Chapter Two

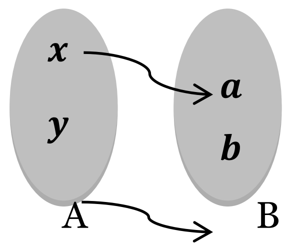
Analytic Functions

[1] Functions of a Complex Variable

Definition:

A function f defined on a set A to a set B is a rule assigns a unique element of B to each element of A; in this case we call f a single function.
i.e.: $f: A \to B$, $A, B \subseteq \mathbb{C}$

$\forall z \in A, \exists!\, w \in B$ s.t $w = f(z) \in B$



Definition:

The domain of f in the above def. is A and the range is the set R of elements of B

Which f associate with elements of A.

Note: The elements in the domain of f are called independent variables and those

In the range of f are called dependent variables.

Definition:

A f rule which assigns more than one number of B to any number of A is called a multiple valued function.

Example:

1. $f(z) = (z)^{1/2}$

Has two roots therefore $f(z)$ is a multiple function.

2. $f(z) = (z)^{3/5} = (z^3)^{1/5}$

Has five roots therefore $f(z)$ is a multiple function. In general, if

$f(z) = \arg z$ then f is a multiple function.

3. If $f(z) = \operatorname{Arg} z$ then f is a single function.

Note:

1. Let $f: Z \to W$, if Z and W are complex, then f is called complex variables

2. Function (a complex function) or a complex valued function of a complex

variable.

3. If A is a set of complex numbers and B is a set of real numbers then f is

Called real−valued function of a complex variable, conversely f is a

Complex−valued function of real variables.

Example: Find the domain of the following functions

1. $\boldsymbol{f(z) = \frac{1}{z}}$

Ans.: $D_f = \mathbb{C} \setminus \{0\}$

2. $\boldsymbol{f(z) = \frac{1}{z^2+1}}$

Ans.: $D_f = \mathbb{C} \setminus \{-i, i\}$

3. $\boldsymbol{f(z) = \frac{z+\bar{z}}{2}}$

Ans.: $D_f = \mathbb{C}$, f is real−valued.

Definition: A complex function $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$

is a positive integer and $a_0, a_1 \ldots a_n \in \mathbb{C}$, is a polynomial of degree n n $(a_n \neq 0)$.

Definition: A function $f(z) = \frac{P(z)}{Q(z)}$, where P and Q are two polynomials, is called a rational function.

Note: $D_f = \mathbb{C} \setminus \{z : Q(z) \neq 0\}$

♦ Suppose that: $w = u + iv$ is the value of a function f at $z = x + iy$

i.e.: $f(z) = f(x + iy) = u + i\,v$

Each of the real numbers u and v depends on the real variables x and y, and it

Follows that $f(z)$ can be expressed in terms of a pair of real−valued functions of real variables x and y.

$$f(z) = u(x, y) + i\, v(x, y)$$

If the polar coordinates r and θ are used instead of x and y, then

$$u + i\, v = f(re^{i\theta})$$

Where $w = u + iv$ and $z = re^{i\theta}$, in that case, we may write

$$f(z) = u(r, \theta) + i\, v(r, \theta)$$

Example: If $f(z) = z^2$, then

$$f(x + iy) = (x + iy)^2 = x^2 - y^2 + i\, 2xy$$

Hence: $u(x, y) = x^2 - y^2$, $v(x, y) = 2xy$, when polar coordinates are used

$$f\left(re^{i\theta}\right) = \left(re^{i\theta}\right)^2$$

$$= r^2 e^{i2\theta}$$

$$= r^2 \cos 2\theta + i\, r^2 \sin 2\theta$$

Therefore: $u(r, \theta) = r^2 \cos 2\theta$

$$v(r, \theta) = r^2 \sin 2\theta$$

Note: If $v(x, y) = 0$ then f is real, i.e. f is real−valued function.

