Chapter One

Differential Equations

Basic concepts of differential equations

A differential equation is an equation between specified derivative on an unknown function, its values, and known quantities and functions. Many physical laws are most simply and naturally formulated as differential equations (or DEs, as we will write for short). For this reason, DEs have been studied by the greatest mathematicians and mathematical physicists since the time of Newton.

Ordinary differential equations are DEs whose unknowns are functions of a single variable; they arise most commonly in the study of dynamical systems and electrical networks. They are much easier to treat that partial differential equations, whose unknown functions depend on two or more independent variables

Differential equations have many applications, including:

1- Physics: Differential equations are used in the study of motion, forces, heat, electromagnetic phenomena, vibrations, waves, alternating currents, and different frequencies.

2- Engineering: Differential equations are used in the study of geometric shapes, areas, volumes, slopes, and remote locations.

3- Chemistry: Differential equations are used in the study of chemical growth, heat transfer, electricity, and chemical equilibrium.

4- Statistics: Differential equations are used in data analysis, regression analysis, and statistical correlations.

5- Many other applications such as medical sciences, economics, financial analysis, accounting, and environmental engineering.

Definition of differential equations

Is an equation content of unknown function and derivative or an equation containing the derivatives of one or more dependent variables w.r.t. one or more independent variables.

i.e. any equation which contains derivatives, either ordinary derivatives or partial derivatives.

$F(x, y, y', y'', \dots \dots \dots y^n) = 0$…….(1)

where x is called the independent variable and y is called the dependent variable. Here are a few more examples of differential equations.

1. $2\frac{dy}{dx} + 3y = 0$
2. $y'' - 4y' + 2y = ex$
3. $\frac{d^4y}{dx^4}+2\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}=sinx$
4. $\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}=0$

The order of a differential equation

The order of a differential equation is the largest derivative present in the differential equation

In the differential equations listed above (1) is first order differential equation, (2) and (4) are second order differential equations, (3) is a fourth order differential equation.

Degree of differential equation

The degree of a differential equation is defined as the power to which the highest order derivative is raised. The equation

$(y''')^2 + (y'') + y = x$ is an example of a second-degree, third-order differential equation.

Example (1.1)

1- $(y'''')^2 + y'' = 0$ $(fourth\ order\ and\ second\ degree)$

2- $\left(\frac{dy}{dx}\right)^3 + 2ytanx = sinx$ $(first\ order\ and\ third\ degree)$

Types of differential equations

There are two types of differential equations:

1. **Ordinary Differential Equation**

A differential equation is called an ordinary differential equation, abbreviated by **ode**, if it has ordinary derivatives in it

$F(x, y, y', y'', \dots \dots \dots y^n) = 0$ …….(2)

Or $F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots\dots\dots\dots \frac{d^ny}{dx^n})$=0

Remark:-

Ordinary derivatives will be written by using either

Leibniz notation $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3} \dots\dots\dots\dots.., \frac{d^ny}{dx^n}$

Or the prime notation $y', y'', y''', y^{(4)} \dots\dots\dots.., y^{(n)}$

Examples of ode :

$1 - \ 2y' + 3y = 0$ Or $2\frac{dy}{dx} + 3y = 0$

2- $y'' - 4y' + 2y = ex$ Or $\frac{d^2y}{dx^2}$−4$\frac{dy}{dx}$+2y=ex

3- y''''+2y'''+y''=$sinx$ Or $\frac{d^4y}{dx^4}$+2$\frac{d^3y}{dx^3}$+$\frac{d^2y}{dx^2}$=$sinx$

Partial Differential Equation

a differential equation is called a partial differential equation, abbreviated by **pde**, if it has partial derivatives in it.

partial derivatives will be written as :

$\frac{\partial z}{\partial x}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^3 y}{\partial x^3}, \dots\dots\dots\dots..$

$\frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^3 z}{\partial y^3}, \dots\dots\dots\dots..$

Examples of pde :

1- $\frac{\partial^2 u}{\partial x^2}$+$\frac{\partial^2 u}{\partial y^2}$=0

2- $\left(\frac{\partial^2 u}{\partial x^2}\right)\left(\frac{\partial^2 u}{\partial y^2}\right) - \left(\frac{\partial^2 u}{\partial x \partial y}\right)$=0

3- $\frac{\partial u}{\partial y}$=$\frac{\partial v}{\partial y}$

Linear differential equation

Is any differential equation that can be written in the following form

$a_n(x)y^{(n)}(t) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'^{(x)} + a_0(x)y(x) = g(x)$….(3)

The important thing to note about linear differential equations is that there are no products of the function, y(x) , and its derivatives and neither the function or its derivatives occur to any power other than the first power. The coefficients $a_0(x), a_1(x), a_2(x), \dots, a_n(x)$ and $g(x)$ can be zero or non-zero functions, constant or non-constant functions, linear or non-linear functions. Only the function, (x) , and its derivatives are used in determining if a differential equation is linear.

If a differential equation cannot be written in the form, (3) then it is called a **non-linear** differential equation. In other words when an equation is not linear in unknown function and its derivatives.

