$$

Since $z = (1 + i)^{1/2}$

$$\therefore re^{i\theta} = \left(\sqrt{2}\right)^{\frac{1}{2}} \left(e^{i\,\frac{\pi}{4} + 2n\pi}\right)^{\frac{1}{2}}$$

$$= \sqrt[4]{2}\, e^{i\,\frac{\pi}{8} + n\pi}$$

$$r = \sqrt[4]{2}\,, \quad \theta = \frac{\pi}{8} + k\pi$$

If $k = 0 \rightarrow z_1 = \sqrt[4]{2}\, e^{i\,\frac{\pi}{8}}$

$$= \sqrt[4]{2}\left(\sqrt{\frac{1+\cos\frac{\pi}{8}}{2}} + i\sqrt{\frac{1-\cos\frac{\pi}{8}}{2}}\right)$$

If $k = 1 \rightarrow z_2 = \sqrt[4]{2}\, e^{i\,\frac{\pi}{8} + \pi}$

$$= \sqrt[4]{2}\left(\cos\left(\frac{\pi}{8} + \pi\right) + i\sin\left(\frac{\pi}{8} + \pi\right)\right)$$

$$= \sqrt[4]{2}\left(-\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right)$$

$$= -\sqrt[4]{2}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$$

Note:

$$\cos\frac{\theta}{2} = \mp\sqrt{\frac{1+\cos\theta}{2}}$$

$$\sin\frac{\theta}{2} = \mp\sqrt{\frac{1-\cos\theta}{2}}$$

Note: Let $m, n \neq 0$ be any integer numbers, let z be any complex number then

$$(z)^{m/n} = \left(z^{\frac{1}{n}}\right)^{m} = \left(\sqrt[n]{r_0}\, e^{\left(\frac{i\theta_0+2k\pi}{n}\right)}\right)^{m}$$

$$= \left(\sqrt[n]{r_0}\right)^{m} e^{i\,\frac{m(\theta_0+2k\pi)}{n}},\ k = 0, \mp1, \mp2, \ldots$$

Example: Solve the equation $z^{2/3} - i = 0$

Solution: $z^{2/3} = i \rightarrow z = (i)^{2/3} = \left(i^{1/3}\right)^{2}$

$$= (i)^{1/3}(i)^{1/3}$$

That is each one has three roots.

Let $w = (i)^{1/3} \rightarrow z = w^2$

Now, we find the roots of w

$$r_0 = 1\ ,\theta_0 = \frac{\pi}{2} + 2k\pi\ , k = 0, \mp1, \mp2, \ldots$$

$$w = re^{i\theta} = 1.\left(e^{i\,\frac{\pi}{2}+2k\pi}\right)^{1/3}$$

$$= e^{i\,\frac{\pi}{6}+\frac{2k\pi}{3}}$$

$$\therefore w_1 = e^{i\,\frac{\pi}{6}} = \cos\left(\frac{\pi}{6} + i\sin\frac{\pi}{6}\right), k = 0$$

$$w_2 = e^{i\,\frac{\pi}{6}+\frac{2\pi}{3}} = e^{i\,\frac{5\pi}{6}}\ , k = 1$$

$$w_3 = e^{i\,\frac{\pi}{6}+\frac{4\pi}{3}} = e^{i\,\frac{3\pi}{2}}\ , k = 2$$

$\therefore z = w^2$

$\therefore z_1 = (w_1)^2 = \left(e^{i\frac{\pi}{6}}\right)^2 = e^{i\frac{\pi}{3}}$

$= cos\frac{\pi}{3} + i\ sin\frac{\pi}{3}$

$= \frac{1}{2} + i\ \frac{\sqrt{3}}{2}$

$z_2 = (w_2)^2 = \left(e^{i\frac{5\pi}{6}}\right)^2 = e^{i\frac{5\pi}{3}}$

$$= \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$$

$$z_3 = (w_3)^2 = \left(e^{i\frac{3\pi}{2}}\right)^2 = e^{i3\pi} = \cos 3\pi + i \sin 3\pi$$

Exercises: solve the equation $z^3 + 1 = 0$

[9] Regions in the Complex Plane

Some definitions and concepts:

Definition: Let z be any point in the z-plane, let $\epsilon > 0$ then

1. $N_\epsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| < \epsilon\}$

This set is called a neighborhood of z_0.

2. $S_\epsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| = \epsilon\}$

This set is called sphere with center z_0.

3. $D_\epsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq \epsilon\}$

This set is called the Disk with center z_0 and radius ϵ.

Definition: Let $U \subseteq \mathbb{C}$, we say that U is open set if

$$\forall\, w \in U, \exists\, N_\epsilon(w) \text{ s.t } N_\epsilon(w) \subseteq U.$$

For example: $\emptyset$, $\mathbb{C}$ are open sets.

Definition: Let $F \subseteq \mathbb{C}$, we say that F is closed set if $\mathbb{C} - F$ is open set.

Definition: An open set $S \subseteq \mathbb{C}$ is connected if each pair of points z_1, z_2 in it can be joined by a polygon line, consisting of a finite number of line segments joined end to

end that lies entirely in S.

Definition: Let $S \subseteq \mathbb{C}$, we say that S is Region if it is open and connected.

Example:

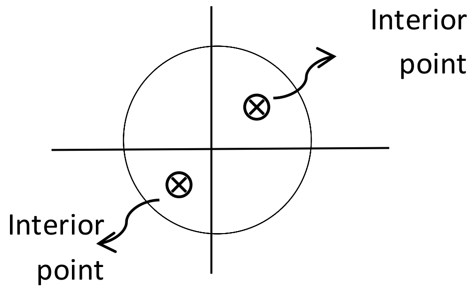
1. $|z| > 1, |z| < 1$ is Region.

