Fourth Lesson

2.5.1 Bernoulli Equation

The Bernoulli equation is given by

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

Let $z = y^{1-n}$. Then

$$\frac{dz}{dx} = (1-n)\,y^{-n}\frac{dy}{dx},$$

giving us

$$\begin{aligned} y^{-n}\frac{dy}{dx} + P(x)y^{1-n} &= Q(x), \\ \frac{1}{1-n}\frac{dz}{dx} + P(x)z &= Q(x), \\ \frac{dz}{dx} + (1-n)\,P(x)z &= (1-n)\,Q(x), \end{aligned}$$

which is linear in z.

Example 2.17. Solve $y' + xy = xy^3$.

Solution. Here, we have $n = 3$. Let $z = y^{-2}$. If $y \neq 0$, then

$$\frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}.$$

Therefore, our equation becomes

$$\begin{aligned} -\frac{y^3 z'}{2} + xy &= xy^3, \\ -\frac{z'}{2} + xy^{-2} &= x, \\ z' - 2xy &= -2x. \end{aligned}$$

We can readily see that $I = e^{-\int 2x\,dx} = e^{-x^2}$. Thus,

$$\begin{aligned} e^{-x^2}z' - 2xe^{-x^2} &= -2xe^{-x^2}, \\ e^{-x^2}z &= e^{-x^2} + C, \\ z &= 1 + Ce^{x^2}, \end{aligned}$$

where C is an arbitrary constant. But $z = y^{-2}$. So

$$y = \pm\frac{1}{\sqrt{1 + Ce^{x^2}}}.$$

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6.16 Solve $y' + xy = xy^2$.

This equation is not linear. It is, however, a Bernoulli differential equation having the form of Eq. (*6.4*) with $p(x) = q(x) = x$, and $n = 2$. We make the substitution suggested by (*6.5*), namely, $z = y^{1-2} = y^{-1}$, from which follow

$$y = \frac{1}{z} \quad \text{and} \quad y' = -\frac{z'}{z^2}$$

Substituting these equations into the differential equation, we obtain

$$-\frac{z'}{z^2} + \frac{x}{z} = \frac{x}{z^2} \quad \text{or} \quad z' - xz = -x$$

This last equation is linear. Its solution is found in Problem 6.10 to be $z = ce^{x^2/2} + 1$. The solution of the original differential equation is then

$$y = \frac{1}{z} = \frac{1}{ce^{x^2/2} + 1}$$

6.17. Solve $y' - \frac{3}{4}y = x^4 y^{1/3}$.

This is a Bernoulli differential equation with $p(x) = -3/x$, $q(x) = x^4$, and $n = \frac{1}{3}$. Using Eq. (*6.5*), we make the substitution $z = y^{1-(1/3)} = y^{2/3}$. Thus, $y = z^{3/2}$ and $y' = \frac{3}{2}z^{1/2}z'$. Substituting these values into the differential equation, we obtain

$$\frac{3}{2}z^{1/2}z' - \frac{3}{x}z^{3/2} = x^4 z^{1/2} \quad \text{or} \quad z' - \frac{2}{x}z = \frac{2}{3}x^4$$

This last equation is linear. Its solution is found in Problem 6.12 to be $z = cx^2 + \frac{2}{9}x^5$. Since $z = y^{2/3}$, the solution of the original problem is given implicitly by $y^{2/3} = cx^2 + \frac{2}{9}x^5$, or explicitly by $y = \pm(cx^2 + \frac{2}{9}x^5)^{3/2}$.

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Solved Problems

In Problems 6.20 through 6.49, solve the given differential equations.

