Complex Numbers

Definition:

A *complex number* ***z*** is an ordered pair (a, b) of real numbers such that

$$\mathbb{C} = \{ \mathbb{R} \times \mathbb{R} \} = \{(a,b): a,b \in \mathbb{R} \}$$

Where $\mathbb{R}$ denotes the Real Numbers set. The real numbers *a, b* are called the real

and imaginary parts of the complex number $z = (a,b)$, that is $a = Re(z)$

and $b = Im(z)$. If $b = Im(z) = 0$ then $z = (a,0) = a$ so that the set of complex

numbers is a natural extension of real numbers, then we have:

$a = (a,0)$ for any real number a. Thus

$$0 = (0,0), \qquad 1 = (1,0), \qquad 2 = (2,0), \ldots$$

A pair $(0, b)$ is called a pure imaginary number and the pair $(0, 1)$ is called the imaginary $\boldsymbol{i}$, that is

$$(0,1) = i$$

Now any complex number z can be written as:

$$(a,0) + (0,b) = (a,b) = z$$

[1] The operations for complex numbers

Let $z_1 = (a_1,b_1), z_2 = (a_2,b_2), then$:

(1) The operation of addition $(z_1 + z_2)$ is defined as follows:

$z_1 + z_2 = (a_1,b_1) + (a_2,b_2) = (a_1 + a_2, b_1 + b_2)$

(2) The operation of multiplication $(z_1.z_2)$ is defined as follows

$z_1.z_2 = (a_1,b_1).(a_2,b_2) = (a_1a_2 - b_1b_2, a_1b_2 + b_1a_2)$

Note

(1) $z = (a, 0) + (0, b) = (a, 0) + (0,1)(b, 0)$

Hence $(a, 0) + (0,1)(b, 0) = (a, b) = z$ where $(0,1) = i$

Then $z = a + ib$

(2) $z^2 = z.z,\ \ z^3 = z.z.z,\ \ z^n = z.\underbrace{z \ldots\ldots z}_{n-\text{times}}$

or $i = \sqrt{-1}$(3)$i^2 = i.i = (0,1).(0,1) = -1$

Then $i^2 = -1,\ \ i = \sqrt{-1}$

[2] Basic Algebraic Properties:

The following algebraic properties hold for all $z_1, z_2, z_3 \in \mathbb{C}$

1. $z_1 + z_2 = z_2 + z_1$
2. $z_1.z_2 = z_2.z_1$

(Commutative laws under addition and multiplication)

3. $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ (Associative under addition)

4. $(z_1.z_2).z_3 = z_1.(z_2.z_3)$ (Associative under multiplication)

5. $z_1.(z_2 + z_3) = z_1.z_2 + z_1.z_3$ (Distribution laws)

6. $z_1 + z_3 = z_3 + z_2$ iff $z_1 = z_3$
7. $z_1.z_2 = z_3.z_2$ iff $z_1 = z_3$

(Cancelation law)

Note: the additive identity $0 = (0,0)$ and the multiplication identity $1 = (1,0)$,

for any complex number. That is

$$z + 0 = 0 + z = z$$

$$1.z = z.1 = z$$

Definition:

The additive inverse z^* of z is a complex number with the property that

................(1)$z + z^* = 0$

It is clear that (1) is satisfied if $z^* = (-x, -y)$, has an additive inverse.

Note: $(-z)$ is the only additive inverse of a given complex number.

Definition:

The multiplication inverse $z^{-1}(z \neq 0)$ of z is a complex number with the

property that

$$(2) z.z^{-1} = z^{-1}.z = 1$$

Such that $z^{-1} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$

Note: the additive and multiplication identity are unique.

Note: if $z_2 \neq 0$, then

$$\frac{z_1}{z_2} = \left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}, \frac{y_1x_2-x_1y_2}{x_2^2+y_2^2}\right)$$

Exercise: show that $z = 0$ iff $Re(z) = 0$ and $Im(z) = 0$.

