Method of solution of Ordinary differential equations

Second Lesson

1. **Construction of differential equation by elementrary aribitrary constant**

Let us have the following n-th order differential equation:

$$f(x, y, y', y'', \cdots, y(n)) = 0$$

This equation has a general solution containing n arbitrary constants:

$$\varphi(x, y, c_1, c_2, \cdots, c_n) = 0$$

On the other hand, by knowing the general solution, we can construct the differential equation for this solution in the following way:

1. We differentiate the general solution with the number of constants, i.e. we differentiate the general solution n times and thus we get n equations.

2. We eliminate the constants by jointly solving the previous equations.

Thus, we get the required differential equation. This can be illustrated by the following example:

Example 1: Find the differential equation that has the following general solution:

$$y = c_1 x + c_2 x_2 \qquad (1)$$

Solution: To find the differential equation, we follow the following steps:

1. We derive the general solution with the number of constants, i.e. we derive the general solution twice because there are two constants c1 and c2, and thus we get the two equations:

$$y' = c_1 + 2c_2 x \qquad (2)$$

$$y'' = 2c_2 \qquad (3)$$

2. We eliminate the constants by jointly solving the two previous equations as follows:

From equation (3), we find that:

$$c_2 = \frac{y''}{2} \qquad (4)$$

Substituting in equation (2), we find:

$$y' = c_1 + 2\frac{y''}{2}x$$

$$\Rightarrow y' = c_1 + y''x$$

$$\Rightarrow c_1 = y' - y''x \qquad (5)$$

Substitute (4) and (5) in the general solution (1) We find:

$$y = (y' - y''x)x + \frac{y''}{2}x^2$$

$$\Rightarrow y = y'x - y''x^2 + \frac{1}{2}y''x^2$$

$$\Rightarrow y = y'x - \frac{1}{2}y''x^2$$

$$\Rightarrow y = y'x - \frac{1}{2}y''x^2$$

This is the required differential equation.

Example 2: Find the differential equation whose general solution is $y = c_1e^{2x} + c_2e^{3x}$

Solution : In this case, to make the solution easier, we use the determinant using the following steps:

In the determinant, we put the first row as the function y, and in the second row we put the first derivative of the function y', and in the third row we put the second derivative of the function y"

$$\begin{vmatrix} y & e^{2x} & e^{3x} \\ y' & 2e^{2x} & 3e^{3x} \\ y'' & 4e^{2x} & 9e^{3x} \end{vmatrix} = 0$$

$$\begin{vmatrix} y & 1 & 1 \\ y' & 2 & 3 \\ y'' & 4 & 9 \end{vmatrix} = 0$$

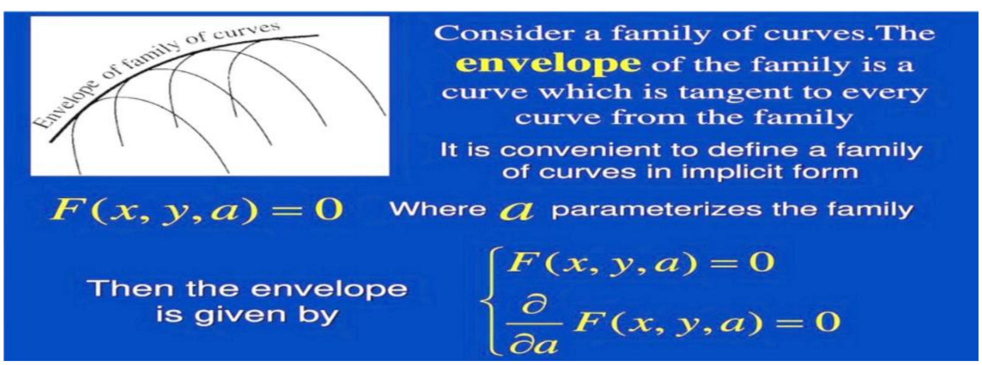
$$y\begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} - 1\begin{vmatrix} y' & 3 \\ y'' & 9 \end{vmatrix} + \begin{vmatrix} y' & 2 \\ y'' & 4 \end{vmatrix} = 0$$

$$y[18 - 12] - [9y' - 3y''] + 4y' - 2y'' = 0$$

$6y - 9y' + 3y'' + 4y' - 2y'' = 0$, which is the differential equation.

Exercise : Find the differential equation whose general solution is $y = c_1e^{2x} + c_2e^{-x} + x$

2. Envelope



A curve which touches each member of a given family of curves is called envelope of that family.

Procedure to find envelope for the given family of curves:

Case 1: Envelope of one parameter family of curves Let us consider y = f(x,α) to be the given family of curves with ‘α’ as the parameter.