6.20. $\frac{dy}{dx} + 5y = 0$

6.21. $\frac{dy}{dx} - 5y = 0$

6.22. $\frac{dy}{dx} - 0.01y = 0$

6.23. $\frac{dy}{dx} + 2xy = 0$

6.24. $y' + 3x^2y = 0$

6.25. $y' - x^2y = 0$

6.26. $y' - 3x^4y = 0$

6.27. $y' + \frac{1}{x}y = 0$

6.28. $y' + \frac{2}{x}y = 0$

6.29. $y' - \frac{2}{x}y = 0$

6.30. $y' - \frac{2}{x^2}y = 0$

6.31. $y' - 7y = e^x$

6.32. $y' - 7y = 14x$

6.33. $y' - 7y = \sin 2x$

6.34. $y' + x^2y = x^2$

6.35. $y' - \frac{3}{x^2}y = \frac{1}{x^2}$

6.36. $y' = \cos x$

6.37. $y' + y = y^2$

6.38. $xy' + y = xy^3$

6.39. $y' + xy = 6x\sqrt{y}$

6.40. $y' + y = y^2$

6.41. $y' + y = y^{-2}$

6.42. $y' + y = y^2e^x$

6.43. $\frac{dy}{dt} + 50y = 0$

6.44. $\frac{dz}{dt} - \frac{1}{2t}z = 0$

6.45. $\frac{dN}{dt} = kN$, ($k =$ a constant)

6.46 $\frac{dp}{dt} - \frac{1}{t}p = t^2 + 3t - 2$

6.47. $\frac{dQ}{dt} + \frac{2}{20 - t}Q = 4$

Solve the following initial-value problems.

6.50. $y' + \frac{2}{x}y = x;\ y(1) = 0$

6.51. $y' + 6xy = 0;\ y(\pi) = 5$

6.52. $y' + 2xy = 2x^3;\ y(0) = 1$

6.53. $y' + \frac{2}{x}y = -x^9y^5;\ y(-1) = 2$

6.54. $\frac{dv}{dt} + 2v = 32;\ v(0) = 0$

6.55. $\frac{dq}{dt} + q = 4\cos 2t;\ q(0) = 1$

6.56. $\frac{dN}{dt} + \frac{1}{t}N = t;\ N(2) = 8$

6.57. $\frac{dT}{dt} + 0.069T = 2.07;\ T(0) = -30$

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2.8. Riccati Equations

Definition. A differential equation of the form

$$\frac{dy}{dx} + p(x)y^2 + q(x)y + r(x) = 0 \qquad (1)$$

is called Riccati differential equation.

If $p(x) \equiv 0$, then equation (1) is linear;
If $r(x) \equiv 0$, then equation (1) is Bernoulli;

If p, q and r are constants, then equation (1) is separable

$$\frac{dy}{py^2 + qy + r} = dx.$$

Theorem. If $y_1 = y_1(x)$ is a particular solution of equation (1), then substitution

$$y = y_1(x) + \frac{1}{u(x)}$$

converts the Riccati equation into a first order linear equation in u.

Example. Solve the following differential equations.
1)

$$\frac{dy}{dx} = (1-x)y^2 + (2x-1)y - x$$

Solution. We observe that the equation is Riccati and a particular solution is $y_1 = 1$. So, from the transformation

$$y = 1 + \frac{1}{u}, \; \frac{dy}{dx} = -\frac{1}{u^2}\frac{du}{dx}$$

we obtain

$$-\frac{1}{u^2}\frac{du}{dx} = (1-x)\left(1 + \frac{2}{u} + \frac{1}{u^2}\right) + (2x-1)\left(1 + \frac{1}{u}\right) - x$$

or

$$\frac{du}{dx} + u = x - 1$$

which is a first order linear differential equation. Integrating factor for linear equation is obtained as

$$\lambda(x) = e^x.$$

So, the general solution of linear equation is

$$u(x) = x - 2 + ce^{-x}.$$

Since $y = 1 + \frac{1}{u}$, general solution of given Riccati equation is obtained as

$$y = \frac{x - 1 + ce^{-x}}{x - 2 + ce^{-x}}.$$

2)

$$xy' - y^2 + (2x+1)y = x^2 + 2x.$$

3)

$$e^{-x}\frac{dy}{dx} + y^2 - 2ye^x = 1 - e^{2x}.$$

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In Exercises 56–59, given that y_1 is a solution of the given equation, use the method suggested by Exercise 55 to find other solutions.