[2] Limits

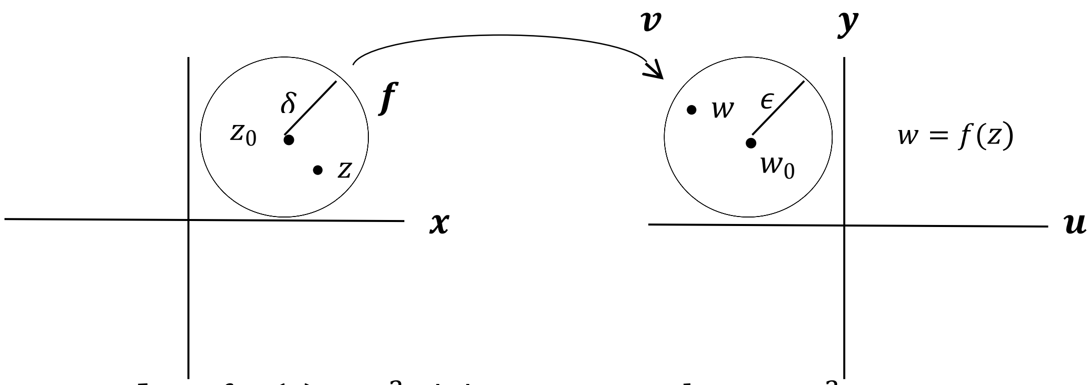
Let f be a function at all points z in some deleted neighborhood of z_0, the statement that the limit of $f(z)$ as z approaches z_0 is a number w_0, or that $\lim_{z \to z_0} f(z) = w_0$

Means that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(z) - f(z_0)| < \epsilon \quad \text{whenever} \quad |z - z_0| < \delta$$

And this means: $z \to z_0$ in $z -$ plane

$w \to w_0$ in $w -$ plane



Example: If $f(z) = z^2,\ |z| < 1,$ prove that $\lim_{z\to 1} z^2 = 1$

Proof: Let $\epsilon > 0$, T.p. $\exists\ \delta > 0$ s.t $|z^2 - 1| < \epsilon$ whenever $0 < |z - 1| < \delta$

$|z^2 - 1| = |z + 1||z - 1| \le (|z| + 1)|z - 1|$

$< 2|z - 1| < \epsilon$

$= |z - 1| < \frac{\epsilon}{2}$

$\therefore$ chose $\delta = \frac{\epsilon}{2}$

$\therefore \lim_{z\to 1} z^2 = 1$

Properties of Limit:

1. If $f(z) = c$ then $\lim_{z\to z_0} f(z) = c$.

2. If $f(z) = z$ then $\lim_{z\to z_0} f(z) = z_0$.

3. $\lim_{z\to z_0}\big(f(z) \mp g(z)\big) = \lim_{z\to z_0} f(z) \mp \lim_{z\to z_0} g(z)$.

4. $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z\to z_0} f(z)}{\lim_{z\to z_0} g(z)}$

5. $\lim_{z\to z_0} f(z).g(z) = \lim_{z\to z_0} f(z).\lim_{z\to z_0} g(z)$

Example: Find limit $f(z)$ if its exist, such that $f(z) = \frac{2xy}{x^2+y^2} + \frac{x^2}{1+y}\ i$

Proof: Assume that limit $f(z)$ exists.

