[4] Derivative

Let f be a function whose domain of definition contains a neighborhood $|z - z_0| < \epsilon$ of a point z_0. The derivative of f at z_0 is the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

And the function f is said to be differentiable at z_0 when $f'(z_0)$ exists. If $\Delta z = z - z_0$, then $\Delta z \to 0$ when $z \to z_0$. Thus

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Theorem: If f is differentiable at z_0, then f is continuous at z_0.

Differentiation Formulas:

In the following formulas, the derivative of a function f at a point z_0 is denoted by either $\frac{d}{dz} f(z)$ or $f'(z_0)$.

1. $\frac{d}{dz}\, c = 0$, c is constant

2. $\frac{d}{dz}\, z = 1$

3. $\frac{d}{dz}\left(c\, f(z)\right) = c\, f'(z)$

4. $\frac{d}{dz}\,[f + g] = \frac{d}{dz}\, f + \frac{d}{dz}\, g = f' + g'$

5. $\frac{d}{dz}\,[f.g] = f.g' + g.f'$

6. $\frac{d}{dz}\left[\frac{f}{g}\right] = \frac{g.f' - f.g'}{g^2}$, $g \neq 0$

7. $\frac{d}{dz}\,(z^n) = n\, z^{n-1}$

8. $(gof)'(z_0) = g'\left(f(z_0)\right).f'(z_0)$

Note: (The Chain rule) If $w = f(z)$ and $W = g(w)$, then $\frac{dW}{dz} = \frac{dW}{dw} \cdot \frac{dw}{dz}$

Example: Find the derivative of $f(z) = (2z^2 + i)^5$

***Solution*:** write $w = 2z^2 + i$ and $W = w^5$ $then$:

$$\frac{d}{dz}(2z^2 + i)^5 = 5w^4 . 4z = 20\, z(2z^2 + i)^4$$

Examples: Find $f'(z)$ $where$ $f(z) = z^2$

Solution:

$f'(z) = 2z$

[5] Cauchy – Riemann Equations (C-R-E)

Theorem: Suppose that $f(z) = u(x, y) + iv(x, y)$ and $f\,'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the first-order partial derivatives of u and v must exist at (x_0, y_0), and they must satisfy the Cauchy-Riemann equations

$$u_x = v_y\,, \qquad u_y = -v_x$$

There is also $f'(z_0) = u_x + iv_x$

Where these partial derivatives are to be evaluated at (x_0, y_0).

Note:

1. $f'(z) = u_x + iv_x$ or $f'(z) = u_y - iv_y$.
2. If $f'(z)$ exists then C-R-Eq. are satisfied, but the converse is not true.

The converse of the above theorem is not necessary true.

Example: $\boldsymbol{f(z) = z^2 = x^2 - y^2 + 2\, ixy}$

Solution:

$u(x, y) = x^2 - y^2 \rightarrow u_x = 2x$

$v(x, y) = 2xy \quad \rightarrow \; v_y = 2x$

$\rightarrow u_x = v_y$

$u_y = -2\text{y}\,,\; v_x = 2y$

$\rightarrow u_y = -v_x$

$\therefore f'(\text{z}) = u_x + iv_x = 2x + i2y = 2(x + iy) = 2z$

<u>Example:</u> $\boldsymbol{f(z) = \bar{z} = x - iy}$

<u>Solution:</u>

$u(x, y) = x \quad \rightarrow \; u_x = 1$

$v(x, y) = -y \rightarrow \; v_y = -1$

$\therefore u_x \neq v_y \rightarrow f$ is not differentiable at z.

<u>Note:</u> The following theorem gives a necessary and sufficient condition to satisfy the converse of the previous theorem.

<u>Theorem:</u> Let $f(z) = u(x, y) + iv(x, y)$, and

1. $u,\; v,\; u_x,\; v_x,\; u_y,\; v_y$ are continuous at $N_\epsilon(z_0)$

2. $u_x = v_y\,,\; u_y = -v_x$

Then f is differentiable at z_0 and

$$f'(z_0) = u_x + iv_x$$

$$f'(z_0) = v_y - iu_y$$

<u>Example:</u> Show that the function $\boldsymbol{f(z) = e^{-y}\cos x + i\, e^{-y}\sin x}$

Is differentiable $\boldsymbol{z}$ for all and find its derivative.

<u>Solution:</u>

Let $u(x, y) = e^{-y}\cos x$

$\rightarrow u_x = -e^{-y}\sin x$

$u_y = -e^{-y}\cos x$

$v(x,y) = e^{-y}\sin x$

$\rightarrow v_x = e^{-y}\cos x$

$v_y = -e^{-y}\sin x$

1. $u_x = v_y$ and $u_y = -v_x$

2. $u,\ v,\ u_x,\ v_x,\ u_y,\ v_y$ are continuous

Then $f'(z)$ exist. To find $f'(z) = u_x + iv_x$

$f'(z) = u_x + iv_x = -e^{-y}\sin x + ie^{-y}\cos x$

$= e^{-y}(i\cos x - \sin x)$

$= ie^{-y}(\cos x + i\sin x)$

$= ie^{-y}e^{ix}$

$= ie^{i\,x-y}$

$= ie^{i(x+iy)}$

$= ie^{iz}$

[6] Polar Coordinates of Cauchy – Riemann Equations

Let $f(z) = u(r,\theta) + iv(r,\theta)$, then Cauchy-Riemann equations are:

and $f'(z_0) = e^{-i\theta}(u_r + i\,v_r).u_r = \frac{1}{r}v_\theta\ ,\ u_\theta = -r\,v_r$

Example: Use C-R equations to show that the functions

1. $f(z) = |z|^2$

2. $f(z) = z - \bar{z}$

are not differentiable at any nonzero point.

