Third Lesson

3.3 Exact Differential Equations

The differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

is ***exact*** on an open rectangle R if there's a function $F = F(x,y)$ such Fx and Fy are continuous, and

$$F_x(x,y) = M(x,y) \text{ and } F_y(x,y) = N(x,y)$$

If and only if $M_y(x,y) = N_x(x,y)$ for all (x,y) in R

Theorem 2.5.1 *If $F = F(x, y)$ has continuous partial derivatives F_x and F_y, then*

$$F(x, y) = c \qquad (c\text{=constant}),$$

is an implicit solution of the differential equation

$$F_x(x, y)\, dx + F_y(x, y)\, dy = 0.$$

Procedure For Solving An Exact Equation

Step 1. Check that the equation

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{2.5.19}$$

satisfies the exactness condition $M_y = N_x$. If not, don't go further with this procedure.

Step 2. Integrate

$$\frac{\partial F(x, y)}{\partial x} = M(x, y)$$

with respect to x to obtain

$$F(x, y) = G(x, y) + \phi(y), \tag{2.5.20}$$

where G is an antiderivative of M with respect to x, and ϕ is an unknown function of y.

Step 3. Differentiate (2.5.20) with respect to y to obtain

$$\frac{\partial F(x, y)}{\partial y} = \frac{\partial G(x, y)}{\partial y} + \phi'(y).$$

Step 4. Equate the right side of this equation to N and solve for ϕ'; thus,

$$\frac{\partial G(x, y)}{\partial y} + \phi'(y) = N(x, y), \quad \text{so} \quad \phi'(y) = N(x, y) - \frac{\partial G(x, y)}{\partial y}.$$

Step 5. Integrate ϕ' with respect to y, taking the constant of integration to be zero, and substitute the result in (2.5.20) to obtain $F(x, y)$.

Step 6. Set $F(x, y) = c$ to obtain an implicit solution of (2.5.19). If possible, solve for y explicitly as a function of x.

Example Solve

$$(4x^3y^3 + 3x^2)\,dx + (3x^4y^2 + 6y^2)\,dy = 0.$$

Solution (Method 1) Here

$$M(x, y) = 4x^3y^3 + 3x^2, \quad N(x, y) = 3x^4y^2 + 6y^2,$$

and

$$M_y(x, y) = N_x(x, y) = 12x^3y^2$$

for all (x, y). Therefore ‘ there’s a function F such that

$$F_x(x, y) = M(x, y) = 4x^3y^3 + 3x^2 \tag{2.5.14}$$

and

$$F_y(x, y) = N(x, y) = 3x^4y^2 + 6y^2 \tag{2.5.15}$$

for all (x, y). To find F, we integrate (2.5.14) with respect to x to obtain

$$F(x, y) = x^4y^3 + x^3 + \phi(y), \tag{2.5.16}$$

where $\phi(y)$ is the “constant” of integration. (Here ϕ is “constant” in that it’s independent of x, the variable of integration.) If ϕ is any differentiable function of y then F satisfies (2.5.14). To determine ϕ so that F also satisfies (2.5.15), assume that ϕ is differentiable and differentiate F with respect to y. This yields

$$F_y(x, y) = 3x^4y^2 + \phi'(y).$$

Comparing this with (2.5.15) shows that

$$\phi'(y) = 6y^2.$$

We integrate this with respect to y and take the constant of integration to be zero because we’re interested only in finding *some* F that satisfies (2.5.14) and (2.5.15). This yields

$$\phi(y) = 2y^3.$$

Substituting this into (2.5.16) yields

$$F(x, y) = x^4y^3 + x^3 + 2y^3. \tag{2.5.17}$$

Now Theorem 2.5.1 implies that

$$x^4y^3 + x^3 + 2y^3 = c$$

is an implicit solution of (2.5.13). Solving this for y yields the explicit solution

$$y = \left(\frac{c - x^3}{2 + x^4}\right)^{1/3}.$$

Example 2.5.4 Solve the equation

$$(ye^{xy}\tan x + e^{xy}\sec^2 x)\,dx + xe^{xy}\tan x\,dy = 0. \tag{2.5.21}$$

Solution We leave it to you to check that $M_y = N_x$ on any open rectangle where $\tan x$ and $\sec x$ are defined. Here we must find a function F such that