Example: verify that

1. $\left(\sqrt{2} - i\right) - i(1 - \sqrt{2}\, i)$

***Solution*:**

$$\sqrt{2} - i - i - \sqrt{2} = -2i$$

2. $(2, -3)(-2, 1)$

***Solution*:**

$$(2, -3)(-2, 1) = (-4 + 3, 2 + 6) = (-1, 8)$$

3. $(\mathbf{3},\mathbf{1})(\mathbf{3},-\mathbf{1})\left(\frac{\mathbf{1}}{\mathbf{5}},\frac{\mathbf{1}}{\mathbf{10}}\right)$

Solution:

$$(3,1)(3,-1)\left(\frac{1}{5},\frac{1}{10}\right) = (9+1,-3+3)\left(\frac{1}{5},\frac{1}{10}\right)$$

$$= (10,0)\left(\frac{1}{5},\frac{1}{10}\right)$$

$$= \left(\frac{10}{5}-0,\ \frac{10}{10}+0\right)$$

$$= (2,1)$$

Example: show that each of the two numbers $z = 1 \mp i$ satisfies the equation

$$z^2 - 2z + 2 = 0$$

Proof**:** **for** $z = 1 + i$

$(1+i)^2 - 2(1+i) + 2 = 1 + 2i - 1 - 2 - 2i + 2 = 0$

for $z = 1 - i$ (H.w)

Example: show that $(1-i)^4 = -4$

Proof**:** $((1-i)^2)^2 = (1-2i-1)^2$

$$= 4i^2 = -4$$

[3] Properties of Complex Numbers:

1. $Im(iz) = Re(z)$

2. $Re(iz) = Im(z)$

3. $\frac{1}{1/z} = z,\ z \neq 0$

4. $(-1)z = -z$

5. $(z_1z_2)(z_3z_4) = (z_1z_3)(z_2z_4)$

6. $\frac{z_1+z_2}{z_3} = \frac{z_1}{z_3} + \frac{z_2}{z_3}, z_3 \neq 0$

Note:

$$(1+z)^n = 1 + nz + \frac{n(n+1)}{2!}z^2 + \frac{n(n-1)(n-2)}{3!}z^3 + \cdots + z^n$$

[4] Vectors

It is natural to associate any nonzero complex number $z = x + iy$ with the directed

Line segment or vector from the origin to the point (x, y) that represents z in the

Complex plane. In fact, we can often refer to z as the point z or the vector z.

In Fig. 1 the number $z = x + iy$ and $-2 + i$ are displayed graphically as both two

Points and radius vector.

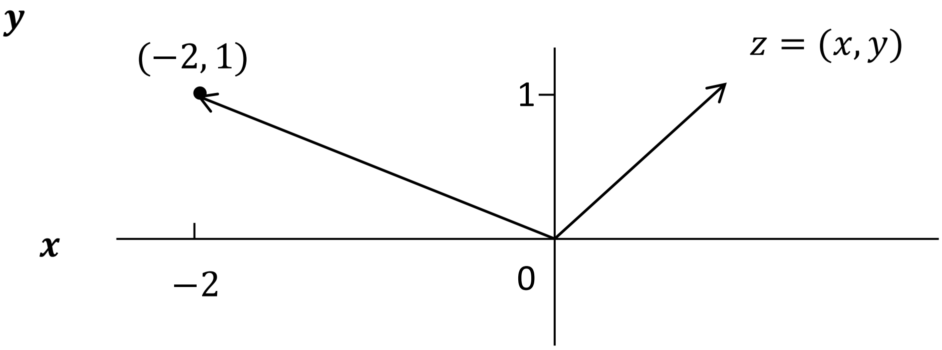
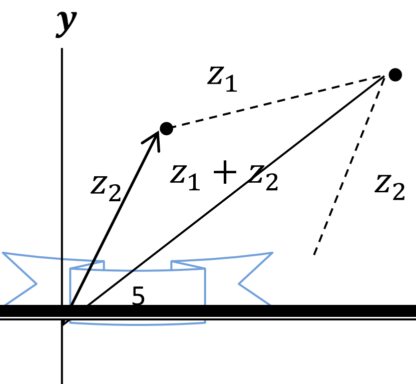


