Theorem: For any square matrix *A*,

(1) $AI=IA=A$.

(2) $A^2 = AA$, $A^3 = A^2A$ *and so on.*

Theorem

(i) $(A+B)^t = A^t + B^t$

(ii) $(A^t)^t = A$

(iii) $(kA)^t = kA^t$, for k a scalar

(iv) $(AB)^t = B^t A^t$.

H.W.

1- Let

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 0 & -3 \\ -1 & -2 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -3 & 0 & 1 \\ 5 & -1 & -4 & 2 \\ -1 & 0 & 0 & 3 \end{pmatrix} \text{ and } D = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

Find: (i) $A+B$, (ii) $A+C$, (iii) $3A-4B$.

Find: (i) AB, (ii) AC, (iii) AD, (iv) BC, (v) BD, (vi) CD.

Find: (i) A^t, (ii) A^tC, (iii) D^tA^t, (iv) B^tA, (v) D^tD, (vi) DD^t.

2- Given the matrices R, S, and T below.

$$R = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 5 \\ 2 & 3 & 1 \end{bmatrix} S = \begin{bmatrix} 0 & -1 & 2 \\ 3 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} T = \begin{bmatrix} -2 & 3 & 0 \\ -3 & 2 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$

Find 2RS - 3ST.

Remark

1) Note that AB≠BA
2) The cancellation law does not hold for matrices as the following example shows:

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} -2 & 7 \\ 5 & -1 \end{bmatrix}$. Then

$$AB = AC = \begin{bmatrix} 8 & 5 \\ 16 & 10 \end{bmatrix}.$$

But B≠C.

3) AB may be zero with neither A nor B equal to zero; that is, if A and B are two nonzero matrices, it is not necessary $AB \neq \mathbf{0}$. That is, the zero property does not hold for matrix multiplication as the following example shows:

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}$. $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Definition

An n×n matrix (square matrix) has an inverse (invertible) if there exists a matrix B such that AB = BA = I_n, where I_n is an n×n identity matrix. B is called the inverse of A and denoted by the symbolA^{-1}.

Observe that the above relation is symmetric; That is, if B is the inverse of A then A is also the inverse of B.

Definition

If A has an inverse we say that A is invertible, otherwise we say that A is **singular matrix (or noninvertible)**.

Example

Given matrices A and B below, verify that they are inverses.

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} B = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$$

Solution:

$$\text{AB} = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \quad \text{BA} = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example: Fide the inverse of the following matrix:

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$$

Solution: Suppose A has an inverse, and it is:

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then AB=I and hence:

$$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

After multiplying the two matrices on the left side, we get

$$\begin{bmatrix} 3a+c & 3b+6 \\ 5a+2c & 5b+2d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating the corresponding entries, we get four equations with four unknowns as follows:

$$3a + c = 1 \quad 3b + d = 0$$

$$5a + 2c = 0 \quad 5b + 2d = 1$$

Using substitution method or elimination method to solve the systems. $a=2$, $b=-1$ $c=-5$, $d=3$ Therefore,

$$A^{-1} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$$

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, find A^{-1} if exist.

Solution:

Let $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $AA^{-1} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$\Rightarrow AA^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} a+2c & b+2d \\ 2a+4c & 2b+4d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$a + 2c = 1 - -\rightarrow E_1$

$2a + 4c = 0 - -\rightarrow E_2$

$b + 2d = 0 - -\rightarrow E_3$

$2b + 4d = 1 - -\rightarrow E_4$

$-2E_1 + E_2 \longrightarrow 0 = -2$ Contradiction (*C!*)

So, the linear systems have no solution. Therefore A has no inverse. That is, A is singular.