$$F_x(x, y) = ye^{xy}\tan x + e^{xy}\sec^2 x \tag{2.5.22}$$

and

$$F_y(x, y) = xe^{xy}\tan x. \tag{2.5.23}$$

It's difficult to integrate (2.5.22) with respect to x, but easy to integrate (2.5.23) with respect to y. This yields

$$F(x, y) = e^{xy}\tan x + \psi(x). \tag{2.5.24}$$

Differentiating this with respect to x yields

$$F_x(x, y) = ye^{xy}\tan x + e^{xy}\sec^2 x + \psi'(x).$$

Comparing this with (2.5.22) shows that $\psi'(x) = 0$. Hence, ψ is a constant, which we can take to be zero in (2.5.24), and

$$e^{xy}\tan x = c$$

is an implicit solution of (2.5.21). ■

Example 2.5.5 Verify that the equation

$$3x^2y^2\,dx + 6x^3y\,dy = 0 \tag{2.5.25}$$

is not exact, and show that the procedure for solving exact equations fails when applied to (2.5.25).

Solution Here

$$M_y(x, y) = 6x^2y \quad \text{and} \quad N_x(x, y) = 18x^2y,$$

so (2.5.25) isn't exact. Nevertheless, let's try to find a function F such that

$$F_x(x, y) = 3x^2y^2 \tag{2.5.26}$$

and

$$F_y(x, y) = 6x^3y. \tag{2.5.27}$$

Integrating (2.5.26) with respect to x yields

$$F(x, y) = x^3y^2 + \phi(y),$$

and differentiating this with respect to y yields

$$F_y(x, y) = 2x^3y + \phi'(y).$$

For this equation to be consistent with (2.5.27),

$$6x^3y = 2x^3y + \phi'(y),$$

or

$$\phi'(y) = 4x^3y.$$

This is a contradiction, since ϕ' must be independent of x. Therefore the procedure fails.

Exercises

In Exercises 1–17 determine which equations are exact and solve them.

1. $6x^2y^2\,dx + 4x^3y\,dy = 0$
2. $(3y\cos x + 4xe^x + 2x^2e^x)\,dx + (3\sin x + 3)\,dy = 0$
3. $14x^2y^3\,dx + 21x^2y^2\,dy = 0$
4. $(2x - 2y^2)\,dx + (12y^2 - 4xy)\,dy = 0$
5. $(x + y)^2\,dx + (x + y)^2\,dy = 0$
6. $(4x + 7y)\,dx + (3x + 4y)\,dy = 0$
7. $(-2y^2\sin x + 3y^3 - 2x)\,dx + (4y\cos x + 9xy^2)\,dy = 0$
8. $(2x + y)\,dx + (2y + 2x)\,dy = 0$
9. $(3x^2 + 2xy + 4y^2)\,dx + (x^2 + 8xy + 18y)\,dy = 0$
10. $(2x^2 + 8xy + y^2)\,dx + (2x^2 + xy^3/3)\,dy = 0$
11. $\left(\frac{1}{x} + 2x\right)\,dx + \left(\frac{1}{y} + 2y\right)\,dy = 0$
12. $(y\sin xy + xy^2\cos xy)\,dx + (x\sin xy + xy^2\cos xy)\,dy = 0$
13. $\frac{x\,dx}{(x^2 + y^2)^{3/2}} + \frac{y\,dy}{(x^2 + y^2)^{3/2}} = 0$
14. $\left(e^x(x^2y^2 + 2xy^2) + 6x\right)\,dx + (2x^2ye^x + 2)\,dy = 0$
15. $\left(x^2e^{x^2+y}(2x^2 + 3) + 4x\right)\,dx + (x^3e^{x^2+y} - 12y^2)\,dy = 0$
16. $\left(e^{xy}(x^4y + 4x^3) + 3y\right)\,dx + (x^5e^{xy} + 3x)\,dy = 0$
17. $(3x^2\cos xy - x^3y\sin xy + 4x)\,dx + (8y - x^4\sin xy)\,dy = 0$

In Exercises 18–22 solve the initial value problem.