Let $y = 0$, we get

$$\lim_{z\to z_0=0} f(z) = \lim_{(x,y)\to(0,0)} f(z) = \lim_{x\to 0} x^2 i = 0$$

Let $x = 0$, we get $\lim f(z) = 0$

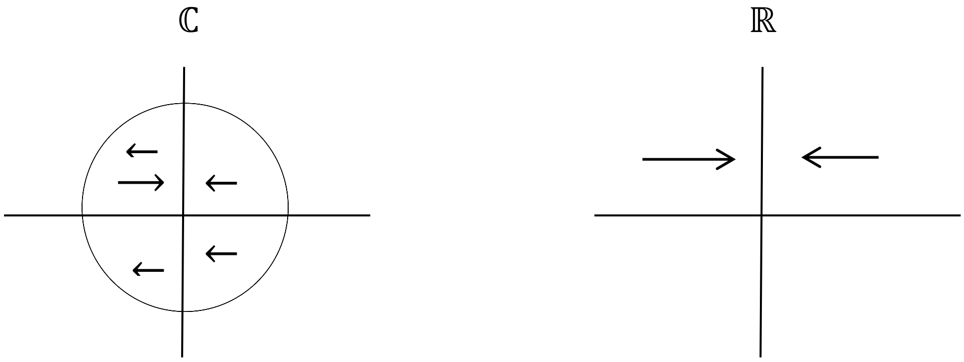
Let $y = x$, then $\lim\limits_{z\to 0} f(z) = \lim\limits_{(x,x)\to(0,0)} f(z) = \lim\limits_{(x,x)\to(0,0)} \left(\frac{2x^2}{2x^2} + \frac{x^2}{1+x} i\right)$

$$\lim_{(x,x)\to(0,0)} \left(1 + \frac{x^2}{1+x} i\right) = 1 + \lim_{(x,x)\to(0,0)} \frac{x^2}{1+x} i = 1 + 0 = 1$$

This is impossible; therefore this limit does not exist.

Exercise: Prove that $\lim\limits_{z\to z_0} z^2 = z_0^2$

Note: The limit in the real numbers is studying the approaches from the right and left, but in the complex numbers is studying from every side of the circle.



Theorem: If $\lim\limits_{z\to z_0} f(z) = w_1$, then $\lim\limits_{z\to z_0} f(z) = w_2$

Then $w_1 = w_2$. (The limit is unique)

Theorem: Let $f(z) = u(x,y) + iv(x,y)$ such that $z = x + iy$,

Then $z_0 = x_0 + y_0$, $w_0 = u_0 + iv_0$,

$$\lim_{z\to z_0} f(z) = w_0 \text{ iff } \lim_{z\to z_0} u(x,y) = u_0, \lim_{z\to z_0} v(x,y) = v_0$$

Note: $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ s.t $a_i \in \mathbb{C}, i = 0,1,\ldots,n$

Then $\lim\limits_{z\to z_0} p(z) = p(z_0)$

Example: Find limit of $f(z)$ if it's exist

1. $\lim\limits_{z\to 3-4i} \frac{4x^2y^2 - 1 + i(x^2 - y^2) - ix}{\sqrt{x^2+y^2}}$

Solution:

$$\lim_{z\to 3-4i}\frac{(4x^2y^2-1)+i(x^2-y^2-x)}{\sqrt{x^2+y^2}}=$$

$$=115\text{-}2i = \lim_{z\to 3-4i}\frac{4x^2y^2-1}{\sqrt{x^2+y^2}} + i\lim_{z\to 3-4i}\frac{x^2-y^2-x}{\sqrt{x^2+y^2}}$$

2. $\lim_{z\to i}\frac{z-i}{z^2+1}$

Solution:

$$\lim_{z\to i}\frac{z-i}{z^2+1} = \lim_{z\to i}\frac{z-i}{z^2-(-1)} = \lim_{z\to i}\frac{z-i}{z^2-i^2} = \lim_{z\to i}\frac{z-i}{(z-i)(z+i)}$$

$$= \lim_{z\to i}\frac{1}{(z+i)} = \frac{1}{2i}$$

3. $\lim_{z\to(-1+i)}\frac{z^2+(3-i)\,z+2-2i}{z+1-i}$

Solution:

Note: $z^2+(3-i)\,z+2-2i=(z+1-i)(z+2)$

$$\therefore \lim_{z\to(-1+i)}\frac{z^2+(3-i)\,z+2-2i}{z+1-i} = \lim_{z\to(-1+i)}\frac{(z+1-i)(z+2)}{(z+1-i)}$$

$$= \lim_{z\to(-1+i)}(z+2)$$

$$= -1+i+2$$

$$= 1+i$$

[3] Continuity

Definition:

A function f is continuous at a point z_0 if all of the three following conditions are satisfied:

1. $\lim_{z\to z_0} f(z)$ Exists,

2. $f(z_0)$ Exists,

3. $\lim_{z\to z_0} f(z) = f(z_0)$

A function of a complex variable is said to be continuous in a region R if it is continuous at each point R.

