# Systems of linear equations

**Definition:** A **system of linear equations** (or **linear system**) is a collection of one or more linear equations involving the same set of variables.

A linear equation in n unknowns is an equation of the form

$a_1x_1 + a_2x_2 + \cdots \dots \dots + a_nx_n = b$

Where $a_1, a_2, \dots \dots \dots, a_n, b$ are real numbers and $x_1, x_2, \dots \dots \dots x_n$ are variables

**Example:**

**(1) The following equations are linear**

(a) $6x_1 - 3x_2 + 4x_3 = -13$

(b) $-3x_1 + 5x_2 + x_3 = 6$

(c) 8x-7y-6z=0

**(2) The following equations are non linear**

(a) $x_1x_2 + sinx_3 = 0$

(b) $e^x - y = 3$

(c) ${x_1}^2 - lnx_2 = 5$

Consider a linear equation in two unknowns x and y is of the form

$ax + by = c$ where a, b, c are real numbers.

**Example:**

Consider the equation

$$2x + y = 4$$

If we substitute $x = -2$ in the equation, we obtain

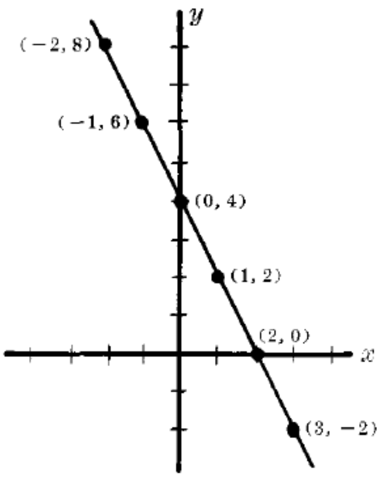
$$2 \cdot (-2) + y = 4 \quad \text{or} \quad -4 + y = 4 \quad \text{or} \quad y = 8$$

Hence $(-2, 8)$ is a solution. If we substitute $x = 3$ in the equation, we obtain

$$2 \cdot 3 + y = 4 \quad \text{or} \quad 6 + y = 4 \quad \text{or} \quad y = -2$$

Hence $(3, -2)$ is a solution. The table on the right lists six possible values for $x$ and the *corresponding* values for $y$, i.e. six solutions of the equation.

| $x$ | $y$ |
|---|---|
| $-2$ | 8 |
| $-1$ | 6 |
| 0 | 4 |
| 1 | 2 |
| 2 | 0 |
| 3 | $-2$ |




Graph of $2x + y = 4$

**Example:**

The equation $6x_1-3x_2+4x_3=-13$ is linear equation of three variables. $x_1 = 2, x_2 = 3, x_3 = -4$ is a solution to the linear equation $6\cdot2-3\cdot3+4\cdot(-4)=-13$. This is not the only solution to the given linear equation, since $x_1 = 3, x_2 = 1, x_3 = -7$ is another solution.

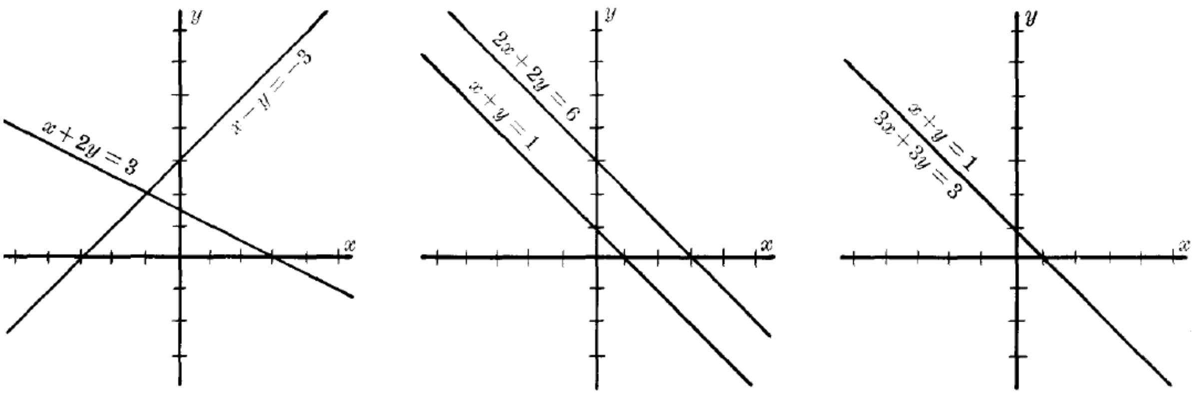
**Example:**

Now consider systems of two linear equations in two unknowns $x$ and y:

| (a) | (b) | (c) |
|---|---|---|
| $x - y = -3$<br>$x + 2y = 3$ | $x + y = 1$<br>$2x + 2y = 6$ | $x + y = 1$<br>$3x + 3y = 3$ |

We study before how to solve these systems, by:

1) graph 2) substitution 3) elimination




(a) The system has exactly one solution.