56. $y' = 1 + x - (1 + 2x)y + xy^2; \quad y_1 = 1$

57. $y' = e^{2x} + (1 - 2e^x)y + y^2; \quad y_1 = e^x$

58. $xy' = 2 - x + (2x - 2)y - xy^2; \quad y_1 = 1$

59. $xy' = x^3 + (1 - 2x^2)y + xy^2; \quad y_1 = x$

Higher-degree first-order equations

The differential equation of first degree can write as a formula:

$$F\left(x, y, \frac{dy}{dx}\right) = 0$$

Or

$$F(x, y, p) = 0, \quad where\ p = \frac{dy}{dx}$$

Higher-degree first-order equations can be written as $F(x,y,dy/dx) = 0$. The most general standard form is

$$p^n + a_{n-1}(x,y)p^{n-2} + \cdots + a_1(x,y)p + a_o(x,y) = 0$$

1. **Equations soluble for *p***

Sometime the LHS of Equation above can be factorized into

$$(p - F_1)(p - F_2) \ldots .(p - F_n) = 0$$

where $Fi = Fi\ (x,y)$. We are then left with solving the n first-degree equations $p = F_i\ (x,y)$. Writing the solutions to these first-degree equations as $G_i\ (x, y) = 0$, the general solution to Equation above is given by the product

$$G_1(x,y)G_2(x,y) \ldots .G_n(x,y) = 0$$

Example1: Solve $(y')^3 - (y')^2 - 2y' = 0$

Sol:

Let $p = y'$, Then equation rewrite as

$$p^3 - p^2 - 2p = 0$$
$$p(p - 2)(p + 1) = 0$$
$$\therefore p = 0 \longrightarrow y = c_1$$
$$p = 2 \longrightarrow y = 2x + c_2$$
$$p = -1 \longrightarrow y = -x + c_3$$

So the general solation as

$$(y - c_1)(y - 2x - c_2)(y + x - c_3) = 0$$

Since, the differential equation is from 1st order, so the general solution must have only one arbitrary constant.

$$(y - c)(y - 2x - c)(y + x - c) = 0$$

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Example2: Solve $(x^3 + x^2 + x + 1)p^2 - (3x^2 + 2x + 1)yp - 2xy^2 = 0$

Sol.

The equation may be factorized to give

$$[(x+1)p - y][(x^2+1)p - 2xy] = 0$$

Turn each bracket in turn we have

$$(x+1)\frac{dy}{dx} - y = 0$$

$$(x^2+1)\frac{dy}{dx} - 2xy = 0$$

Which can give the solution

$$y - c(x+1) = 0 \ \& \ y - c(x^2+1) = 0$$

So, the general solution is

$$[y - c(x+1)][y - c(x^2+1)] = 0$$

2. Equations soluble for x

Equations that can be solved for x, i.e. such that they may be written in the form

$$x = F(y, p)$$

can be reduced to first-degree first-order equations in p by differentiating both sides with respect to y, so that

$$\frac{dx}{dy} = \frac{1}{p} = \frac{dF}{dy} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial y}$$

This results in an equation of the form $G(y, p) = 0$, which can be used together with $x = F(y, p)$ to eliminate p and give the general solution. Note that often a singular solution to the equation will be found at the same time

Example1: Solve $6y^2p^2 + 3xp - y = 0$

Sol.