Example: Find the inverse, if it exists, of

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating corresponding terms, we see that this is true only if

$$\begin{array}{lll} a - b + c = 1 & d - e + f = 0 & g - h + i = 0 \\ 2b - c = 0 & 2e - f = 1 & 2h - i = 0 \\ 2a + 3b = 0 & 2d + 3e = 0 & 2g + 3h = 1 \end{array}$$

Use substitution or elimination methods to solve these systems.

$$a = 3, b = -2, c = -4$$
$$d = 3, e = -2, f = -5$$
$$g = -1, h = 1, i = 2.$$

Therefore: $A^{-1} = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$

(1.1) Determinants

Definition

For any square matrix A, the **determinant** of A is a real number denoted by **det(A)** or $|A|$. If A is a square matrix of order n, then det(A) is called a **determinant of order *n*.**

Remark

1) The determinant of a 1 x 1 matrix A=[a] is the number **a** itself. det(A)=a.
2) The determinant of the second-order square matrix
   $A = \begin{bmatrix} a11 & a12 \\ a21 & a22 \end{bmatrix}$, is

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Example: **Find:** $\begin{vmatrix} -1 & 2 \\ -3 & -4 \end{vmatrix}$

Solution:

det(A) =(-1)(14) – 2(-3) = 4+6=10

3) Third-Order Determinants:
   The determinant of 3×3 matrix can be obtained by:

$$\begin{vmatrix} a1 & b1 & c1 \\ a2 & b2 & c2 \\ a3 & b3 & c3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) .$$

Note that each 2 x 2 matrix can be obtained by deleting, in the original matrix, the row and column containing its coefficient.

$$a_1 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Remark: If we think of the sign $(-1)^{i+j}$ as being located in position (i,j) of an $n\times n$ matrix, then the signs form a checkerboard pattern that has a (+) in the (1,1) position. The patterns for $n=3$ and $n=4$ are as follows:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

Example: Find

$$\begin{vmatrix} 2 & -2 & 0 \\ -3 & 1 & 2 \\ 1 & -3 & -1 \end{vmatrix}$$

Solution:

We can choose any row or column to expand. We will choose the first row because it has zero: ☺

$$\begin{vmatrix} 2 & -2 & 0 \\ -3 & 1 & 2 \\ 1 & -3 & -1 \end{vmatrix} = 2\left[(-1)^{1+1}\begin{vmatrix} 1 & 2 \\ -3 & -1 \end{vmatrix}\right] + (-2)\left[(-1)^{1+2}\begin{vmatrix} -3 & 2 \\ 1 & -1 \end{vmatrix}\right] + 0$$

$$= (2)(1)[(1)(-1) - (-3)(2)] + (-2)(-1)[(-3)(-1) - (1)(2)]$$

$$= (2)(5) + (2)(1) = 12$$

Example: Evaluate

$$\begin{vmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 3 & 0 & 0 & -3 \\ 2 & 0 & -2 & 3 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 3 & 0 & 0 & -3 \\ 2 & 0 & -2 & 3 \end{vmatrix} = (-1)^{3+1}(3)\begin{vmatrix} 2 & -3 & 4 \\ 2 & 1 & 3 \\ 0 & -2 & 3 \end{vmatrix} + (-1)^{3+2}(0)\begin{vmatrix} 1 & -3 & 4 \\ -4 & 1 & 3 \\ 2 & -2 & 3 \end{vmatrix}$$

$$+ (-1)^{3+3}(0)\begin{vmatrix} 1 & 2 & 4 \\ -4 & 2 & 3 \\ 2 & 0 & 3 \end{vmatrix} + (-1)^{3+4}(-3)\begin{vmatrix} 1 & 2 & -3 \\ -4 & 2 & 1 \\ 2 & 0 & -2 \end{vmatrix}$$

$$=(3)\begin{vmatrix} 2 & -3 & 4 \\ 2 & 1 & 3 \\ 0 & -2 & 3 \end{vmatrix} + (3)\begin{vmatrix} 1 & 2 & -3 \\ -4 & 2 & 1 \\ 2 & 0 & -2 \end{vmatrix}$$