(b) The system has no solutionns.

(c)The system has an infinite number of solutions.

In **general a system of *m* equations** in $n$ unknowns $x_1, x_2, x_3, \ldots, x_n$ is of the form

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \quad \ldots\ldots\ldots\ldots(1)$$

The numbers $a_{ij}$ are called the **coefficients** of $x_j$ and $b_i$ is called the **constant term** for each $i$.

<u>**Solution to a linear system**</u> is a sequence of ***n*** numbers $s_1, _2, \cdots, s_n$, which has the property that each equation in the system is satisfied when $x_1 = s_1, x_2 = s_2, \cdots, x_n = s_n$ are substituted in the system.

<u>**Definition:**</u>

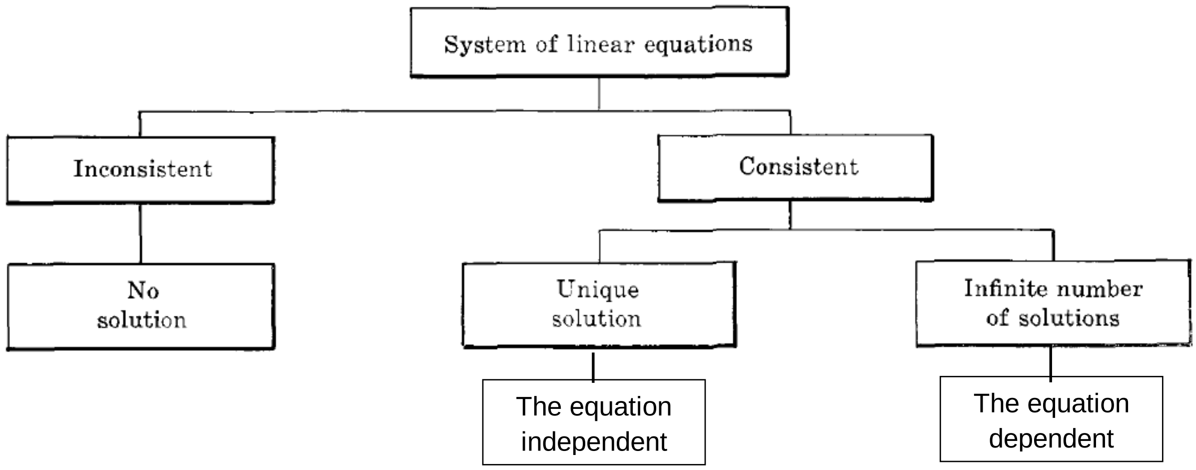
**(1)** A system which each constant term $b_i$ is zero called **homogenous system**.

The solution $x_1 = x_2 = \cdots x_n = 0$ to the homogenous system is called **trivial solution.** A solution $x_1 = s_1, _2 = s_2, \cdots x_n = s_n$ to a homogenous system in which not all the $s_i = 0$ is called **nontrivial solution**.

**(2)** It is possible for a system of linear equation to have exactly **one solution, an infinite number of solutions or no solution** if a system of equations has at least one solution, it is said to be **consistent**.

**(3)** If it has no solution, it is said to be **inconsistent**.

**(4)** If a consistent system of equations has exactly one solution, the equations of the system are said to be **independent**. If it has an infinite number of solutions, the equations are called **dependent**.




## (2.1) Represent linear system in augmented matrix

We can represent the above system of linear equations (1) by:

$$AX=B$$

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{bmatrix}_{m\times n} \qquad X=\begin{bmatrix} x_1 \\ x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$

$$B=\begin{bmatrix} b_1 \\ b_1 \\ \vdots \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

Where A is the **coefficient matrix** of the linear system (1).

The **augmented matrix** can be obtained by: [A|B] as follows:

$$C=\left[\begin{array}{ccccc|c} a_{11} & a_{12} & \cdots & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} & b_m \end{array}\right]_{m\times(n+1)}$$

## Example:

(1) Express the following system as AX = B and find the augmented matrix.

$$2x + 3y - 4z = 5$$
$$3x + 4y - 5z = 6$$
$$5x - 6z = 7$$

**Solution:**

$$\begin{bmatrix} 2 & 3 & -4 \\ 3 & 4 & -5 \\ 5 & 0 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

The augmented matrix is: $\left[\begin{array}{ccc|c} 2 & 3 & -4 & 5 \\ 3 & 4 & -5 & 6 \\ 5 & 0 & -6 & 7 \end{array}\right]$

**(2)** The matrix $\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 2 & 0 & 1 & 1 \\ 3 & 1 & 0 & 1 \end{array}\right]$ is the augmented matrix of the following linear system

$$\begin{aligned} x + 3y - z &= 0 \\ 2x \qquad + z &= 1 \\ 3x + y \qquad &= 1 \end{aligned}$$

**Theorem:**

**(1)** A homogenous system of $m$ equations in $n$ unknown always has a nontrivial solution if $m<n$.

**(2)** If $A$ is an $n\times n$ matrix, then the homogenous system $AX=0$ has a nontrivial solution $\Leftrightarrow$ $A$ is singular $\Leftrightarrow$ $|A| = 0$.