This equation can be solved for x explicitly to give $3x = (y/p) - 6y^2p$. Differentiating both sides with respect to y, we find

$$3\frac{dx}{dy} = \frac{3}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 6y^2\frac{dp}{dy} - 12py$$

which factorizes to give

$$(1 + 6yp^2)\left(2p + y\frac{dp}{dy}\right) = 0$$

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Setting the factor containing dp/dy equal to zero gives a first-degree first-order equation in p, which may be solved to give $py^2 = c$. Substituting for p in the differential equation given then yields the general solution of this equation

بوضع الحد الذي يحتاوي dp/dy مساوي الى صفر, يعطي معادله من الدرجه الاولى ل p والتي ممكن ان تحل لتعطي $py^2 = c$. وبتعويض قيمة في المعادله التفاضليه تعطي حل عام لهذه المعادله:

$$y^3 = 3cx + 6c^2$$

If we now consider the first factor in the primary solution of the differential equation after factories, we find $6p^2y = -1$ as a possible solution. Substituting for p in the differential equation we find the singular solution

اذا اخذنا بالاعتبار العامل الاول في الحل الابتدائي للمعادله التفاضليه بعد التحليل, نحن نجد $6p^2y = -1$ كحل محتمل. وبتعويض p في المعادله التفاضليه نجد الحل المنفرد

$$8y^3 + 3x^2 = 0$$

Note that the singular solution contains no arbitrary constants and cannot be found from the general solution the differential equation by any choice of the constant c.

Solution method. *Write the equation in the form $x = F(y,p)$ and differentiate both sides with respect to y. Rearrange the resulting equation into the form G(y,p)=0 , which can be used together with the original ODE to eliminate p and so give the general solution. If G(y,p) can be factorized then the factor containing dp/dy should be used to eliminate p and give the general solution. Using the other factors in this fashion will instead lead to singular solutions.*

طريقة الحل: اكتب المعادله بالصيغة واشتق الطرفين بالنسبه ل y $x = F(y,p)$. اعد ترتيب الداله الناتجه بالصيغه $G(y,p)=0$. والتي مممكن ان تستخدام مع المعادله التفاضليه الاعتيادية لاستبعاد p وهكذا الحصول على الحل العام.
اذا $G(y,p)$ ممكن ان تحلل , ثم الحد الذي يحتوي dp/dy يجب ان يستخدام لاستبعاد p واعطاء الحل العام. باستخدام الحد الثاني بنفس الطريقة سيؤدي بدلا من ذلك إلى حلول منفرده.

3 Equations soluble for *y*

Equations that can be solved for y, i.e. such that they may be written in the form

$$y = F(x,p)$$

can be reduced to first-degree first-order equations in p by differentiating both sides with respect to x, so that

$$\frac{dy}{dx} = \frac{1}{p} = \frac{dF}{dx} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x}$$

This results in an equation of the form $G(x,p) = 0$, which can be used together with $y = F(x,p)$ to eliminate p and give the general solution. Note that often a singular solution to the equation will be found at the same time.

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Example1: Solve $xp^2 + 2xp - y = 0$

Sol.

This equation can be solved for x explicitly to give $y = xp^2 + 2xp$. Differentiating both sides with respect to x, we find

$$\frac{dy}{dx} = p = 2xp\frac{dp}{dx} + p^2 + 2x\frac{dp}{dx} + 2p$$

which factorizes to give

$$(p+1)\left(p + 2x\frac{dp}{dx}\right) = 0$$

To obtain the general solution of the differential equation, we first consider the factor containing dp/dx. This first-degree first-order equation in p has the solution $xp^2 = c$, which we then use to eliminate p from the differential equation. We therefore find that the general solution to the differential equation is

$$(y-c)^2 = 4cx.$$

If we now consider the first factor in the equation above, we find this has the simple solution $p = -1$. Substituting this into the differential equation then gives

$$x + y = 0$$

which is a singular solution to the differential equation.

UNSOLVED EXAMPLES:

Solve the following ODEs:

EXAMPLE-1: $xp^2 + x = 2yp$

EXAMPLE-2: $x(1 + p^2) = 1$

EXAMPLE-3: $x^2p^2 + xyp - 6y^2 = 0$

EXAMPLE-4: $y = px + p^3$