Step 1: Differentiate w.r.t to the parameter α partially, and find the value of the parameter

Step 2: By substituting the value of parameter α in the given family of curves, we get the required envelope.

Case 2: Envelope of two parameter family of curves.

Let us consider $y = f(x, \alpha, \beta)$ to be the given family of curves, and a relation connecting the two parameters α and $\beta, g(\alpha, \beta) = 0$

Step 1: Consider α as independent variable and β depends α . Differentiate $y = f(x, \alpha, \beta)$ and $g(\alpha, \beta) = 0$, w.r. to the parameter α partially.

Step 2: Eliminating the parameters α, β from the equations resulting from step 1 and g(α, β) = 0, we get the required envelope.

Example for case1 : Find the envelope of $y = mx + am^p$ where m is the parameter and a, p are constants

Solution : Differentiate $y = mx + am^p$ (1) with respect to the parameter m, we get,

$0 = x + pam^{p-1}$

$\Rightarrow m = \left(\frac{-x}{pa}\right)^{\frac{1}{p-1}}$ (2)

Using (2) eliminate m from (1)

$y = \left(\frac{-x}{pa}\right)^{\frac{1}{p-1}} x + a \left(\frac{-x}{pa}\right)^{\frac{1}{p-1}}$

$\Rightarrow y^{p-1} = \left(\frac{-x}{pa}\right) x^{p-1} + a^{p-1} \left(\frac{-x}{pa}\right)^{p}$

i.e. $ap^p y^{p-1} = -x^p p^{p-1} + (-x)^p$

which is the required equation of envelope of (1).

Example for case 2 : Find the envelope of family of straight lines $ax + by = 1$, where a and b are parameters connected by the relation ab = 1

Solution : $ax + by = 1$ (1)

ab = 1 (2)

Differentiating (1) with respect to a (considering 'a' as independent variable and 'b' depends on a).

$x + \frac{db}{da} y = 0$

i.e. $\frac{db}{da} = -\frac{x}{y}$ (3)

Differentiating (2) with respect to a

$b + a\frac{db}{da} = 0$

i.e. $\frac{db}{da} = -\frac{b}{a}$ (4)

From (3) and (4), we have

$\frac{x}{y} = \frac{b}{a}$

i.e. $\frac{ax}{1} = \frac{by}{1} = \frac{ax+by}{2} = \frac{1}{2}$

$\therefore a = \frac{1}{2x}$ and $b = \frac{1}{2y}$ (5)

Using (5) in (2), we get the envelope as 4xy = 1

Exercises :

1. Determine the envelope of $n\theta - y\,cos\theta = a\theta$, where θ being the parameter.
2. Find the envelope of family of straight lines $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$, where a and b are parameters connected by the relation $\sqrt{a} + \sqrt{b} = 1$

3. First Order Ordinary Differential Equations

The complexity of solving de's increases with the order. We begin with first order de's.

3.1 Separable Equations

A first order ode has the form $F(x, y, y') = 0$. In theory, at least, the methods of algebra can be used to write it in the form (We use the notation $\frac{dy}{dx} = G(x, y)$ and $dy = G(x, y)dx$ interchangeably)
$y' = G(x, y)$. If $G(x, y)$ can be factored to give $G(x, y) = M(x)N(y)$, then the equation is called ***separable***. To solve the separable equation $y' = M(x)N(y)$, we rewrite it in the form $f(y)y' = g(x)$. Integrating both sides gives $\int f(y)y'dx = \int g(x)dx$,

$$\int f(y)\,dy = \int f(y)\frac{dy}{dx}\,dx .$$

Example 1. Solve $2xy + 6x + (x^2 - 4)\,y' = 0$

Solution. Rearranging, we have

$(x^2 - 4)\,y' = -2xy - 6x,$
$= -2x(y + 3),$
$\frac{y'}{y+3} = -\frac{2x}{x^2-4}, x \neq \pm 2$
$\ln(|y + 3|) = -\ln(|x^2 - 4|) + C,$
$ln(|y + 3|) + ln(|x^2 - 4|) = C,$
where C is an arbitrary constant. Then
$|(y + 3)(x^2 - 4)| = A,$
$(y + 3)(x^2 - 4) = A,$
$y + 3 = A/(x^2 - 4)$,
where A is a constant (equal to $\pm e^C$) and $x \neq \pm 2$. Also $y = -3$ is a solution (corresponding to $A = 0$) and the domain for that solution is R.