## (2.2) Solution of linear system by using the Gauss-Jordan method

**Definition**: a system is said to b in **strict triangular form** if in the kth equation the coefficients of the first k-1 variables are all zero and the coefficient of $x_k$ is nonzero (k=1,…..,n)

**Example: Solve the following systems**

**(1)** $\mathbf{3x_1 + 2x_2 + x_3 = 1}$

$\mathbf{x_2 - x_3 = 2}$

$\mathbf{2x_3 = 4}$

**Solution:** using back substitution, we obtain:

$\therefore x_3 = 2$

$\therefore x_2 - 2 = 2 \rightarrow x_2 = 4$

Then

$3x_1 + 2.4 + 2 = 1 \rightarrow x_1 = -3$

The solution of the given system is (-3, 4, 2)

**(2)** $\mathbf{2x_1 - x_2 + 3x_3 - 2x_4 = 1}$

$\mathbf{x_2 - 2x_3 + 3x_4 = 2}$

$\mathbf{4x_3 + 3x_4 = 3}$

$\mathbf{4x_4 = 4}$

**Solution:** using back substitution, we obtain:

$4x_4 = 4 \rightarrow x_4 = 1$

$4x_3 + 3.1 = 3 \rightarrow x_3 = 0$

$x_2 - 2.0 + 3.1 = 2 \rightarrow x_2 = -1$

$2x_1 — 1 + 3.0 - 2.1 = 1 \rightarrow x_1 = 1$

The solution is (1,0,-1,1)

**Definition:**

The two linear systems are equivalent if and only if they have the same solution set.

**Theorem**

Let **Ax=b** and **Cx=d** be two linear systems, each of m equations in n unknowns .if the augmented matrices **[A:b]** and **[C:d]** are row equivalent

,then the linear systems are equivalent ;that is ,they have exactly the same solutions.

**Corollary**

If A and C are row equivalent m×n matrices, then the homogenous system Ax=0 and Cx=0 are equivalent.

**Remark**

1-The set of solutions to this system gives precisely the set of solutions to Ax=b; that is, the linear systems Ax=b and Cx=d are equivalent.

2-The method where [C:d] is in row echelon form is called **Gaussian elimination.**

3-The method where [C:d] is in reduced row echelon form is called **Gauss-Jordan reduction.**

**Gaussian elimination consists of two steps:**

**Step 1.** The transformation of the augmented matrix [A:b] to the matrix [C:d] in row echelon form using elementary row operations.

**Step 2.** Solution of the linear system corresponding to the augmented matrix [C:d] using **back substitution**.

4-For the case in which A is in n×n, and the linear system **Ax=b** has a unique solution, the matrix **[C:d]** has the following form:

$$\begin{bmatrix} 1 & c_{12} & c_{13} & \cdots & c_{1n} & \vdots\, d_1 \\ 0 & 1 & c_{23} & \cdots & c_{2n} & \vdots\, d_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & c_{n-1\,n} & \vdots\, d_{n-1} \\ 0 & 0 & 0 & 0 & 1 & \vdots\, d_n \end{bmatrix}$$

The final matrix is called **the row-echelon form**.

This augmented matrix represents the linear system:

$$x_1 + c_{12}x_2 + c_{13}x_3 + \cdots + c_{1n}x_n = d_1$$

$$x_2 + c_{23}x_3 + \cdots + c_{2n}x_n = d_2$$

.

.

$$x_{n-1} + c_{n-1\,n}x_n = d_{n-1}$$

$$x_n = d_n$$

Back substitution proceeds from the nth equation upward, solving for one variable from each equation:

$$x_n = d_n$$

$$x_{n-1} = d_{n-1} = c_{n-1\,n}x_n$$

.

.

$$x_2 = d_2 - c_{23}x_3 - \cdots - c_{2n}x_n$$

$$x_1 = d_1 - c_{12}x_2 - c_{13}x_3 - \cdots - c_{1n}x_n$$

**Remark**

1- If $d_{k+1} = 1$ then

$\text{Cx} = \text{d has no solution , since at least one equation is not satisfied}$

2- If $d_{k+1} = 0 \; which\ implies\ that\ d_{k+2} = \cdots \ldots = \; d_m = 0$

This merely indicates that Cx=d has infinitely many solution.