Theorem: If f, g are continuous functions at z_0 then

1. $f + g$ is continuous.

2. $f . g$ is continuous.

3. $\frac{f}{g}$, $g(z_0) \neq 0$ is continuous.

4. fog is continuous at z_0 if f is continuous at $g(z_0)$.

Example: $f(z) = z^2$ is continuous in complex plane since $\forall\, z_0 \in \mathbb{C}$

1. $f(z_0) = z_0^2$

2. $\lim_{z \to z_0} f(z) = z_0^2$

3. $\lim_{z \to z_0} f(z) = f(z_0)$

Example: Is $f(z) = \frac{z^2-1}{z-1}$ continuous at $z = 1$?

Solution: f is not continuous since $f(1)$ not exist

$$f(z_0) = \frac{z_0^2-1}{z_0-1} = \frac{(z_0-1)(z_0+1)}{z_0-1} = z_0 + 1$$

$\therefore \lim_{z \to 1} f(z) = 2$

But $f(1) = \frac{0}{0}$

$\therefore \lim_{z \to 1} f(z) \neq f(1)$

Theorem: $f(z) = u(x,y) + iv(x,y)$ is continuous at z_0 iff $u(x,y)$ and $v(x,y)$ are continuous at (x_0, y_0).

Proof: Let f be continuous at z_0 , then $\lim_{z \to z_0} f(z) = f(z_0)$

That means:

$$\lim_{z \to z_0} (u(x,y) + iv(x,y)) = u(x_0,y_0) + i\, v(x_0,y_0)$$

$$\to \lim_{z \to z_0} u(x,y) + i \lim_{z \to z_0} v(x,y) = u(x_0,y_0) + i\, v(x_0,y_0)$$

$$\therefore \lim_{z \to z_0} u(x,y) = u(x_0,y_0)$$

$$\lim_{z\to z_0} v(x,y) = v(x_0,y_0)$$

are continuous at z_0. $\therefore u, v$

Example: Is $f(x+iy) = x^2 + y^2 + ixy$ continuous at $(1,1)$?

Solution: $u(x,y) = x^2 + y^2,\ v(x,y) = xy$

By the above theorem

$$u(1,1) = 2\,, \qquad \lim_{\substack{x\to 1\\ y\to 1}} u(x,y) = 2 = u(1,1)$$

$$v(1,1) = 1\,, \qquad \lim_{\substack{x\to 1\\ y\to 1}} v(x,y) = 1 = v(1,1)$$

are continuous at $(1,1)\therefore u, v$

is continuous at $(1,1).\therefore f(z)$

Example: Find the limit if it's exists $\lim_{z\to 0}\frac{\bar{z}}{z}$

Solution:

$$\lim_{z\to 0}\frac{\bar{z}}{z} = \lim_{z\to 0}\frac{x-iy}{x+iy}$$

1. If $y = 0 \to \lim_{x\to 0}\frac{x}{x} = 1$

2. If $x = 0 \to \lim_{y\to 0}\frac{-iy}{iy} = -1$

The limit is not exist.$\therefore$

Example: Discuss the continuity of $f(z) = \begin{cases} \frac{z-i}{z^2-1} & if\ \ z\neq i,-i \\ 2i & if\ \ z=\mp i \end{cases}$

Solution: Note f is not continuous at $z = \mp i$.

