

Second: Continuous Distributions

- 1) Uniform or Rectangular distribution:-
This dist. particularly useful in theoretical statistics because it is convenient (d.f.) to deal with mathematically.

$$f(x, a, b) = \begin{cases} \frac{1}{b-a}, & a \leq x < b \\ 0, & \text{otherwise} \end{cases}$$

$X \sim U(a, b)$ where a, b satisfy, $-\infty < a < b < \infty$

$$\mu_x = \frac{a+b}{2}, \quad \sigma_x^2 = \frac{(b-a)^2}{12}, \quad M_x(t) = \frac{e^{bt} - e^{at}}{(b-a)t}$$

Ex: Is uniform dist. p.d.f.?

$$\int_{-\infty}^{\infty} f(x) dx = 1 \Rightarrow \int_a^b \frac{1}{b-a} dx = 1 \Rightarrow \frac{1}{b-a} [x]_a^b = 1$$

$$\frac{1}{b-a} [b-a] = 1 \therefore \text{p.d.f.}$$

- 2) Gamma distribution:-

$$f(x, \alpha, \beta) = \begin{cases} \frac{1}{\Gamma(\alpha) \beta^\alpha} x^{\alpha-1} e^{-x/\beta}, & 0 \leq x < \infty \\ 0, & \text{otherwise} \end{cases}$$

where the parameters $\alpha, \beta > 0$, $\Gamma(\alpha)$ is gamma fun. which defined as:-

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx > 0 \quad \text{by integration by parts we get:}$$

$$\Gamma(\alpha) = (\alpha-1)!$$

Second

Continuous Distributions

Normal distribution

مثال، جدول

The normal dist. is one of the widely used in application of statistical methods.

$$f(x, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

$\pi = 3.14, e = 2.71$

where the parameters μ, σ^2 satisfy $-\infty < \mu < \infty, \sigma > 0 \Rightarrow X \sim N(\mu, \sigma^2)$

The mode of a normal dist occurs at $x = \mu$ and inflection points occurs at $(\mu - \sigma)$ and $(\mu + \sigma)$ we can also used the notation $\phi_{\mu, \sigma^2}^*(x)$ for the normal dist. and $\Phi_{\mu, \sigma^2}(x)$ for the cumulative dist

$$\mu_x = E(X), \text{ var}(X) = \sigma^2, \mu_x(t) = e^{\mu t + \sigma^2 t^2 / 2}$$

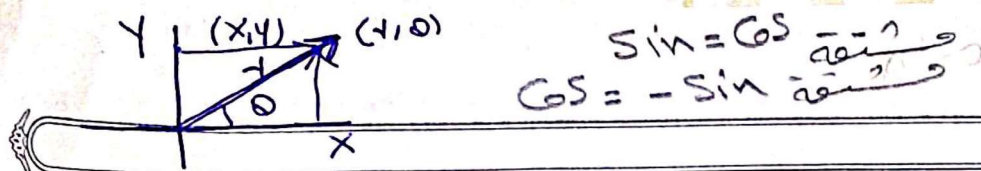
Ex: Is the normal dist. p.d.f.?

$$\int_{\text{all } x} f(x) dx = 1 \Rightarrow \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \stackrel{?}{=} 1$$

$$\text{Let } A = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1$$

$$\text{Let } y = \frac{x-\mu}{\sigma} \Rightarrow dy = \frac{1}{\sigma} dx \Rightarrow dx = \sigma dy$$

$$A = \frac{1}{\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy$$



We wish to show that $A = 1$, by showing A^2 is 1 and then reasoning that $A = 1$, we may put

$$A^2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz$$

$$A^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y^2+z^2)} dy dz$$

we change the variables to polar coordinates by the substitutions $y = r \sin \theta$, $z = r \cos \theta$

$$[A^2 = \frac{1}{2\pi} \int_0^{\infty} \int_0^{2\pi} (r) e^{-\frac{1}{2}r^2} d\theta dr] \text{ ch.}$$

$$A^2 = \frac{1}{2\pi} \int_0^{\infty} r e^{-\frac{1}{2}r^2} dr \int_0^{2\pi} d\theta = \frac{1}{2\pi} \int_0^{\infty} r e^{-\frac{1}{2}r^2} dr \cdot 2\pi$$

$$A^2 = \int_0^{\infty} r e^{-\frac{1}{2}r^2} dr = -[e^{-\frac{1}{2}r^2}]_0^{\infty} = 1$$

Q6) $1 = 1 \therefore$ The Normal dist is p.d.f

Ex:- Drive the M_x , σ_x^2 , $M_x(t)$ for Normal dist.?