Figure 1

When $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the sum $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

Corresponds to the point $(x_1 + x_2, y_1 + y_2)$, it is also corresponds to a vector with

Those coordinate as its components. Hence $z_1 + z_2$ may be obtained vectorially as shown in Fig. 2.



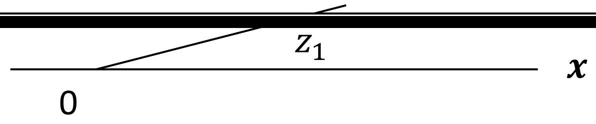


Figure 2

The distance between two points (x_1, y_1) and (x_2, y_2) is $|z_1 - z_2|$,

this is clear from Fig. 3, since $|z_1 - z_2|$ is the length of the vector representing the number $z_1 - z_2 = z_1 + (-z_2)$,

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

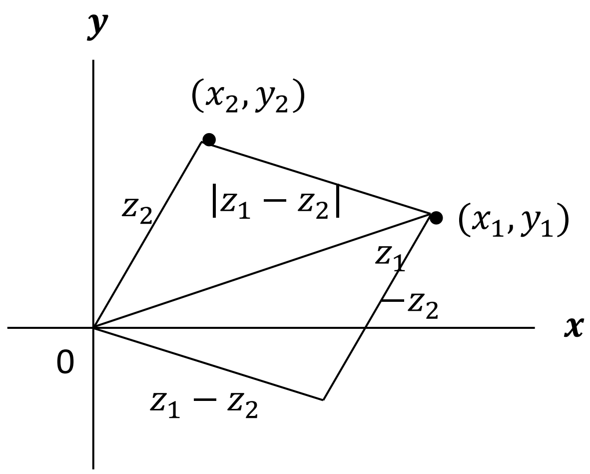


Figure 3

<u>Example:</u> the equation $|z - 1 + 3i| = 2$ represents the circle whose center is

$z_0 = (1, -3)$ and whose radius is $R = 2$.

, where z_0 represents the center of circle with radius R.$|z - z_0| = R$

<u>Definition:</u> (The Absolute Value)

The absolute value of a complex number $z = x + iy$ is defined by $\sqrt{x^2 + y^2}$

And also by $|z|$, such that $|z| = \sqrt{x^2 + y^2}$

We notice that the absolute value $|z|$ is a distance from $(0,0)$ to (x, y).

The statement $|z_1| < |z_2|$ means that z_1 is closer to $(0,0)$ than z_2.

The distance between z_1 and z_2 is given by $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Example: $|z - i| = 3$

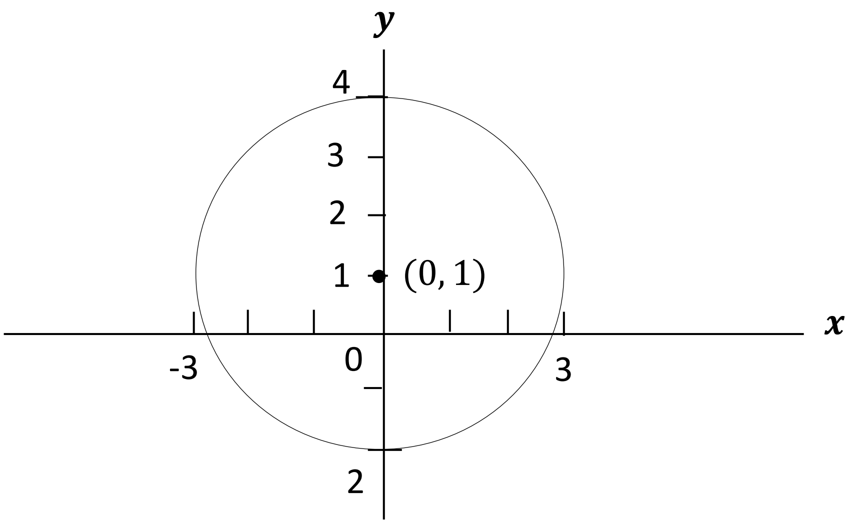
Solution: we refer to $|z - i| = 3$ as $|x + iy - i| = 3$