(Since $f(\mp i)$ is undefined)

$$f(z) = 2i \ \text{ and } \lim_{z\to -i} f(z) = \lim_{z\to -i}\frac{z-i}{(z-i)(z+i)} = \lim_{z\to -i}\frac{1}{(z+i)} = \frac{1}{2i}$$

But f is not defined at $z = -i$, therefore f is not continuous at $z = i$, that is f is continuous at $\{z \in \mathbb{C}\backslash\{-i, i\}\}$

Example: Discuss the continuity of $f(z) = \begin{cases} \frac{z^2+4}{z+2i} & if\ \ z\neq -2i \\ -4i & if\ \ z=\mp i \end{cases}$

Solution: f is continuous at $\forall\, z \neq -2i$.

When $z = -2i$

$$\lim_{z\to -2i} f(z) = f(-2i) = -4i$$

$$\lim_{z\to -2i} f(z) = \lim_{z\to -2i} \frac{(z-2i)(z+2i)}{(z+2i)} = -4i$$

But f is not defined at $z = -2i$

is not continuous at $z = -2i. \therefore f$

Then is f is continuous at $\{z \in \mathbb{C} : z \neq -2i\}$

Exercise: Discuss the continuity of $f(z) = \begin{cases} \frac{z+2i}{z^2+4} & if \;\; z \neq \mp 2i \\ \frac{1}{4} i & if \;\; z = -2i \end{cases}$

[4] Derivative

Let f be a function whose domain of definition contains a neighborhood $|z - z_0| < \epsilon$ of a point z_0. The derivative of f at z_0 is the limit

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

And the function f is said to be differentiable at z_0 when $f'(z_0)$ exists. If $\Delta z = z - z_0$, then $\Delta z \to 0$ when $z \to z_0$. Thus

$$f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta\, z}$$

Theorem: If f is differentiable at z_0, then f is continuous at z_0.

Differentiation Formulas:

In the following formulas, the derivative of a function f at a point z_0 is denoted by either $\frac{d}{dz} f(z)$ or $f'(z_0)$.

1. $\frac{d}{dz}\, c = 0$, c is constant

2. $\frac{d}{dz}\, z = 1$

3. $\frac{d}{dz}\left(c\, f(z)\right) = c\, f'(z)$

4. $\frac{d}{dz}[f+g] = \frac{d}{dz}f + \frac{d}{dz}g = f' + g'$

5. $\frac{d}{dz}[f.g] = f.g' + g.f'$

6. $\frac{d}{dz}\left[\frac{f}{g}\right] = \frac{g.f' - f.g'}{g^2}$, $g \neq 0$

7. $\frac{d}{dz}(z^n) = n\,z^{n-1}$

8. $(gof)'(z_0) = g'(f(z_0))\,.\,f'(z_0)$

Note: (The Chain rule) If $w = f(z)$ and $W = g(w)$, then $\frac{dW}{dz} = \frac{dW}{dw}\,.\,\frac{dw}{dz}$

Example: Find the derivative of $f(z) = (2z^2 + i)^5$

***Solution*:** write $w = 2z^2 + i$ and $W = w^5$ *then*:

$$\frac{d}{dz}(2z^2 + i)^5 = 5w^4.4z = 20\,z(2z^2 + i)^4$$

Examples: Find $f'(z)$ *where* $f(z) = z^2$

Solution:

$f'(z) = 2z$

[5] Cauchy – Riemann Equations (C-R-E)

Theorem: Suppose that $f(z) = u(x, y) + iv(x, y)$ and $f\,'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the first-order partial derivatives of u and v must exist at (x_0, y_0), and they must satisfy the Cauchy-Riemann equations

$$u_x = v_y\,, \qquad u_y = -v_x$$

There is also $f'(z_0) = u_x + iv_x$

Where these partial derivatives are to be evaluated at (x_0, y_0).