18. $(4x^3y^2 - 6x^2y - 2x - 3)\,dx + (2x^4y - 2x^3)\,dy = 0, \quad y(1) = 3$
19. $(-4y\cos x + 4\sin x\cos x + \sec^2 x)\,dx + (4y - 4\sin x)\,dy = 0, \quad y(\pi/4) = 0$
20. $(y^3 - 1)e^x\,dx + 3y^2(e^x + 1)\,dy = 0, \quad y(0) = 0$
21. $(\sin x - y\sin x - 2\cos x)\,dx + \cos x\,dy = 0, \quad y(0) = 1$
22. $(2x - 1)(y - 1)\,dx + (x + 2)(x - 3)\,dy = 0, \quad y(1) = -1$

INTEGRATING FACTORS

In general, Eq. (*5.1*) is not exact. Occasionally, it is possible to transform (*5.1*) into an exact differential equation by a judicious multiplication. A function $I(x, y)$ is an *integrating factor* for (*5.1*) if the equation

$$I(x, y)[M(x, y)\,dx + N(x, y)\,dy] = 0 \tag{5.7}$$

is exact. A solution to (*5.1*) is obtained by solving the exact differential equation defined by (*5.7*). Some of the more common integrating factors are displayed in Table 5-1 and the conditions that follow:

If $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) \equiv g(x)$, a function of x alone, then

$$I(x, y) = e^{\int g(x)dx} \tag{5.8}$$

If $\dfrac{1}{M}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) \equiv h(y)$, a function of y alone, then

$$I(x, y) = e^{-\int h(y)dy} \tag{5.9}$$

5.21. **Convert $y' = 2xy - x$ into an exact differential equation.**

Rewriting this equation in differential form, we have

$$(-2xy + x)dx + dy = 0 \tag{1}$$

Here $M(x, y) = -2xy + x$ and $N(x, y) = 1$. Since

$$\frac{\partial M}{\partial y} = -2x \quad \text{and} \quad \frac{\partial N}{\partial x} = 0$$

are not equal, (*1*) is not exact. But

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{(-2x) - (0)}{1} = -2x$$

is a function of x alone. Using Eq. (*5.8*), we have $I(x, y) = e^{\int -2x\,dx} = e^{-x^2}$ as an integrating factor. Multiplying (*1*) by e^{-x^2}, we obtain

$$(-2xye^{-x^2} + xe^{-x^2})\,dx + e^{-x^2}dy = 0 \tag{2}$$

which is exact.

5.22. **Convert $y^2\,dx + xy\,dy = 0$ into an exact differential equation.**

Here $M(x, y) = y^2$ and $N(x, y) = xy$. Since

$$\frac{\partial M}{\partial y} = 2y \quad \text{and} \quad \frac{\partial N}{\partial x} = y$$

are not equal, (*1*) is not exact. But

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2y - y}{y^2} = \frac{1}{y}$$

is a function of y alone. Using Eq. (*5.9*), we have as an integrating factor $I(x, y) = e^{-\int (1/y)dy} = e^{-\ln y} = 1/y$. Multiplying the given differential equation by $I(x, y) = 1/y$, we obtain the exact equation $y\,dx + x\,dy = 0$.

5.23. Convert $y' = \dfrac{xy^2 - y}{x}$ into an exact differential equation.

Rewriting this equation in differential form, we have

$$y(1 - xy)\,dx + x\,dy = 0 \qquad (1)$$

Here $M(x, y) = y(1 - xy)$ and $N(x, y) = x$. Since

$$\frac{\partial M}{\partial y} = 1 - 2xy \quad \text{and} \quad \frac{\partial N}{\partial x} = 1$$

are not equal, (*1*) is not exact. Equation (*5.10*), however, is applicable and provides the integrating factor

$$I(x, y) = \frac{1}{x[y(1 - xy)] - yx} = \frac{-1}{(xy)^2}$$

Multiplying (*1*) by $I(x, y)$, we obtain

$$\frac{xy - 1}{x^2 y}dx - \frac{1}{xy^2}dy = 0$$

which is exact.

3.4 Linear First Order Differential Equation

METHOD OF SOLUTION

A first-order *linear* differential equation has the form

$$y' + p(x)y = q(x) \qquad (6.1)$$

An integrating factor for Eq. (*6.1*) is

$$I(x) = e^{\int p(x)\,dx} \qquad (6.2)$$

which depends only on x and is independent of y. When both sides of (*6.1*) are multiplied by $I(x)$, the resulting equation

$$I(x)y' + p(x)I(x)y = I(x)q(x) \qquad (6.3)$$

is exact. This equation can be solved by the method described in Chapter 5. A simpler procedure is to rewrite (*6.3*) as

$$\frac{d(yI)}{dx} = Iq(x)$$

integrate both sides of this last equation with respect to x, and then solve the resulting equation for y.

