Chapter Four
Distributions of functions of Random Variables

As the title of this chapter indicates, we are intersetel in finding the distributions of function of random variables. More precisely (أدق), for given r.vs, say $X_1, X_2, \dots, X_n$ denote (n) r.v's having the joint pdf $f(x_1, x_2, \dots, x_n)$, and that $Y_1 = U_1(X_1, X_2, \dots, X_n)$, $Y_2 = U_2(X_1, X_2, \dots, X_n), \dots, Y_K = U_K(X_1, X_2, \dots, X_n)$ we want in general to find the Joint dist. of $Y_1, \dots, Y_K$ where $K \leq n$ as well as the marginal dist of any combination of $Y_1, Y_2, \dots, Y_K$.

In this chapter three techniques for finding the dist. of functions of r.vs will be presented.
These three techniques are called:

1) The transformation techniques.
2) The cumulative distribution function technique
3) The Moment generating function techniques.

(ايجاد توزيعات لدوال المتغيرات العشوائية)

(x_1, x_2) $A \longrightarrow B$ (y_1, y_2)

1) Transformation Techniques of (2) random var.

a) Transformation of two discrete r.v's.

Let X_1 & X_2 are two d.r.v.s with J.p.m.f. $f(x_1, x_2) > 0$, such that A is space of (x_1, x_2)

let

$$\left.\begin{array}{l} Y_1 = u_1(x_1, x_2) \\ Y_2 = u_2(x_1, x_2) \end{array}\right\} \text{be (1-1) transformation of } A \text{ onto } B \text{ (the space of } (y_1, y_2))$$

$$\left.\begin{array}{l} x_1 = u_1^{-1}(y_1, y_2) \\ x_2 = u_2^{-1}(y_1, y_2) \end{array}\right\} \text{be (1-1) transformation of } B \text{ to } A$$

Then the J.p.m.f. of (y_1, y_2) is :-

$$g(y_1, y_2) = f[\underbrace{u_1^{-1}(y_1, y_2)}_{x_1}, \underbrace{u_2^{-1}(y_1, y_2)}_{x_2}].$$

Example:- Given a J.P.M.f of X_1 & X_2

$$f(x_1, x_2) = \begin{cases} \left(\frac{2}{3}\right)^{x_1 + x_2} \left(\frac{1}{3}\right)^{2 - x_1 - x_2}, & (x_1, x_2) = (0,0), (0,1), (1,0), (1,1) \\ 0, & \text{o/w} \end{cases}$$

Find the J.P.M.f. of $y_1 = x_1 - x_2$, $y_2 = x_1 + x_2$ & Marginal of y_1 $g(y_1, y_2)$

Sol

$$u_1 = y_1 = x_1 - x_2$$
$$u_2 = y_2 = x_1 + x_2$$

1) إيجاد قيم x_1 & x_2 بدلالة y_2 & y_1

لاستخراج قيم نقاط $y_1 \& y_2$ نأخذ قيم $x_1 \& x_2$ ونعوضها بأحد هذه المعادلات

مثلاً نأخذ $(x_1, x_2) = (0,0)$ ونعوضها في المعادلة (*)

(*) بالجمع

$$y_1 = x_1 - x_2$$
$$y_2 = x_1 + x_2$$
$$x_1 = \frac{y_1 + y_2}{2}$$

بالطرح

$$y_1 = x_1 - x_2$$
$$y_2 = x_1 + x_2$$
$$y_2 - y_1 = 2x_2$$
$$x_2 = \frac{y_2 - y_1}{2}$$

وهكذا $y_1 = 0 - 0$

$y_2 = 0 + 0$

** $\therefore (y_1, y_2) = (0,0)$

$\Rightarrow g(y_1, y_2)$

$$g(y_1,y_2) = \begin{cases} \left(\frac{2}{3}\right)^{\left(\frac{y_1+y_2}{2}\right)+\left(\frac{y_2-y_1}{2}\right)} \left(\frac{1}{3}\right)^{2-\left(\frac{y_1+y_2}{2}\right)-\left(\frac{y_2-y_1}{2}\right)} & \text{for } (y_1,y_2) = (0,0), (-1,1), (1,1), (0,2) \;^{**} \\ 0 & o/w \end{cases}$$

$$g(y_1,y_2) = \begin{cases} \left(\frac{2}{3}\right)^{y_2} \left(\frac{1}{3}\right)^{2-y_2}, & (y_1,y_2) = (0,0), (-1,1), (1,1), (0,2) \\ 0, & o/w \end{cases}$$

$$g(y_1) = \sum_{\forall y_2} g(y_1,y_2), \quad y_2 = 0, 1, 1, 2$$

$$= \sum_{\forall y_2} \left(\frac{2}{3}\right)^{y_2} \left(\frac{1}{3}\right)^{2-y_2}$$

$$= \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^1 \left(\frac{1}{3}\right)^{2-1} + \left(\frac{2}{3}\right)^1 \left(\frac{1}{3}\right)^{2-1} + \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^{2-2}$$

$$= \frac{1}{9} + \frac{2}{9} + \frac{2}{9} + \frac{4}{9}$$

$$= 1$$

$X \to f(x) = \text{Poisson}$

X بدلالة $\boxed{Y} \to g(y) = ?$

Example: Let X have Poisson p.d.f. find the dist. of $\boxed{Y = 4X}$ by using transformation techniques?

Sol

$$f(x;\lambda) = \begin{cases} \dfrac{e^{-\lambda}\lambda^{x}}{x!} & , x = 0,1,2,\dots,\infty \\ 0 & , o/w \end{cases}$$

$y = 4x$, $A = \{x ; x = 0,1,2,\dots,\infty\}$, $f(x) > 0$
sample space of X

$x = \dfrac{y}{4}$, $B = \{y ; y = 0,4,8,\dots,\infty\}$, $g(y) > 0$
sample space of Y

X	$y = 4X$
0	0
1	4
2	8
⋮	⋮
∞	∞

$$g(y) = P_r(Y = y) = P_r(4X = y)$$

$$= P_r\left(X = \frac{y}{4}\right)$$

$$\therefore g(y) = \begin{cases} \dfrac{e^{-\lambda}\lambda^{\frac{y}{4}}}{(y/4)!} & , y = 0,4,8,\dots,\infty \\ 0 & , o/w \end{cases}$$

① اوجد قيمة الـ X بدلالة الـ y ,
② اعوضها بالدالة الاصلية .