Note:

1. $f'(z) = u_x + iv_x$ or $f'(z) = u_y - iv_y$.
2. If $f'(z)$ exists then C-R-Eq. are satisfied, but the converse is not true.

The converse of the above theorem is not necessary true.

Example: $f(z) = z^2 = x^2 - y^2 + 2\,ixy$

Solution:

$u(x,y) = x^2 - y^2 \rightarrow\ u_x = 2\text{x}$

$v(x,y) = 2xy \qquad \rightarrow\ v_y = 2x$

$\rightarrow u_x = v_y$

$u_y = -2\text{y}\ ,\ \ v_x = 2y$

$\rightarrow u_y = -v_x$

$\therefore f'(\text{z}) = u_x + iv_x = 2x + i2y = 2(x+iy) = 2z$

Example: $f(z) = \bar{z} = x - iy$

Solution:

$u(x,y) = x \quad \rightarrow\ u_x = 1$

$v(x,y) = -y \rightarrow\ v_y = -1$

$$\therefore u_x \neq v_y \rightarrow f \text{ is not differentiable at } z.$$

Note: The following theorem gives a necessary and sufficient condition to satisfy the converse of the previous theorem.

Theorem: Let $f(z) = u(x,y) + iv(x,y)$, and

1. $u,\ v,\ u_x,\ v_x,\ u_y,\ v_y$ are continuous at $N_\epsilon(z_0)$

2. $u_x = v_y\ ,\ u_y = -v_x$

Then f is differentiable at z_0 and

$$f'(z_0) = u_x + iv_x$$

$$f'(z_0) = v_y - iu_y$$

Example: Show that the function $f(z) = e^{-y}\cos x + i\, e^{-y}\sin x$

Is differentiable z for all and find its derivative.

Solution:

Let $u(x,y) = e^{-y}\cos x$

$\rightarrow u_x = -e^{-y}\sin x$

$u_y = -e^{-y}\cos x$

$v(x,y) = e^{-y}\sin x$

$\rightarrow v_x = e^{-y}\cos x$

$v_y = -e^{-y}\sin x$

1. $u_x = v_y$ and $u_y = -v_x$

2. $u,\ v,\ u_x,\ v_x,\ u_y,\ v_y$ are continuous

Then $f'(z)$ exist. To find $f'(z) = u_x + iv_x$

$f'(z) = u_x + iv_x = -e^{-y}\sin x + ie^{-y}\cos x$

$= e^{-y}(i\cos x - \sin x)$

$= ie^{-y}(\cos x + i\sin x)$

$= ie^{-y}e^{ix}$

$= ie^{i\,x-y}$

$= ie^{i(x+iy)}$

$= ie^{iz}$

[6] Polar Coordinates of Cauchy – Riemann Equations

Let $f(z) = u(r,\theta) + iv(r,\theta)$, then Cauchy-Riemann equations are:

and $f'(z_0) = e^{-i\theta}(u_r + i\,v_r).u_r = \frac{1}{r}v_\theta\ ,\ u_\theta = -r\,v_r$

Example: Use C-R equations to show that the functions

1. $\boldsymbol{f(z) = |z|^2}$

2. $\boldsymbol{f(z) = z - \bar{z}}$

are not differentiable at any nonzero point.

***Solution*:**

1. $|z|^2 = x^2 + y^2$

$u(x, y) = x^2 + y^2$, $v(x, y) = 0$

$u_x = 2x$, $v_x = 0$

$u_y = 2y$, $v_y = 2x$

C-R equations are not satisfied, therefore f' is not exist.

2. $z - \bar{z} = (x + iy) - (x - iy)$

$= x + iy - x + iy$

$= 2y\, i$

$u(x, y) = 0$, $v(x, y) = 2y$

$u_x = 0$, $v_x = 0$

$u_y = 0$, $v_y = 2$

C-R equations are not satisfied, hence f' is not exist.

