b) Transformation of Continuous type:

In This section we shall examine the same Problem when the r.v's are the Continuous type.
To find the dist. of Y, we apply the Law.

$$g(y) = f(x) \cdot \left|\frac{dx}{dy}\right| = f(x) \cdot |J|$$

where $|J|$ is Jacobian

Ex) Assume that X be a r.v. having p.d.f

$$f(x) = \begin{cases} 2x & , 0<x<1 \\ 0 & , o/w \end{cases}$$

Find the p.d.f. of $\boxed{Y = 8X^3}$?

Sol//

① إيجاد ال y

x	$y = 8x^3$
0	0
1	8

$A = \{x ; 0<x<1\}, f(x) > 0$

$B = \{y ; 0<y<8\}, g(y) > 0$

$y = 8x^3$

$x^3 = \frac{y}{8} \Rightarrow x = \sqrt[3]{\frac{y}{8}} \Rightarrow x = \frac{\sqrt[3]{y}}{2}$

② إيجاد قيمة X بدلالة y

$g(y) = f(x) \cdot |J|$

$|J| = \left|\frac{dx}{dy}\right| = \frac{\partial(\sqrt[3]{y}/2)}{\partial y} = \left(\frac{\sqrt[3]{y}}{2}\right)'$

③ إيجاد قيمة $|J|$

$= \frac{1}{2}(y^{\frac{1}{3}})' = \frac{1}{2} \cdot \frac{1}{3} y^{\frac{1}{3}-1}$

$$|J| = \frac{1}{6} \cdot y^{\frac{1-3}{3}} = \frac{1}{6} y^{\frac{-2}{3}} = \boxed{\frac{1}{6\sqrt[3]{y^2}}}$$

$$g(y) = \cancel{2} \cdot \frac{\sqrt[3]{y}}{\cancel{2}} \cdot \frac{1}{6\sqrt[3]{y^2}} = \frac{1}{6\sqrt[3]{y}}$$

نعوض بالقانون

$g(y) = f(x) \cdot |J|$

$$\therefore g(y) = \begin{cases} \frac{1}{6\sqrt[3]{y}} & , 0<y<8 \\ 0 & , o|w \end{cases}$$

Example: Let $f(x) = \begin{cases} 1 & , 0<x<1 \\ 0 & , o|w \end{cases}$

Find the p.d.f of $Y = -2\ln x$?

Sol

X	$y = -2\ln x$
0	$-2\ln 0 = \infty$
1	$-2\ln 1 = 0$

$y = -2\ln x$

$\ln x = -\frac{y}{2}$

$x = e^{-\frac{y}{2}}$

$\therefore \boxed{x = e^{-y/2}}$

$A = \{x;\ 0<x<1\},\ f(x)>0$

$B = \{y;\ 0<y<\infty\},\ g(y)>0$

$g(y) = f(x) \cdot |J|$

$$= 1 \cdot \left|\frac{dx}{dy}\right| = 1 \left|\frac{\partial e^{-\frac{y}{2}}}{\partial y}\right|$$

$$= 1 \cdot \left|e^{-y/2}\left(-\frac{1}{2}\right)\right| = \frac{1}{2} e^{-y/2}$$

$$\therefore g(y) = \begin{cases} \frac{1}{2} e^{-y/2} & , 0<y<\infty \\ 0 & , o|w \end{cases}$$

$\therefore Y \sim \chi^2_{(2)}$

Example 2. Let $f(x) = \begin{cases} (x+1)/2 & , -1 \leqslant x \leqslant 1 \\ 0 & , o/w \end{cases}$

Find the density function of $Y = X^2$?

x	$y = x^2$
-1	$(-1)^2 = 1 \simeq 0$
1	$(1)^2 = 1$

$y = x^2$

$x = \mp \sqrt{y}$

الـ-1 قيمة نأخذ فقط موجب

$A = \{x ; -1 \leqslant x \leqslant 1\} , f(x) > 0$

$B = \{y ; 0 \leqslant y \leqslant 1\} , g(y) > 0$

$g(y) = f(x) . |J| = f(x) . \left| \frac{\partial x}{\partial y} \right|$

$f(x) = \frac{\cancel{\sqrt{y}}+1}{2} + \frac{-\cancel{\sqrt{y}}+1}{2} = \frac{2}{2} = \boxed{1}$

هنا اخذ قيمة $x = \sqrt{y}$ و $-\sqrt{y}$ لأن قيم $-1 \leqslant x \leqslant 1$ يعني x لديها قيم بالسالب

$|J| = \left| \frac{\partial x}{\partial y} \right| = \frac{\partial \sqrt{y}}{\partial y} = \partial (y)^{\frac{1}{2}} = \frac{1}{2} y^{\frac{1}{2} - 1}$

$= \frac{y^{-\frac{1}{2}}}{2} = \frac{-\sqrt{y}}{2} = \boxed{\frac{1}{2\sqrt{y}}}$

$\therefore g(y) = 1 . \frac{1}{2\sqrt{y}} = \frac{1}{2\sqrt{y}}$

$\therefore g(y) = \begin{cases} \frac{1}{2\sqrt{y}} & , 0 \leqslant y \leqslant 1 \\ 0 & , o/w \end{cases}$

$Y_1 = X_1 + X_2 = [0, 2)$ $\qquad$ $Y_2 = X_1 - X_2 = [-1, 1]$

$0 \quad X_1 \quad 1 \quad X_2 \quad 2$ $\qquad$ $-1 \quad X_2 \quad 0 \quad X_1 \quad 1$

Ex) Let $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & o/w \end{cases}$

Find the Joint of $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$

Then find the Marginal of Y_1 & Y_2 ?