$|x + i(y-1)| = 3 \rightarrow \sqrt{x^2 + (y-1)^2} = 3$

$x^2 + (y-1)^2 = 9 \Leftrightarrow (x - x_0)^2 + (y - y_0)^2 = r^2$

The complex number corresponding to the points lying on the circle with

Center $(0,1)$ and radius 3



Note: the real numbers $|z|$, $Re(z)$ and $Im(z)$ are related by the equation:

$$|z|^2 = (Re(z))^2 + (Im(z))^2$$

As follows

$|z| = \sqrt{x^2 + y^2} \rightarrow |z|^2 = x^2 + y^2 = (Re(z))^2 + (Im(z))^2$

Since $y^2 \geq 0$, we have

$|z|^2 \geq x^2 = \left(Re(z)\right)^2 = |Re(z)|^2$

And since $|z| \geq 0$, we get

$|z| \geq |Re(z)| \geq Re(z)$

Similarly $|z| \geq |Im(z)| \geq Im(z)$.

[5] Complex Conjugates

The complex conjugate of $z = x + iy$ is defined by $\bar{z} = x - iy$

The number is $\bar{z}$ represented by the point $(x, -y)$, which is the reflection in the

real axis of the point (x, y) representing z (Fig. 4), note that

$$\bar{\bar{z}} = z \text{ and } |\bar{z}| = |z|, \qquad \text{for all } z$$

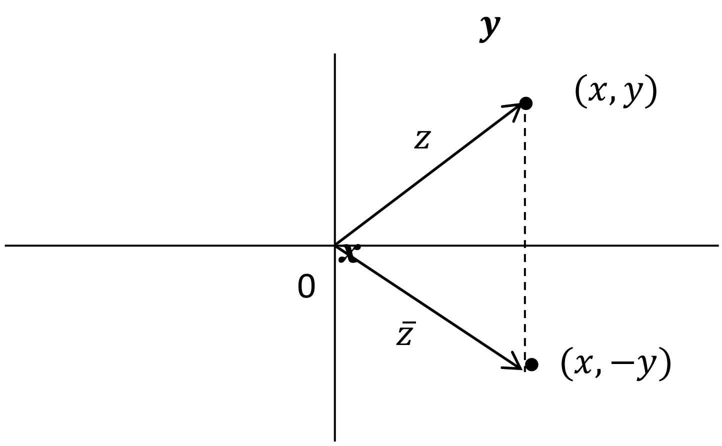


Figure 4

Some Properties of Complex Conjugates:

1. $\bar{\bar{z}} = z$

2. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$, $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

3. $\overline{z_1 . z_2} = \bar{z}_1 . \bar{z}_2$

4. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, $z_2 \neq 0$

Note:

1. $z + \bar{z} = x + iy + x - iy = 2x = 2Re(z)$

$\frac{z+\bar{z}}{2} Re(z) =$

2. $z - \bar{z} = x + iy - x + iy = 2iy = 2Im(z)$

$\frac{z-\bar{z}}{2} Im(z) =$

Some Properties of the absolute value

1. $|z_1 z_2| = |z_1||z_2|$

2. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$

3. $|z_1 + z_2| \leq |z_1| + |z_2|$

4. $|z_1 + z_2 + \cdots z_n| \leq |z_1| + |z_2| \cdots |z_n|$

5. $||z_1| - |z_2|| \leq |z_1 + z_2|$

6. $||z_1| - |z_2|| \leq |z_1 - z_2|$

Example: If a point z lies on the unite circle $|z| = 1$ about the origin, show that $|z^2 + z + 1| \leq 3$ and $|z^3 - 2| \geq ||z|^3 - 2|$