Example 2. Solve the IVP $sin(x)\,dx + y\,dy = 0, where\ y(0) = 1.$
Solution. Note: $sin(x)\,dx + y\,dy = 0$ is an alternate notation meaning the same as $\sin(x) + y\,dy/dx = 0.$
We have

$y\,dy = -\sin(x)\,dx,$
$\int y\,dy = \int -\sin(x)\,dx\,,$
$\frac{y^2}{2} = cos(x) + C_1,$
$y = \sqrt{2\,cos(x) + C_2},$
where C_1 is an arbitrary constant and $C_2 = 2C_1$. Considering $y(0) = 1$, we have
$1 = \sqrt{2 + C_2} \Rightarrow 1 = 2 + C_2 \Rightarrow C_2 = -1.$
Therefore, $y = \sqrt{2\,cos(x) - 1}$ on the domain $(-\pi/3, \pi/3)$, since we need $\cos(x) \geq$ 1/2 and $\cos(\pm \pi/3) = ½$

Example 3. Solve $y^4 y' + y' + x^2 + 1 = 0.$
Solution. We have
$y^4 + 1\, y' = -x^2 - 1,$
$\frac{Y^5}{5} + y = -\frac{x^3}{3} - x + C$
where C is an arbitrary constant. This is an implicit solution which we cannot easily solve explicitly for y in terms of x.

3.2 Homogeneous Equations

Definition (Homogeneous function of degree *n*)

A function $F(x, y)$ is called *homogeneous of degree n* if $F(\lambda x, \lambda y) = \lambda^{\boldsymbol{n}} F(x, y)$. For a polynomial, homogeneous says that all of the terms have the same degree.

Example 1: The following are homogeneous functions of various degrees:

$3x^6 + 5x^4y^2$ $3x^6 + 5x^3y^2$	,homogeneous of degree ,not homogeneous
$x\sqrt{x^2 + y^2}$	, homogeneous of degree 2
$\sin\left(\frac{y}{x}\right)$	, homogeneous of degree 0
$\frac{1}{x+y}$	, homogeneous of degree -1. $*$

If F is homogeneous of degree n and G is homogeneous of degree k, then F/G is homogeneous of degree $n - k$.

Proposition

If F is homogeneous of degree 0, then F is a function of y/x.
Proof. We have $F(\lambda x, \lambda y) = F(x, y)$ for all λ. Let $\lambda = 1/x$. Then $F(x, y) = F(1, y/x)$.

Procedure Consider $M(x, y)\,dx + N(x, y)\,dy = 0$. Suppose M and N are both homogeneous and of the same degree. Then

$\frac{dy}{dx} = -\frac{M}{N}$
This suggests that $v = y/x$ (or equivalently, $y = vx$) might help. In fact, write
$-\frac{M(x,y)}{N(x,y)} = R\left(\frac{y}{x}\right).$
Then
$\frac{dy}{dx} = R\left(\frac{y}{x}\right) = R(v)$
$v + x\,dv\,dx$

Therefore,

$$x\frac{dv}{dx} = R(v) - v,$$

$$\frac{dv}{R(v) - v} = \frac{dx}{x}$$

which is separable. We conclude that if *M* and *N* are homogeneous of the same degree, setting *y* = *vx* will give a separable equation in *v* and *x*.

Example 2. Solve $xy^2\,dy = x^3 + y^3\,dx$.

Solution. Let *y* = *vx*. Then *dy* = *v dx* + *x dv*, and our equation becomes

$$xv^2x^2\,(v\,dx + x\,dv) = x^3 + v^3x^2\,dx,$$

$$x^3v^3\,dx + x^4v^2\,dv = x^3\,dx + v^3x^3\,dx.$$

Therefore, $x = 0$ or $v^2\,dv = dx/x$. So we have

$$\frac{v^3}{3} = ln(|x|) + C = ln(|x|) + \underbrace{ln(|A|)}_{C} = ln(|Ax|) = ln(Ax)$$

where the sign of *A* is the opposite of the sign of *x*. Therefore, the general solution is

$y = x\left(3\,ln(Ax)\right)^{1/3}$, where *A* is a nonzero constant. Every *A* > 0 yields a solution on the domain (0, ∞); every *A* < 0 yields a solution on (-∞, 0).

In addition, there is the solution *y* = 0 on the domain R.

Example 3.

Solve $dy = 2t(y^2 + 9)\,dt$.