$$= (3)\left(2\begin{vmatrix} 1 & 3 \\ -2 & 3 \end{vmatrix} - 2\begin{vmatrix} -3 & 4 \\ -2 & 3 \end{vmatrix}\right) + (3)\left((-2)\begin{vmatrix} -4 & 1 \\ 2 & -2 \end{vmatrix} + 2\begin{vmatrix} 1 & -3 \\ 2 & -2 \end{vmatrix}\right)$$

$$= (3)(2)(3+6) - (3)(2)(-9+8) + (3)(-2)(8-2) + (3)(2)(-2+6)$$

$$= 54 + 6 - 36 + 24 = 48.$$

H.W: Evaluate

$$\begin{vmatrix} 0 & -1 & 0 & 2 \\ -5 & -6 & 0 & -3 \\ 4 & 5 & -2 & 6 \\ 0 & 3 & 0 & -4 \end{vmatrix}$$

<u>Theorem:</u> A matrix is invertible if and only if its determinant is not zero. Then a matrix is singular if its determinant is zero.

<u>Theorem</u>: If a **2x2** matrix A is invertible then we can obtain the inverse of matrix by using the determinant of A as follows: (i) interchanging the elements on the main diagonal, (ii) taking the negative **of** the other elements, and (iii) dividing each element by the determinant of the original matrix.

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d/|A| & -b/|A| \\ -c/|A| & a/|A| \end{pmatrix} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

<u>Example</u> : Find the inverse of the matrix $A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$

$|A|=2 \times 5 - 3 \times 4 = -2$

$$A^{-1} = \begin{pmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{pmatrix}$$

Theorem: (Determinant properties)

(1) If a matrix B results from a matrix A by interchanging two rows (columns) of A, then $B =-|A|$.

(2) If two rows (column) of A are equal, then $|A| =0$.

(3) If all the elements of a row (or column) are zeros, then the value of the determinant is zero.

(4) If all elements of a row (or column) of a determinant are multiplied by some scalar number k, the value of the new determinant is k times of the given determinant.

(5) Let A and B be two matrix, then det(AB) = det(A)*det(B) or

$|AB| = |A|\,|B|$

(6) If A is a non-singular matrix, then the determinant of Inverse of matrix can be defined as $|A^{-1}| = \frac{1}{|A|}$.

(7) The determinant of a matrix and its transpose are equal; that is,

$|A| = |A^T|$.

(8) If a matrix $A=[aij]n\times n$ is upper (lower) triangular, then $A=a11\cdot a22\cdot\cdots\cdot ann$ Product of the elements on the main diagonal.

(9) In a determinant each element in any row (or column) consists of the sum of two terms, then the determinant can be expressed as sum of two determinants of same order. For example,

$$\begin{vmatrix} a & b & \alpha+x \\ c & d & \beta+y \\ e & f & \gamma+z \end{vmatrix} = \begin{vmatrix} a & b & \alpha \\ c & d & \beta \\ e & f & \gamma \end{vmatrix} + \begin{vmatrix} a & b & x \\ c & d & y \\ e & f & z \end{vmatrix}$$

(10) If B=[b_{ij}] is obtained from A=[a_{ij}] by adding to each element of the rth row (column) of A, k times the corresponding element sth row (column),s≠r, of A then det(B)= det(A)

Examples

1- Consider the matrix $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \mathbf{1} & \mathbf{3} \\ \mathbf{3} & \mathbf{1} & \mathbf{2} \end{bmatrix}$

Then prove that det(A)=det($\boldsymbol{A^T}$)

Solution $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 3 & 2 \end{bmatrix}$

det(A)=det(A^T)=6

2- Let A= $\begin{bmatrix} \mathbf{2} & \mathbf{-1} \\ \mathbf{3} & \mathbf{2} \end{bmatrix}$

Then $|\boldsymbol{A}| = \begin{vmatrix} \mathbf{2} & \mathbf{-1} \\ \mathbf{3} & \mathbf{2} \end{vmatrix} = \mathbf{7}$, $\begin{vmatrix} \mathbf{3} & \mathbf{2} \\ \mathbf{2} & \mathbf{-1} \end{vmatrix} = \mathbf{-7}$