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx =$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$M_x(t) = \frac{1}{\sqrt{2\pi}\sigma^2} \int_{-\infty}^{\infty} e^{\frac{2\sigma^2 tx - x^2 + 2\mu x - \mu^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2} [x^2 - 2tx(\mu + \sigma^2 t) + \mu^2]} dx$$

(X دالة عشوائية)

If we complete the square, by add and substitute half square of coefficient X.

$$M_X(t) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2} [x^2 - 2x(\mu + \sigma^2 t) + \mu^2 + (\mu + \sigma^2 t)^2 - (\mu + \sigma^2 t)^2]} dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2} [x^2 - 2x(\mu + \sigma^2 t) + (\mu + \sigma^2 t)^2]} e^{-\frac{\mu^2 + (\mu + \sigma^2 t)^2}{2\sigma^2}} dx$$

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} \left\{ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{[x - (\mu + \sigma^2 t)]^2}{2\sigma^2}} dx \right\} \quad \text{P.d.f}$$

$$M'_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} \cdot (\mu + \sigma^2 t)$$

$$M_X = M'_X(0) = e^{\mu(0) + \frac{\sigma^2(0)^2}{2}} (\mu + \sigma^2(0)) = \mu$$

$$M''_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} \cdot \sigma^2 + (\mu + \sigma^2 t) e^{\mu t + \frac{\sigma^2 t^2}{2}} (\mu + \sigma^2 t)$$

$$E(X^2) = M''_X(0) = \sigma^2 + \mu^2$$

$$\text{Var}(X) = M''_X(0) - [M'_X(0)]^2$$

$$= \sigma^2 + \mu^2 - \mu^2$$

$$= \sigma^2$$

Standard Normal distribution (توزيع طبيعي قياسي)

If the normal r.v. has mean zero and variance one instead of μ and σ^2 , it is called standard normal distribution,

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, \quad -\infty < z < \infty$$

here $z = \frac{x - \mu}{\sigma}$, $x \sim N(0, 1)$

$$\mu_x = 0, \quad \sigma_x^2 = 1, \quad f_x(z) = e^{-\frac{z^2}{2}}$$

Theorem: If $x \sim N(\mu_x, \sigma_x^2)$, then $\frac{x - \mu_x}{\sigma_x} \sim N(0, 1)$

Theorem: If $x \sim N(\mu, \sigma^2)$, then

$$P(a < x < b) = \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right)$$

Proof: $P(a < x < b) = \int_a^b f(x) dx$

$$= \int_a^b \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x - \mu)^2}{2\sigma^2}} dx = \int_{\frac{a - \mu}{\sigma}}^{\frac{b - \mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$= \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right)$$

Remark:

$$\Phi(-x) = 1 - \Phi(x)$$

$$\frac{x - \mu}{\sigma} = z$$

$$\sigma z = x - \mu$$

$$x = \sigma z + \mu$$

$$dx = \sigma dz$$

$B = \frac{1}{\lambda}$, $\alpha = 1$ التوزيع الأسي

Exponential distribution : (التوزيع الأسي)

The exponential dist. has been used as a model for lifetimes.

An exponential dist. is a special case of Gamma dist., when $\alpha = 1$, $\beta = \frac{1}{\lambda}$ ($\lambda > 0$); the new dist. is called exponential dist. denoted by "exp. (λ)" its p.f. function is as follows: $X \sim \text{exp.}(\lambda)$

$$f(x, \lambda) = \begin{cases} \lambda e^{-\lambda x} & , 0 \leq x < \infty \\ 0 & , \text{otherwise} \end{cases}$$

Also, the m.g.f of X is:

$$M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-1} ; M_X = E(X) = (1)\left(\frac{1}{\lambda}\right) = \frac{1}{\lambda}$$

$$\sigma_X^2 = \text{Var}(X) = (1)\left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

$$G(x, \beta) = \frac{1}{\Gamma(\beta)} \int_0^x t^{\beta-1} e^{-t} dt$$

ملاحظة: قيم X في توزيع (الغاما) $X \sim \text{Gamma}(\alpha, \beta)$ لا تكون أبد من الصفر، أما للقيم الغمرى (الالبية) فاه

الذات تساوي صفر و هو المقادير

$$P(-a \leq X \leq a) = P(X \leq a) - \underbrace{P(X \leq -a)}_{\text{zero}} = P(X \leq a)$$