***Proof*:** $|z^2 - z + 1| = |(z^2 + 1) + z| \leq |z^2 + 1| + |z|$

$\leq |z^2| + 1 + |z|$

$= |z|^2 + 1 + |z|$

$= 1^2 + 1 + 1$

$= 3$

$\rightarrow |z^2 - z + 1| \leq 3$

Note:

1. $(|x| - |y|)^2 \geq 0$

$\rightarrow |x|^2 + |y|^2 - 2|x||y| \geq 0$

$\rightarrow\ x^2 + y^2 \geq 2|x||y| \quad \dots (*)$

2. z is real iff $\bar{z} = z$

3. z is either real or pure imaginary iff $(\bar{z}\,)^2 = z^2$

4. if $|z_2| \neq |z_3|$ then $\left|\frac{z_1}{z_2+z_3}\right| \leq \frac{|z_1|}{||z_2|-|z_3||}$

Example: If a point z lies on the unite circle $|z| = 2$ then show that

$$\leq \frac{1 \qquad 1}{3|z^4-4z^3+3|}$$

Proof: $|z^4 - 4z^3 + 3| = |(z^2 - 1)(z^2 - 3)|$

$$= |z^2 - 1||z^2 - 3|$$

$$\geq ||z|^2 - 1|\,||z|^2 - 3|$$

$$= |4 - 1|\,|4 - 3|$$

$$= 3$$

$\therefore\ |z^4 - 4z^3 + 3| \geq 3$

$\leq \frac{1}{3} \rightarrow \quad \frac{1}{|z^4-4z^3+3|}$

Exercises:

1. Find $|z|$ where

   a.z=3-4i

   b. $z = -2 + \sqrt{12}\, i$

2. If z=x+iy then show that

   a. $\frac{1}{z} = \frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2}$

   b. $\overline{iz} = -i\bar{z}$

Note: $x = \frac{z+\bar{z}}{2} = R(z),\ \ y = \frac{z-\bar{z}}{2i} = Im(z)$

$x^2 - y^2 = 1$

$\left(\frac{z+\bar{z}}{2}\right)^2 \quad \left(\frac{z-\bar{z}}{2i}\right)^2 \quad 1$

$= 1\frac{z^2+2z\bar{z}+\bar{z}^2}{4} - \frac{z^2-2z\bar{z}+\bar{z}^2}{4i^2}$

$= 1\ \frac{z^2+2z\bar{z}+\bar{z}^2}{4} + \frac{z^2-2z\bar{z}+\bar{z}^2}{4}$

$\rightarrow 2z^2 + 2\bar{z}^2 = 4$

$\rightarrow 2(z^2 + \bar{z}^2) = 4$

$\rightarrow z^2 + \bar{z}^2 = 2$

[6] Polar Form of Complex Numbers: (Exponential Form)

Let r and θ be polar coordinates of the point (x, y) that corresponds to a nonzero complex number $z = x + iy$,

$$x = r\cos\theta \quad , \quad y = r\sin\theta$$

The number z can be written in polar form as

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$\frac{y}{x}$, $x \neq 0,\ r^2 = x^2 + y^2,\ i\theta = \cos\theta + i\sin\theta tan\theta =$