3- Find $\begin{vmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{-1} & \mathbf{0} & \mathbf{7} \\ \mathbf{1} & \mathbf{2} & \mathbf{3} \end{vmatrix}$

Solution $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 7 \\ 1 & 2 & 3 \end{vmatrix} = 0$

4- Find $\begin{vmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \end{vmatrix}$

Solution $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 0$

5- Let $\boldsymbol{A} = \begin{bmatrix} \mathbf{2} & \mathbf{6} \\ \mathbf{1} & \mathbf{12} \end{bmatrix}$ **.Find** $|\boldsymbol{A}|$

Solution $\begin{vmatrix} 2 & 6 \\ 1 & 12 \end{vmatrix} = 18$

Or $\begin{vmatrix} 2 & 6 \\ 1 & 12 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 \\ 1 & 12 \end{vmatrix} = 2(12 - 3) = 18$

Or$\begin{vmatrix} 2 & 6 \\ 1 & 12 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 \\ 1 & 12 \end{vmatrix} = (2)(3)\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = 6(4-1) = 18$

6- Let $\boldsymbol{A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 3 \\ 2 & 8 & 6 \end{bmatrix}. Find\ |A|}$

<u>Solution</u> $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 5 & 3 \\ 2 & 8 & 6 \end{vmatrix} = 0$

Or$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 5 & 3 \\ 2 & 8 & 6 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 1 & 5 & 3 \\ 1 & 4 & 3 \end{vmatrix} = 0$

Or$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 5 & 3 \\ 2 & 8 & 6 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 1 & 5 & 3 \\ 1 & 4 & 3 \end{vmatrix} = (2)(3)\begin{vmatrix} 1 & 2 & 1 \\ 1 & 5 & 1 \\ 1 & 4 & 1 \end{vmatrix} = 0$

7- We have $\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 5 & 0 & 9 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{vmatrix}$

Obtained by adding twice the second row to the first row.

Then $\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 5 & 0 & 9 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{vmatrix} = 4$

8- Compute $\begin{vmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 0 & 4 \end{vmatrix}$

<u>Solution</u> $\begin{vmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 0 & 4 \end{vmatrix} = -4$

9- Let $\boldsymbol{A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\ then\ prove}$ $\det(AB) = \det(A)\det(B)$

<u>Solution</u> AB=$\begin{bmatrix} \mathbf{4} & \mathbf{3} \\ \mathbf{10} & \mathbf{5} \end{bmatrix}$

det(AB)=-10 ,det(A)=-2 ,det(B)=5

det(AB)=det(A)det(B)=-10

10- **If** $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{4} \end{bmatrix}$ **and** $\mathbf{A^{-1}} = \begin{bmatrix} \mathbf{-2} & \mathbf{1} \\ \mathbf{\frac{3}{2}} & \mathbf{\frac{-1}{2}} \end{bmatrix}$ **then prove** $|A^{-1}| = \frac{1}{|A|}$

<u>Solution</u> $|A| = -2$, $|A^{-1}| = \frac{-1}{2}$ $\quad \therefore \ |A^{-1}| = \frac{1}{|A|}$

H.W.

1. **Find the determinant of each matrix:**

(i) $\begin{pmatrix} 2 & 1 & 1 \\ 0 & 5 & -2 \\ 1 & -3 & 4 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 & -2 & -4 \\ 2 & 5 & -1 \\ 0 & 6 & 1 \end{pmatrix}$ (iii) $\begin{pmatrix} 4 & -5 \\ 0 & 2 \end{pmatrix}$ (iv) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

2. Determine those values of k for which $\begin{vmatrix} k & k \\ 4 & 2k \end{vmatrix} = 0$.

3. **Find the inverse of each matrix:** (i) $\begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$ (ii) $\begin{pmatrix} 2 & -3 \\ 1 & 3 \end{pmatrix}$