Chapter -2-

"Methods to solve O.D.E. of the first order"

Introduction

In this chapter we will studied a solution for O.D.E. of the first order.

The general form of O.D.E. of the first order and degree is:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

$$or \quad M(x,y)dx + N(x,y)dy = 0$$

$$or \quad \frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)} = f(x,y)$$

Ex.

$$1-(y-x)+x^2\frac{dy}{dx} = 0$$

$$2-(y-x)+xy\frac{dy}{dx} = 0$$

$$3-(x^2y+2x)dx+(3x-cosx)dy$$

$$4-\frac{dy}{dx} = \frac{y}{x}+2sinx$$

Remark

1- There is no rule to solve all O.D.E.

2- We will be solving the following equations:

1- Equations with separated and separable variables

2- Exact equations and integrating factors

3- Linear equations

4- Bernoulli's Equation

1- Equations with separated and separable variables

Definition

A first order O.D.E. of the form:

$$\frac{dy}{dx} = f_1(x)f_2(y)$$

Is said to be separable or to have separable variables

Ex.

1- $\frac{dy}{dx} = y^2xe^{3x+4y}$ Separable variables

2- $\frac{dy}{dx} = y + sinx$ not Separable variables

Ex. Solve the following equations:

1- $\boldsymbol{xydy + (2x^2 - 1)(y + 2)dx = 0}$

Solution

$$\frac{y}{y+2}dy + \frac{2x^2 - 1}{x}dx = 0$$

$$\frac{y + 2 - 2}{y+2}dy + [2x - \frac{1}{x}]dx = 0$$

$$[1 - \frac{2}{y+2}]dy + [2x - \frac{1}{x}]dx = 0$$

$$y - 2\ln(y + 2) + x^2 - lnx + c = 0$$

2- $\boldsymbol{xdx + ydy = 0}$

Solution

$$\frac{x^2}{2} + \frac{y^2}{2} = \frac{c_1}{2}$$

$x^2 + y^2 = c$ (circles with center at the origin)

3- $\frac{dy}{dx} = -\frac{x}{y}$, $y(4) = -3$

Solution

ydy=-xdx

$$\frac{y^2}{2} = \frac{x^2}{2} + \frac{c_1}{2}$$

$$x^2 + y^2 = c$$

16+9=c $\rightarrow \therefore c = 25$

$$x^2 + y^2 = 25$$

$$y = \mp\sqrt{25 - x^2}$$

$$y = \varphi_1(x) = \sqrt{25 - x^2} \;, \qquad y = \varphi_2(x) = -\sqrt{25 - x^2}$$

Exercises

Solve the following:

1- $(1 + x)dy - ydx = 0$

2- $(e^{2y} - y)cosx\frac{dy}{dx} = e^y sin2x$, $y(0) = 0$

2- Exact O.D.E.

Exact differential:- The exact differential for f(x,y) in x,y has the form:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Exact O.D.E. The differential equation M(x,y)dx+N(x,y)dy=0 is called exact if M(x,y)dx+N(x,y)dy is exact differential for a function f(x,y)

i.e. O.D.E.is called exact if there exist a function f(x,y) s.t.

$$\frac{\partial f}{\partial x} = M \;\; and \;\; \frac{\partial f}{\partial y} = N$$

$$\therefore df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Mdx + Ndy$$

Then we will get the solution of exact O.D.E. by integrating df=0

f(x,y)=c is a solution of exact O.D.E.∴

<u>Def.:-</u> A differential expression Mdx+Ndy is an exact differential in a region R of the xy-plane if it corresponds to the differential of some function f(x,y) defined in R;A first differential equation Mdx+Ndy=0 is said to be an exact equation if the expression on the left –hand side is an exact differential.

<u>Theorem:-</u> Let M(x,y) and N(x,y) be continuous and have continuous first partial derivative in a rectangular R defined by $a < x < b, c < y < d$.