This implies that for any complex number $z = x + iy$, we have

$$|z| = \sqrt{x^2 + y^2} = \sqrt{r^2} = r$$

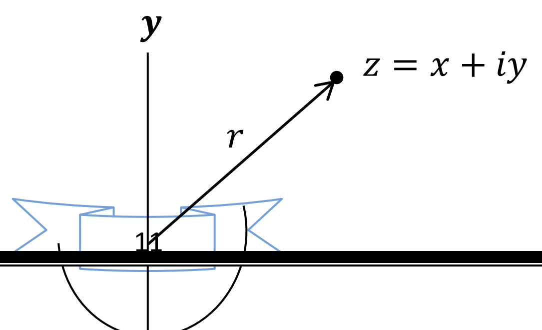
In fact r is the length of the vector represent z. In particular, since $z = x + iy$

We may express z in polar form by

$$z = r\cos\theta + i\,r\sin\theta = r(\cos\theta + i\sin\theta)$$

The real number θ represents the angle, measured in radians, that z makes with

The positive real axis (Fig. 5).



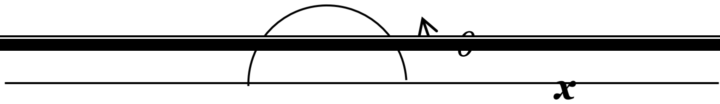


Figure 5

Each value of θ is called an argument of z and the set of all such values is denoted by $\arg z = \theta$.

Note: $arg\, z$ is not unique.

Definition: The principal value of $\arg z$ (Arg z)

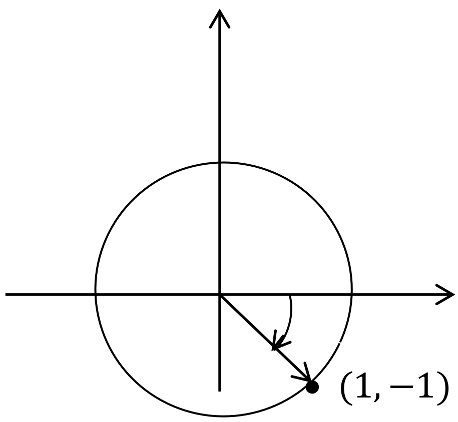
If $-\pi < \theta < \pi$ and satisfy

$\arg z = \operatorname{Arg} z + 2n\pi,\ n = 0, \mp 1, \mp 2, \ldots$

Then this value of θ (which is unique) is called the principal value of $\arg z$ and

denoted by Arg z.

Example: Write $z = 1 - i$ in polar form



Solution: $r = \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2}$

$x = r\cos\theta \rightarrow 1 = \sqrt{2}\cos\theta \rightarrow \cos\theta = \frac{1}{\sqrt{2}}$

$$y = r\sin\theta \rightarrow -1 = \sqrt{2}\sin\theta \rightarrow \sin\theta = \frac{-1}{\sqrt{2}}$$

$\tan\theta = \frac{y}{x} = \frac{-1}{1} = -1$

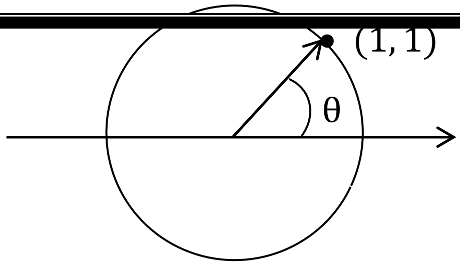
$\theta = \tan^{-1}(-1) = \frac{-\pi}{4}$

$z = 1 - i = \sqrt{2}\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right)$

$= \sqrt{2}\left(\cos\left(\frac{-\pi}{4} + 2n\pi\right) + i\sin\left(\frac{-\pi}{4} + 2n\pi\right)\right)$

Example: Write $z = 1 + i$ in polar form

Solution: $r = \sqrt{2}\ ,\ \ \tan\theta = \frac{y}{x} = 1$



$\rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4}$

$\therefore \theta = arg\, z = \frac{\pi}{4} + 2n\pi$

$\therefore 1 + i = \sqrt{2}\left(\cos\left(\frac{\pi}{4} + 2n\pi\right) + i\sin\left(\frac{\pi}{4} + 2n\pi\right)\right)$