Sol

$$\begin{aligned} Y_1 &= X_1 + X_2 \\ Y_2 &= X_1 - X_2 \end{aligned} \qquad\qquad \begin{aligned} Y_1 &= X_1 + X_2 \\ Y_2 &= X_1 - X_2 \end{aligned}$$

$$X_1 = \frac{Y_1 + Y_2}{2} \qquad\qquad X_2 = \frac{Y_1 - Y_2}{2}$$

$A = \{(x_1, x_2) ; 0 < x_1 < 1, 0 < x_2 < 1\}, f(x_1, x_2) > 0$

$B = \{(y_1, y_2) ; 0 < y_1 2, -1 < y_2 < 1\}, g(y_1, y_2) > 0$

x_1	x_2	y_1	y_2
0	0	0	0
1	1	1	-1
		1	1
		2	0

$g(y_1, y_2) = f(x_1, x_2) \cdot |J|$

$$|J| = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

$$= -\frac{1}{4} - \frac{1}{4} = \left| -\frac{1}{2} \right| = \frac{1}{2}$$

$g(y_1, y_2) = 1 \cdot \frac{1}{2} = \frac{1}{2}$

$\therefore \quad g(y_1, y_2) = \begin{cases} \frac{1}{2}, & 0 < y_1 < 2, \ -1 < y_2 < 1 \\ 0, & o/w \end{cases}$

2) The Cumulative distribution function technique

The Joint dist. of r.v's $X_1, X_2, \dots, X_n$ is given as $f(x_1, \dots, x_n)$, then the Joint dist. of r.v's of $Y_1, Y_2, \dots, Y_k$ can be determined, where $U_j(x_1, \dots, x_n)$, $j = 1, 2, \dots, k$, $k \leq n$, by using c.d.F technique, then

$$F(y_1, y_2, \dots, y_k) = P_r(Y_1 \leq y_1, Y_2 \leq y_2, \dots, Y_k \leq y_k)$$

$$= P_r\left[u_1(x_1, \dots, x_n) \leq y_1, \dots, u_k(x_1, \dots, x_n) \leq y_k\right]$$

If Y is univariate r.v, then :-

$$F(y) = P_r(Y \leq y)$$

$$F(y) = P_r\left[u(x_1, x_2, \dots, x_n) \leq y\right], \text{ then}$$

$$f(y) = F'(y), \text{ where } f(y) \text{ is p.d.f. of } Y$$

Ex) Let $f(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1 \\ 0, & o/w \end{cases}$

Find the distribution of $\boxed{Y = X^2}$ by using c.d.F ?

Sol

$$F(y) = P_r(Y \leq y) = P_r(X^2 \leq y)$$ 1) نطلع $F(y)$

$$= P_r(|x| \leq \sqrt{y})$$

([illegible]) $$= P_r(-\sqrt{y} \leq x \leq \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2}\, dx$$

$$= \frac{1}{2} x \Big|_{-\sqrt{y}}^{\sqrt{y}} = \frac{1}{2}(\sqrt{y} + \sqrt{y}) = \frac{1}{2} \cdot 2\sqrt{y} = \sqrt{y}$$

$$\therefore F(y) = \sqrt{y}$$

③ استخرج $f(y)$ من الـ $F(y)$ ؟

Now, باشتقاقه

$f(y) = F'(y)$

$f(y) = \sqrt{y}\,' = (y^{\frac{1}{2}})' = \frac{1}{2} y^{-\frac{1}{2}} = \frac{1}{2\sqrt{y}}$

الآن نحدد حدود الـ y من جدول كالآتي

④ استخرج حدود الـ (y)

X	$Y = X^2$
-1	$(-1)^2 = 1 \simeq 0$
1	$1^2 = 1$

$$f(y) = \begin{cases} \frac{1}{2\sqrt{y}} & , \quad 0 < y < 1 \\ 0 & , \quad o.w \end{cases}$$

Ex) Let $f(x) = \begin{cases} \frac{x}{6} & , \quad x = 1,2,3 \\ 0 & , \quad o.w \end{cases}$

Find the dist. function of $Y = X^2$ by using C.d.F. technique.