This equation may be rewritten as

$$\frac{dy}{y^2 + 9} - 2t\,dt = 0$$

which is separable in variables *y* and *t*. Its solution is

$$\int \frac{dy}{y^2 + 9} - \int 2t\,dt = c$$

or, upon evaluating the given integrals,

$$\frac{1}{3}\arctan\left(\frac{y}{3}\right) - t^2 = c$$

Solving for *y*, we obtain

$$\arctan\left(\frac{y}{3}\right) = 3(t^2 + c)$$

$$\frac{y}{3} = \tan\,(3t^2 + 3c)$$

or

$$y = 3\,\tan\,(3t^2 + \mathrm{k})$$

with $k = 3c$.

Example 4.

Solve $y' = \dfrac{2y^4 + x^4}{xy^3}$.

This differential equation is not separable. Instead it has the form $y' = f(x, y)$, with

$$f(x, y) = \frac{2y^4 + x^4}{xy^3}$$

where
$$f(tx, ty) = \frac{2(ty)^4 + (tx)^4}{(tx)(ty)^3} = \frac{t^4(2y^4 + x^4)}{t^4(xy^3)} = \frac{2y^4 + x^4}{xy^3} = f(x, y)$$

so it is homogeneous. Substituting Eqs. (*4.6*) and (*4.7*) into the differential equation as originally given, we obtain

$$v + x\frac{dv}{dx} = \frac{2(xv)^4 + x^4}{x(xv)^3}$$

which can be algebraically simplified to

$$x\frac{dv}{dx} = \frac{v^4 + 1}{v^3} \quad \text{or} \quad \frac{1}{x}dx - \frac{v^3}{v^4 + 1}dv = 0$$

This last equation is separable; its solution is

$$\int \frac{1}{x}dx - \int \frac{v^3}{v^4 + 1}dv = c$$

Integrating, we obtain in $\ln |x| - \frac{1}{4}\ln (v^4 + 1) = c$, or

$$v^4 + 1 = (kx)^4 \qquad (1)$$

where we have set $c = -\ln |k|$ and then used the identities

$$\ln |x| + \ln |k| = \ln |kx| \quad \text{and} \quad 4 \ln |kx| = \ln (kx)^4$$

Finally, substituting $v = y/x$ back into (*1*), we obtain

$$y^4 = c_1x^8 - x^4 \quad (c_1 = k^4) \qquad (2)$$

Exercise 1

A function $g(x, y)$ is *homogeneous of degree n* if $g(tx, ty) = t^n g(x, y)$ for all t. Determine whether the following functions are homogeneous, and, if so, find their degree:

(*a*) $xy + y^2$, (*b*) $x + y \sin (y/x)^2$, (*c*) $x^3 + xy^2e^{x/y}$, and (*d*) $x + xy$.

(*a*) $(tx)(ty) + (ty)^2 = t^2(xy + y^2)$; homogeneous of degree two.

(*b*) $tx + ty\sin\left(\dfrac{ty}{tx}\right)^2 = t\left[x + y\sin\left(\dfrac{y}{x}\right)^2\right]$; homogeneous of degree one.

(*c*) $(tx)^3 + (tx)(ty)^2e^{tx/ty} = t^3(x^3 + xy^2e^{x/y})$; homogeneous of degree three.

(*d*) $tx + (tx)(ty) = tx + t^2xy$; not homogeneous.

In Problems 4.23 through 4.45, solve the given differential equations or initial-value problems.

4.23. $x\,dx + y\,dy = 0$

4.24. $x\,dx - y^3\,dy = 0$

4.25. $dx + \frac{1}{y^4}dy = 0$

4.26. $(t+1)\,dt - \frac{1}{y^2}dy = 0$

4.27. $\frac{1}{x}dx - \frac{1}{y}dy = 0$

4.28. $\frac{1}{x}dx + dy = 0$

4.29. $x\,dx + \frac{1}{y}dy = 0$

4.30. $(t^2+1)\,dt + (y^2+y)\,dy = 0$

4.31. $\frac{4}{t}dt - \frac{y-3}{y}dy = 0$

4.32. $dx - \frac{1}{1+y^2}dy = 0$

Exercise 2

In Problems 4.46 through 4.54, determine whether the given differential equations are homogenous and, if so, solve them.

4.46. $y' = \frac{y-x}{x}$

4.47. $y' = \frac{2y+x}{x}$

4.48. $y' = \frac{x^2+2y^2}{xy}$

4.49. $y' = \frac{2x+y^2}{xy}$

4.50. $y' = \frac{x^2+y^2}{2xy}$

4.51. $y' = \frac{2xy}{y^2-x^2}$

4.52. $y' = \frac{y}{x+\sqrt{xy}}$

4.53. $y' = \frac{y^2}{xy+(xy^2)^{1/3}}$

4.54. $y' = \frac{x^4+3x^2y^2+y^4}{x^3y}$