6.1. Find an integrating factor for $y' - 3y = 6$.

The differential equation has the form of Eq. (*6.1*), with $p(x) = -3$ and $q(x) = 6$, and is linear. Here

$$\int p(x)\, dx = \int -3\, dx = -3x$$

so (*6.2*) becomes

$$I(x) = e^{\int p(x)\, dx} = e^{-3x} \qquad (1)$$

6.2. Solve the differential equation in the previous problem.

Multiplying the differential equation by the integrating factor defined by (*1*) of Problem 6.1, we obtain

$$e^{-3x}y' - 3e^{-3x}y = 6e^{-3x} \quad \text{or} \quad \frac{d}{dx}(ye^{-3x}) = 6e^{-3x}$$

Integrating both sides of this last equation with respect to x, we have

$$\int \frac{d}{dx}(ye^{-3x})\, dx = \int 6e^{-3x} dx$$

$$ye^{-3x} = -2e^{-3x} + c$$

$$y = ce^{3x} - 2$$

6.3. Find an integrating factor for $y' - 2xy = x$.

The differential equation has the form of Eq. (*6.1*), with $p(x) = -2x$ and $q(x) = x$, and is linear. Here

$$\int p(x)\, dx = \int (-2x)\, dx = -x^2$$

so (*6.2*) becomes

$$I(x) = e^{\int p(x)\, dx} = e^{-x^2} \qquad (1)$$

6.4. Solve the differential equation in the previous problem.

Multiplying the differential equation by the integrating factor defined by (*1*) of Problem 6.3, we obtain

$$e^{-x^2}y' - 2xe^{-x^2}y = xe^{-x^2} \quad \text{or} \quad \frac{d}{dx}[ye^{-x^2}] = xe^{-x^2}$$

Integrating both sides of this last equation with respect to x, we find that

$$\int \frac{d}{dx}(ye^{-x^2})\, dx = \int xe^{-x^2} dx$$

$$ye^{-x^2} = -\frac{1}{2}e^{-x^2} + c$$

$$y = ce^{x^2} - \frac{1}{2}$$

6.9. Solve $y' - 5y = 0$.

Here $p(x) = -5$ and $I(x) = e^{\int(-5)\,dx} = e^{-5x}$. Multiplying the differential equation by $I(x)$, we obtain

$$e^{-5x}y' - 5e^{-5x}y = 0 \quad \text{or} \quad \frac{d}{dx}(ye^{-5x}) = 0$$

Integrating, we obtain $ye^{-5x} = c$ or $y = ce^{5x}$.

Note that the differential equation is also separable.

In Problems 6.20 through 6.49, solve the given differential equations.

6.20. $\dfrac{dy}{dx} + 5y = 0$

6.21. $\dfrac{dy}{dx} - 5y = 0$

6.22. $\dfrac{dy}{dx} - 0.01y = 0$

6.23. $\dfrac{dy}{dx} + 2xy = 0$

6.24. $y' + 3x^2y = 0$

6.25. $y' - x^2y = 0$

6.26. $y' - 3x^4y = 0$

6.27. $y' + \dfrac{1}{x}y = 0$

6.28. $y' + \dfrac{2}{x}y = 0$

6.29. $y' - \dfrac{2}{x}y = 0$

6.30. $y' - \dfrac{2}{x^2}y = 0$

6.31. $y' - 7y = e^x$

6.32. $y' - 7y = 14x$

6.33. $y' - 7y = \sin 2x$

6.34. $y' + x^2y = x^2$

6.35. $y' - \dfrac{3}{x^2}y = \dfrac{1}{x^2}$

6.36. $y' = \cos x$

6.37. $y' + y = y^2$

6.38. $xy' + y = xy^3$

6.39. $y' + xy = 6x\sqrt{y}$

6.40. $y' + y = y^2$

6.41. $y' + y = y^{-2}$

6.42. $y' + y = y^2e^x$

6.43. $\dfrac{dy}{dt} + 50y = 0$

6.44. $\dfrac{dz}{dt} - \dfrac{1}{2t}z = 0$

6.45. $\dfrac{dN}{dt} = kN$, $(k = \text{a constant})$

6.46 $\dfrac{dp}{dt} - \dfrac{1}{t}p = t^2 + 3t - 2$

6.47. $\dfrac{dQ}{dt} + \dfrac{2}{20-t}Q = 4$