Sol

$F(y) = P_r(Y \le y) = P_r(X^2 \le y) = P_r(|X| \le \sqrt{y})$

$= P_r(-\sqrt{y} \le X \le \sqrt{y})$ (من هنا نستخرج حدود التكامل للدالة الأصلية)

لعدم وجود السالب في حدود X $= P_r(X \le \sqrt{y})$

$= \sum_{x=1}^{\sqrt{y}} \frac{x}{6} = \frac{1}{6} \sum_{x=1}^{\sqrt{y}} x$ قانون مجموع الأعداد الطبيعية $\frac{N(N+1)}{2}$

$= \frac{1}{6} \cdot \frac{\sqrt{y}(\sqrt{y}+1)}{2} = \frac{\sqrt{y}(\sqrt{y}+1)}{12}$ ✻

$N(N+1)$

لأن نفرغ قيم الـ القد

X	$Y = X^2$
1	1
2	4
3	9

قيم الـ $f(y)$

$$F(Y) = \begin{cases} 0 & , y < 1 \\ 1/6 & , 1 \leq y < 4 \\ 1/2 & , 4 \leq y < 9 \\ 1 & , y \geq 9 \end{cases}$$

هذه الطريقة نأخذها من تعويض قيم الـ $X = 1, 2, 3$ في $F(Y)$ ✳

$$f(y) = \begin{cases} 1/6 & , y = 1 \\ 1/3 & , y = 4 \\ 1/2 & , y = 9 \\ 0 & , o.w \end{cases}$$

هنا نشتق الـ $F(y)$

$\frac{1}{6} - 0 = \frac{1}{6}$

$\frac{1}{2} - \frac{1}{6} = \frac{3-1}{6} = \frac{1}{3}$

$1 - \frac{1}{2} = \frac{1}{2}$

Ex) Suppose $f(x) = \begin{cases} 1 & , 0 < x < 1 \\ 0 & , o.w \end{cases}$

Find the p.d.f of $Y = \sqrt{X}$ by c.d.f technique?

Sol:

$F(y) = P_r(Y \leq y) = P_r(\sqrt{X} \leq y) = P_r(X \leq y^2)$ نشتق لجد الـ (F)

$= \int_0^{y^2} 1 \, dx = x \Big|_0^{y^2} = [y^2 - 0] = y^2$

X	$Y = \sqrt{X}$
0	0
1	1

$f(y) = F'(y) = (y^2)' = 2y$

$\therefore f(y) = \begin{cases} \boxed{2y} & , 0 < y < 1 \\ 0 & o.w \end{cases}$

Ex) Let $X \sim N(0,1)$ find the p.d.f. of $Y = X^2$ by using C.d.F techniques ?

Sol//

$$X \sim \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$F(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$

$$= 2P(0 \le X \le \sqrt{y})$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{y}} e^{-\frac{x^2}{2}} dx$$

$$Y \sim \frac{dP(Y \le y)}{dy} = \frac{dP(Y \le y)}{d\sqrt{y}} \frac{d\sqrt{y}}{dy} = \sqrt{\frac{2}{\pi}} e^{-\frac{y^2}{2}} \frac{1}{2\sqrt{y}}$$

$$= \frac{e^{-\frac{y}{2}}}{\sqrt{2\pi y}}$$

3- The Moment generating function techniques:-

The another method for determining the dist. of function of r.v's which we shall find to be particularly useful in certain instance for given r.v's $X_1, \dots, X_n$ with given density $f(X_1, \dots, X_n)$ and given functions $g_1(X_1, \dots, X_n), \dots, g_K(X_1, \dots, X_n)$ find the Joint dist. of $Y_1 = g_1(X_1, \dots, X_n), \dots, Y_K = g_K(X_1, \dots, X_n)$.

Now, the Joint M.g.f of $Y_1, \dots, Y_K$, if it is exists is $M_{\cancel{Y_1}, \cancel{Y_2}, \dots, Y_K}(t_1, \dots, t_K) = E\left[e^{t_1 Y_1 + t_2 Y_2 + \dots + t_K Y_K}\right]$

$$= \int_{\text{all } y_1} \cdots \int_{\text{all } y_K} e^{t_1 Y_1 + \dots + t_K Y_K} f(X_1, \dots, X_n) \prod_{i=1}^{n} dx_i$$

$$= \int_{\forall y_1} \cdots \int_{\forall y_K} e^{t_1 g_1(X_1, \dots, X_n) + \dots + t_K g_K(X_1, \dots, X_n)} f(X_1, \dots, X_n) \prod_{i=1}^{n} dx_i$$

Ex) Suppose X has standard normal dist and let $Y = X^2$, find the dist. of Y by Moment generating fun. technique ?

Sol

$$M_Y(t) = E\left[e^{tY}\right] = E\left[e^{tX^2}\right]$$

$$= \int_{-\infty}^{\infty} e^{tX^2} f(x)\,dx = \int_{-\infty}^{\infty} e^{tX^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}X^2}\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}X^2(1-2t)}\,dx$$