2. Let $|z| = 0$ is not Region, since it is connected but not open set.

3. $\mathbb{R} \subset \mathbb{C}$ is connected but not open, since $\forall\, r \in \mathbb{R}, \exists\, N_\epsilon(r)$ contain some of complex points.

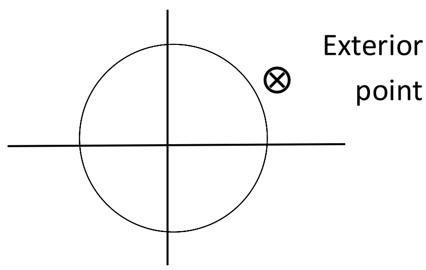
Definition: Let $z_0 \in S$, we say that z_0 is interior point if there exist a neighborhood $N_\epsilon(z_0)$ s.t $N_\epsilon(z_0) \subseteq S$.

Example: $|z| < 1$



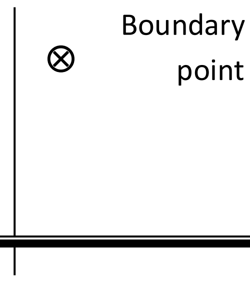
Definition: Let $z_0 \in S$, we say that z_0 is exterior point if there exist a neighborhood $N_\epsilon(z_0)$ s.t $N_\epsilon(z_0) \cap S = \emptyset$.

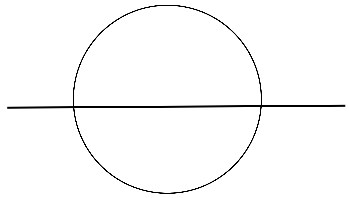
Example: $|z| > 1$



Definition: Let $z_0 \in S$, we say that z_0 is Boundary point if $\forall\, N_\epsilon(z_0)$ contain points

from inside S and outside it.

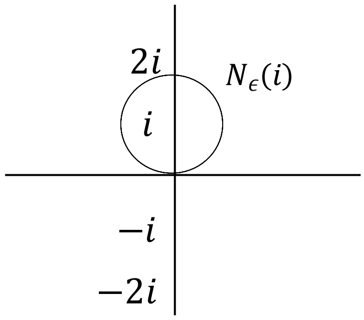




Note: S is close set iff it contains all the boundary points.

Example: $S = \{ \mp i, \mp 2i\}$, is S open set ?

Note $N_\epsilon(i) \nsubseteq S$, therefore S is not open.

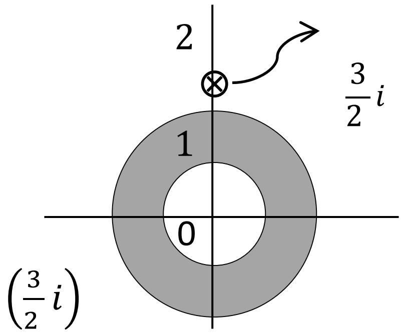


Example: $S = \{z \in \mathbb{C} : 1 < |z| < 2\}$

0 is exterior point of S

1, 2 are boundary points of S

is interior point of S

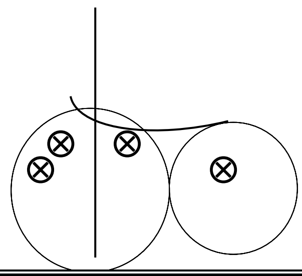


Example: $D = \{z \in \mathbb{C} : 2 < |z| \leq 3\}$

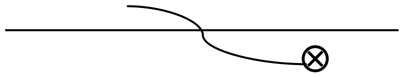
is not open set since it contain all the boundary points. D

Example: $S = \{z \in \mathbb{C} : |z| < 1\} \cup \{z \in \mathbb{C} : |z - 2| \leq 1\}$

S is connected set.



But if



, $S = \{z \in \mathbb{C} : |z| < 1\} \cup \{z \in \mathbb{C} : |z - 2| < 1\}$

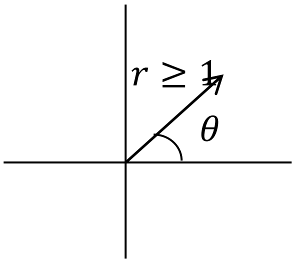
Then S is not a connected set.

Definition: Let $S \subseteq \mathbb{C}$, we say that S is bounded set if $\exists$ Disk D,

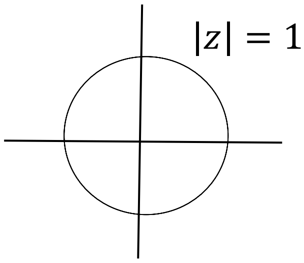
such that $S \subseteq D$.$D = \{z : |z| \leq \mathbb{R}\}$

Example: $S = \left\{z \in \mathbb{C} : r \geq 1,\ 0 \leq \theta \leq \frac{\pi}{4}\right\}$

is not bounded set since $\nexists$ Disk contain S.S



Example: $|z| = 1$ is bounded set



Example: $S = \{\mp i, \mp 2i\}$

1. S is not open set since every point of S is boundary point.

2. S is close set since every point of S is boundary point.

3. S is not connected set.

4. S is not bounded set.

Definition: Let $z_0 \in S$, we say that z_0 is limit point if

$$N_\epsilon(z_0) \cap (S - z_0) \neq \emptyset$$

Example: $S = \left\{z \in \mathbb{C} : z = \frac{1}{n},\ n = 1, 2, \ldots\right\}$, 0 is the only limit point