***Solution*:**

1. $|z|^2 = x^2 + y^2$

$u(x,y) = x^2 + y^2$, $v(x,y) = 0$

$u_x = 2\text{x}$, $v_x = 0$

$u_y = 2y$, $v_y = 2x$

C-R equations are not satisfied, therefore f' is not exist.

2. $z - \bar{z} = (x + iy) - (x - iy)$

$= x + iy - x + iy$

$= 2y\, i$

$u(x,y) = 0$, $v(x,y) = 2y$

$u_x = 0$, $v_x = 0$

$u_y = 0$, $v_y = 2$

C-R equations are not satisfied, hence f' is not exist.

<u>Example</u>: Use C-R equations to show that $f'(z)$ and $f''(z)$ are exist everywhere $f(z) = z^3$

***<u>Solution</u>*:**

$f(z) = z^3 = (x + iy)^3$

$= x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3$

$= x^3 + 3i\, x^2 y - 3xy^2 - iy^3$

$= x^3 - 3xy^2 + i\,(3x^2 y - y^3)$

$u(x,y) = x^3 - 3xy^2 \rightarrow u_x = 3x^2 - 3y^2$

$u_y = -6xy$

$v(x,y) = 3x^2 y - y^3 \rightarrow v_x = 6xy$

$v_y = 3x^2 - 3y^2$

$\therefore u_x = v_y$, $u_y = -v_x$

C-R equations are satisfied$\therefore$

$f'(z) = u_x + iv_x$

$= 3x^2 - 3y^2 + i\,6xy$

$= 3(x^2 + i^2y^2 + 2i\,xy) = 3(x + iy)^2 = 3z^2$

$f''(z) = u'_x + iv'_x$

$= 6x + i\,6y$

$= 6(x + iy)$

$= 6z$

Example: **Let $f(z) = z^3$, write f in polar form and then find $f'(z)$**

Solution: $f(z) = z^3 = \left(re^{i\theta}\right)^3 = r^3e^{3i\theta}$

$= r^3\cos 3\theta + i\,r^3\sin 3\theta$

$u(r,\theta) = r^3\cos 3\theta \to u_r = 3r^2\cos 3\theta$

$u_\theta = -3r^3\sin 3\theta$

$v(r,\theta) = r^3\sin 3\theta \to v_r = 3r^2\sin 3\theta$

$v_\theta = 3r^3\cos 3\theta$

Now, $u_r = \frac{1}{r}\,v_\theta\,, \quad u_\theta = -rv_r$

$f'(z) = e^{-i\theta}[u_r + i\,v_r]$

$= e^{-i\theta}[3r^2\cos 3\theta + i3r^2\sin 3\theta]$

$= 3r^2e^{-i\theta}[\cos 3\theta + i\sin 3\theta]$

$= 3r^2e^{-i\theta}e^{3\theta i}$

[7] Analytic Functions

Definition:

A function f is said to be analytic at z_0 if $f'(z_0)$ exists and $f'(z)$ exists at each point z in the same neighborhood of z_0.

Note: f is analytic in a region R if it is analytic at every point in R.

Definition:

If f is analytic at each point in the entire plane, then we say that f is an entire function.

Example: $f(z) = z^2$, is an entire function since it is a polynomial.

Definition: If f is analytic at every point in the same neighborhood of z_0 but f is not analytic at z_0, then z_0 is called singular point

Example: Let $f(z) = \frac{1}{z}$, then $f'(z) = \frac{-1}{z^2}$ $(z \neq 0)$

Then f is not analytic at $z_0 = 0$, which is a singular point.

Note: If f is analytic in D, then f is continuous through D and C-R equations are satisfied.

Note: A sufficient conditions that f be analytic in $\mathbb{R}$ are that C-R equations are satisfied and $u_x,\ v_x,\ u_y,\ v_y$ are continuous in $\mathbb{R}$.

[8] Harmonic Functions

Definition:

A function h of two variables x and y is said to be harmonic in D if the first partial derivatives are continuous in D and $h_{xx} + h_{yy} = 0$ (Laplace equation)

Example: Show that $u(x, y) = 2x(1 - y)$ is harmonic in some domain D.

***Solution*:**

$u_x = 2(1 - y) \rightarrow u_{xx} = 0$

$u_y = -2x \rightarrow u_{yy} = 0$

$\therefore u_{xx} + u_{yy} = 0$

Since u, u_x, u_y are continuous and satisfied Laplace equation then the function is harmonic.