3- Every unknown may have a determined value, indicating that the solution is unique.

**Examples**

**1-Solve the following linear system**

**X+2y+3z=9**

**2x-y+z=8**

**3x-z=3**

**Solution**

The linear system has the augmented matrix

$$[A\!:b] = \begin{bmatrix} 1 & 2 & 3 & :9 \\ 2 & -1 & 1 & :8 \\ 3 & 0 & -1 & :3 \end{bmatrix}$$

Transforming the matrix to row echelon form, we obtain

$$[C:d] = \begin{bmatrix} 1 & 2 & 3 & :9 \\ 2 & 1 & 1 & :2 \\ 0 & 0 & 1 & :3 \end{bmatrix}$$

Using back substitution, we now have

z=3

y=2-z=2-3=-1

x=9-2y-3z=9+2-9=2

Thus the solution is x=2, y=-1, z=3 which is unique.

**2- Let [C:d] =** $\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & :\mathbf{6} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{-1} & :\mathbf{7} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & :\mathbf{7} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{2} & :\mathbf{9} \end{bmatrix}$

then $\mathrm{x}_4 = 9 - 2x_5$

$x_3 = 7 - 2x_4 - 3x_5 = -11 + x_5$

$x_2 = 7 - 2x_3 - 3x_4 + x_5 = 2 + 5x_5$

$x_1 = 6 - 2x_2 - 3x_3 - 4x_4 - 5x_5 = -1 - 10x_5$

$x_5 = any\ real\ number$

The system is consistent and all solutions are of the form

$x_1 = -1 - 10r$

$x_2 = 2 + 5r$

$x_3 = -11 + r$

$\mathrm{x}_4 = 9 - 2r$

$x_5 = r\ any\ real\ number$

The given linear system has infinitely many solutions

**3- If [C:d] =** $\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & :\mathbf{5} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & :\mathbf{6} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$

Cx=d has no solution since the last equation is

$0x_1 + 0x_2 + 0x_3 + 0x_4 = 1$

0=1

H.W.

**Solve the system**

**1) x+2y+3z=6**

**2x-3y+2z=14**

**3x+y-z=-2**

2) **x+y+z+w=0**

**x+y=0**

**x+2y+z=0**

**<u>Remark</u>**

The following row operations give the equivalent linear systems.

**Row Operations:**

1. Any two rows in the augmented matrix may be interchanged.
2. Any row may be multiplied by a non-zero constant.
3. A constant multiple of a row may be added to another row.

One can easily see that these three row operation may make the system look different, but they do not change the solution of the system.

Gauss-Jordan method: this method reduces the system into a series of equivalent systems by employing the row operations. This row reduction continues until the system is expressed in what is called the reduced row echelon form as follows:

1. Write the augmented matrix.
2. Interchange rows if necessary to obtain a non-zero number in the first row, first column.
3. Use a row operation to make the entry in $a_{11}$= 1.
4. Use row operations to make all other entries as zeros in column one, i.e $a_{21}$=0 $a_{23}$= 0,…

5. Interchange rows if necessary to obtain a nonzero number in $a_{22}$. Use a row operation to make this entry 1. Use row operations to make all other entries as zeros in column two.

6. Repeat step 5 for row 3, column 3. Continue moving along the main diagonal until you reach the last row, or until the number is zero.

(7) All zero rows, if there are any; appear at the bottom of the matrix

(8) If a column contains a leading one, then all other entries in that column are zero.

The final matrix is called **the reduced row-echelon form**.

**Example:**

Use Gauss-Jordan method to solve the following linear system:

$$x + 3y = 7$$

$$3x + 4y = 11$$

**Solution:**

The augmented matrix for the system is as follows

$$\begin{bmatrix} 1 & 3 & | & 7 \\ 3 & 4 & | & 11 \end{bmatrix}$$

To make the position of 3 in the second row = 0, we multiply the first row by -3, and add to the second row

$$\begin{bmatrix} 1 & 3 & | & 7 \\ 0 & -5 & | & -10 \end{bmatrix}$$

To make the position of -5 =1, we divide the second row by -5, we get

$$\begin{bmatrix} 1 & 3 & | & 7 \\ 0 & 1 & | & 2 \end{bmatrix}$$

Finally, to make the position of 3 = 0, we multiply the second row by - 3 and add to the first row, and we get

$$\begin{bmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & 2 \end{bmatrix} \begin{bmatrix} x = 1 \\ y = 2 \end{bmatrix}$$