<u>Example</u>: Use C-R equations to show that $f'(z)$ and $f''(z)$ are exist everywhere $f(z) = z^3$

***<u>Solution</u>*:**

$f(z) = z^3 = (x + iy)^3$

$= x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3$

$= x^3 + 3i\, x^2 y - 3xy^2 - iy^3$

$= x^3 - 3xy^2 + i\,(3x^2 y - y^3)$

$u(x, y) = x^3 - 3xy^2 \rightarrow u_x = 3x^2 - 3y^2$

$u_y = -6xy$

$v(x, y) = 3x^2 y - y^3 \rightarrow v_x = 6xy$

$v_y = 3x^2 - 3y^2$

$\therefore u_x = v_y$, $u_y = -v_x$

C-R equations are satisfied $\therefore$

$f'(z) = u_x + iv_x$

$$= 3x^2 - 3y^2 + i\,6xy$$

$$= 3(x^2 + i^2y^2 + 2i\,xy) = 3(x+iy)^2 = 3z^2$$

$$f''(z) = u'_x + iv'_x$$

$$= 6x + i\,6y$$

$$= 6(x+iy)$$

$$= 6z$$

<u>Example:</u> Let $f(z) = z^3$, write f in polar form and then find $f'(z)$

<u>*Solution*</u>: $f(z) = z^3 = \left(re^{i\theta}\right)^3 = r^3e^{3i\theta}$

$$= r^3\cos 3\theta + i\,r^3\sin 3\theta$$

$$u(r,\theta) = r^3\cos 3\theta \rightarrow u_r = 3r^2\cos 3\theta$$

$$u_\theta = -3r^3\sin 3\theta$$

$$v(r,\theta) = r^3\sin 3\theta \rightarrow v_r = 3r^2\sin 3\theta$$

$$v_\theta = 3r^3\cos 3\theta$$

Now, $u_r = \frac{1}{r}\,v_\theta$, $u_\theta = -rv_r$

$$f'(z) = e^{-i\theta}[u_r + i\,v_r]$$

$$= e^{-i\theta}[3r^2\cos 3\theta + i3r^2\sin 3\theta]$$

$$= 3r^2e^{-i\theta}[\cos 3\theta + i\sin 3\theta]$$

$$= 3r^2e^{-i\theta}e^{3\theta i}$$

<u>[7] Analytic Functions</u>

<u>Definition:</u>

A function f is said to be analytic at z_0 if $f'(z_0)$ exists and $f'(z)$ exists at each point z in the same neighborhood of z_0.

<u>Note:</u> f is analytic in a region R if it is analytic at every point in R.

<u>Definition:</u>

If f is analytic at each point in the entire plane, then we say that f is an entire function.

Example: $f(z) = z^2$, is an entire function since it is a polynomial.

Definition: If f is analytic at every point in the same neighborhood of z_0 but f is not analytic at z_0, then z_0 is called singular point

Example: Let $f(z) = \frac{1}{z}$, then $f'(z) = \frac{-1}{z^2}$ $(z \neq 0)$

Then f is not analytic at $z_0 = 0$, which is a singular point.

Note: If f is analytic in D, then f is continuous through D and C-R equations are satisfied.

Note: A sufficient conditions that f be analytic in $\mathbb{R}$ are that C-R equations are satisfied and $u_x,\ v_x,\ u_y,\ v_y$ are continuous in $\mathbb{R}$.

[8] Harmonic Functions

Definition:

A function h of two variables x and y is said to be harmonic in D if the first partial derivatives are continuous in D and $h_{xx} + h_{yy} = 0$ (Laplace equation)

Example: Show that $u(x, y) = 2x(1 - y)$ is harmonic in some domain D.

***Solution*:**

$u_x = 2(1 - y) \rightarrow u_{xx} = 0$

$u_y = -2x \qquad \rightarrow u_{yy} = 0$

$\therefore u_{xx} + u_{yy} = 0$

Since u, u_x, u_y are continuous and satisfied Laplace equation then the function is harmonic.