Example (1.2):

1. $4ty' + 2e^t y = sint$ (linear)
2. $y''' + cosy = 0$ (non-linear)
3. $(1 - y)'' + 4y = lnx$ (non-linear)
4. $y'' + x^2 y' = 0$ (linear)
5. $yy'' + x^2 y' = 0$ (non-linear)

Solution of differential equation

A solution of a differential equation is an expression for the dependent variable (y) in terms of the independent one(s) (x) which satisfies the relation. The general solution includes all possible solutions and typically includes arbitrary constants (in the case of an ode) or arbitrary functions (in the case of a pde)

In other words solution to a differential equation on an interval $\alpha<x<\beta$ is any function $y = y(x)$, which satisfies the differential equation in question on the interval

Example (1.3):

Example Show that $y(x) = x^{-\frac{3}{2}}$ is a solution to $4x^2y'' + 12xy' + 3y = 0$ for $x > 0$.

Solution We'll need the first and second derivative to do this.

$$y'(x) = -\frac{3}{2}x^{-\frac{5}{2}} \qquad y''(x) = \frac{15}{4}x^{-\frac{7}{2}}$$

Put these function into the differential equation.

$$4x^2\left(\frac{15}{4}x^{-\frac{7}{2}}\right)+12x\left(-\frac{3}{2}x^{-\frac{5}{2}}\right)+3\left(x^{-\frac{3}{2}}\right)=0$$

$$15x^{-\frac{3}{2}}-18x^{-\frac{3}{2}}+3x^{-\frac{3}{2}}=0$$

$$0=0$$

So, $y(x)=x^{-\frac{5}{2}}$ does satisfy the differential equation and hence is a solution.

Example (1.4)

prove that $y = e^3x$ is a solution of $\boldsymbol{y'-3y=0}$

Solution:-We'll need the first derivative to do this

$y'=3e3x$

Substitute above function in given differential equation obtain: $3e3x-3e3x=0$

So, $y = e^3x$ does satisfy the differential equation and hence is asolution.

Example (1.5)

Show that $y = x^2$ is a solution of $x^2y'' - 3xy' + 4y = 0$?

Solution:- We'll need the first and second derivative to do this

$$y' = 2x\ , y'' = 2$$

Substituted in given differential equation obtain: $2x2-6x2+4x2=0$

$\therefore y = x^2$ *is a solution of given* differential *equation*

Exercises

1- Show that :

a- $y = \sqrt{1-x2}$ *is a solution of* $yy' + x = 0$ *on*$(-1,1)$.

b- $y = \frac{1}{16}x^4$ *is a solution of* $y' = xy^{1/2}$

2- $y = e^{5x}$ *is a solution of* $y'' - y' + y = 0$?

3- $y = xe^x$ *is a solution of* $y'' - 2y' + y = 0$?

Initial Condition(s)

Initial condition(s) are a condition, or set of conditions, on the solution that will allow us to determine which solution that we are after. Initial conditions (often abbreviated i.c.'s) are of the form,

$$y(x_0) = y_0 \ \frac{and}{or} \ y^{(k)}(x_0) = y_k$$

So, in other words, initial conditions are values of the solution and/or its derivative(s) at specific points.

Note: The **number** of initial conditions that are required for a given differential equation will depend upon the **order** of the differential equation as well as see.

Example(1.6):

Example $y(x) = x^{-\frac{3}{2}}$ is a solution to

$4x^2y'' + 12xy' + 3y = 0$, $y(4) = \frac{1}{8}$, and $y'(4) = -\frac{3}{64}$.

Solution As we saw in previous example the function is a solution and we can then note that

$$y(4) = 4^{-\frac{3}{2}} = \frac{1}{\left(\sqrt{4}\right)^3} = \frac{1}{8}$$

$$y'(4) = -\frac{3}{2}4^{-\frac{5}{2}} = -\frac{3}{2}\frac{1}{\left(\sqrt{4}\right)^5} = -\frac{3}{64}$$

and so this solution also meets the initial conditions of $y(4) = \frac{1}{8}$ and $y'(4) = -\frac{3}{64}$

Initial Value Problem (or IVP)

An Initial Value Problem (or IVP) is a differential equation along with an appropriate number of initial conditions.

Example The following is an IVP.

$$4x^2y'' + 12xy' + 3y = 0 \qquad y(4) = \frac{1}{8}, \quad y'(4) = -\frac{3}{64}$$

Example Here's another IVP.

$$2t y' + 4y = 3 \qquad y(1) = -4$$

Types of the solution for differential equation

1. **The general solution** to a differential equation is the most general form that the solution can take and doesn't take any initial conditions into account i.e

contains **a constants** same as the order of DE.

Example $y(t) = (3/4) + (c/t^2)$ is the general solution to

$$2ty' + 4y = 3$$

2. **The particular solution** to a differential equation is the specific solution that not only satisfies the differential equation, but also satisfies the given initial condition(s).

Example 6 What is the particular solution to the following IVP?

$$2ty' + 4y = 3 \qquad y(1) = -4$$

Solution This is actually easier to do than it might at first appear. From the previous example we already know (well that is provided you believe my solution to this example...) that all solutions to the differential equation are of the form.

$$y(t) = \frac{3}{4} + \frac{c}{t^2}$$

All that we need to do is determine the value of c that will give us the solution that we're after. To find this all we need do is use our initial condition as follows.

$$-4 = y(1) = \frac{3}{4} + \frac{c}{1^2} \qquad \Rightarrow \qquad c = -4 - \frac{3}{4} = -\frac{19}{4}$$

So, the actual solution to the IVP is.

$$y(t) = \frac{3}{4} - \frac{19}{4t^2}$$