Then a necessary and sufficient condition that Mdx+Ndy be an exact differential is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ [$\text{O.D.E. Mdx} + \text{Ndy} = 0$ is exact iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$]

<u>Ex.:-</u> Solve the following equations

(1) ydx+xdy=0

<u>Solution</u>

$$\frac{\partial M}{\partial y} = 1 \quad , \frac{\partial \text{N}}{\partial \text{x}} = 1$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial \text{N}}{\partial \text{x}}$$

The O.D.E. is exact∴

$\exists f(x,y) \ s.t. df = \frac{\partial f}{\partial x}\text{dx} + \frac{\partial \text{f}}{\partial \text{y}}\text{dy} = \text{Mdx} + \text{Ndy}$∴

$\frac{\partial f}{\partial x} = \text{M} = \text{y}, \frac{\partial \text{f}}{\partial \text{y}} = \text{N} = \text{x}$∴

Integrating $\frac{\partial f}{\partial x}$ $w.r.t.x$

$$\int_x \frac{\partial f}{\partial x}dx = \int_x ydx + g(y) \qquad (g(y) is\ arbitrary\ function)$$

f(x,y)=xy+g(y)

$$\frac{\partial f}{\partial y} = x + g'(y)$$

$$\frac{\partial f}{\partial y} = N = x$$

$x + g'(y) = x$∴

$$\therefore g'(y) = 0$$

$$integrating\ g'(y) w.r.t.y$$

$$\therefore g(y) = 0$$

f(x,y)=xy∴

The solution of O.D.E. is f(x,y)=c

xy=c is the general solution ∴

(2) $\boldsymbol{(6x^2 + 4xy + y^2)dx + (2x^2 + 2xy - 3y^2)dy = 0}$

<u>**Solution**</u>

$$\frac{\partial M}{\partial y} = 4x + 2y \quad , \frac{\partial \mathrm{N}}{\partial \mathrm{x}} = 4\mathrm{x} + 2\mathrm{y}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial \mathrm{N}}{\partial \mathrm{x}}$$

The O.D.E. is exact∴

$\exists f(x,y)\ s.t. df = \frac{\partial f}{\partial x}\mathrm{dx} + \frac{\partial \mathrm{f}}{\partial \mathrm{y}}\mathrm{dy} = \mathrm{Mdx} + \mathrm{Ndy}$∴

$\frac{\partial f}{\partial x} = \mathrm{M} = 6\mathrm{x}^2 + 4\mathrm{xy} + \mathrm{y}^2$, $\frac{\partial \mathrm{f}}{\partial \mathrm{y}} = \mathrm{N} = 2\mathrm{x}^2 + 2\mathrm{xy} - 3\mathrm{y}^2$∴

Integrating $\frac{\partial f}{\partial x}\ w.r.t.x$

$$f(x,y) = \int_x \frac{\partial f}{\partial x} dx = \int_x (6\mathrm{x}^2 + 4\mathrm{xy} + \mathrm{y}^2) dx + g(y)$$

$$= 2\mathrm{x}^3 + 2\mathrm{x}^2\mathrm{y} + \mathrm{xy}^2 + g(y) \quad (g(y) is\ arbitrary\ function)$$

$$\frac{\partial f}{\partial y} = 2x^2 + 2xy + g'(y) = N = 2x^2 + 2xy - 3y^2$$

$$\therefore g'(y) = -3y^2$$

integrating $g'(y)$ *w.r.t.* y

$$\therefore g(y) = -y^3$$

the general solution of given equation is∴

f(x,y)=c

$$\therefore 2x^3 + 2x^2y + xy^2 - y^3 = c$$

Exercises

(1) are the following equations exact?

1) $(y^2 - x)dx + (x^2 - y)dy = 0$

2) xcosydx+ycosxdy=0

(2) solve the following equations:-

1)$2xydx + (x^2 - 1)dy = 0$

2) $(3x^2 + 2y\sin 2x)dx + (2\sin^2 x + 3y^2)dy = 0$

3) $\left(3x^2 + \frac{2y}{x}\right)dx + \left(2\ln 3x + \frac{3}{y}\right)dy = 0 , x > 0 , y \neq 0$