Note: If $X \sim N(0, 1)$ then:

$$1) P(X \leq a) = \int_{-\infty}^a \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \underline{N(a)}$$

$$2) P(X > a) = 1 - P(X \leq a) = 1 - N(a)$$

$$3) P(X \leq -a) = 1 - P(X \leq a) = \underline{1 - N(a)} = \underline{N(-a)}$$

$$4) P(a \leq X \leq b) = P(X \leq b) - P(X \leq a) = N(b) - N(a)$$

$$5) P(X \leq a) = P(X \leq a) = N(a)$$

لا نحتاج إلى دالة الكثافة
بل فقط دالة التوزيع

$$6) P(X > -a) = 1 - P(X \leq -a)$$

$$= 1 - N(-a)$$

$$= 1 - [1 - N(a)] = 1 - 1 + N(a) = N(a)$$

$$7) P(|X| \leq a) = P(-a \leq X \leq a)$$

$$= P(X \leq a) - P(X \leq -a)$$

$$= N(a) - N(-a)$$

$$= N(a) - [1 - N(a)]$$

$$= N(a) - 1 + N(a)$$

$$= \underline{2N(a) - 1}$$

Beta distribution: $(a=b=1)$ is (uniform) dist.

Let X_1 and X_2 be two stochastically independent r.v's, that have Gamma dist., then the marginal p.d.f. of $Y = \frac{X_1}{X_1 + X_2}$ is Beta dist., which is:

$$f(x, a, b) = \begin{cases} \frac{1}{B(a, b)} x^{a-1} (1-x)^{b-1}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

where a and b are two parameters satisfy $a, b > 0$. The function

$$B(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx \text{ is called}$$

Beta function and the value of it is:

$$B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}$$

The beta dist. reduces to the uniform dist. if $a=b=1$, the M.g.f. of beta dist. does not have a simple form.

$$M_X = \frac{a}{a+b}, \quad \sigma_X^2 = \frac{ab}{(a+b+1)(a+b)^2}$$

$$M_X(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \frac{B(j+a, b)}{B(a, b)}$$

$$B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}$$

Ex: IS beta dist. p.d.f?

$$\int_{\text{all } x} f(x) dx = 1 \Rightarrow \int_0^1 \frac{1}{B(a,b)} x^{a-1} (1-x)^{b-1} dx = 1$$

$$\frac{1}{B(a,b)} \int_0^1 x^{a-1} (1-x)^{b-1} dx = \frac{1}{B(a,b)}$$

$$\frac{1}{B(a,b)} B(a,b) = 1 \Rightarrow 1 = 1$$

∴ The beta dist. is p.d.f.

Ex: Derive $\underline{M_x}$, σ_x^2 for beta dist.?

$$M_x = \int_{\text{all } x} x f(x) dx = \int_0^1 x \frac{1}{B(a,b)} x^{a-1} (1-x)^{b-1} dx$$

$$= \frac{1}{B(a,b)} \int_0^1 x^a (1-x)^{b-1} dx$$

$$= \frac{1}{B(a,b)} B(a+1, b)$$

$$M_x = \frac{\frac{\Gamma(a+1) \Gamma(b)}{\Gamma(a+b+1)}}{\frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}} = \frac{\Gamma(a+1) \Gamma(a+b)}{\Gamma(a+b+1) \Gamma(a)} = \frac{a! (a+b-1)!}{(a+b)! (a-1)!}$$

$$= \frac{a}{a+b}$$

$$E(x^2) = \int_{\text{all } x} x^2 f(x) dx$$

$$= \int_0^1 x^2 \frac{1}{B(a,b)} x^{a-1} (1-x)^{b-1} dx$$

$$\beta(a, b)$$

$$\frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}$$

$$= \frac{\Gamma(a+2) \Gamma(a+b)}{\Gamma(a+b+2) \Gamma(a)} \cdot \frac{a(a+1)}{(a+b+1)(a+b)}$$

$$\text{Var}(X) = E(X^2) - M_X^2$$

$$\sigma_X^2 = \frac{a(a+1)}{(a+b+1)(a+b)} - \frac{a^2}{(a+b)^2}$$

$$= \frac{a(a+1)(a+b) - a^2(a+b+1)}{(a+b)^2(a+b+1)} = \frac{ab}{(a+b)^2(a+b+1)}$$

Chi-square distribution: $\beta = 2$, $\alpha = \frac{\gamma}{2}$

Consider the special case of the Gamma dist. in which $\alpha = \frac{\gamma}{2}$ and $\beta = 2$, then the r.v. X is said to be chi-square dist. where $\gamma =$

$$f(x, \gamma) = \begin{cases} \frac{1}{\Gamma(\frac{\gamma}{2}) 2^{\frac{\gamma}{2}}} x^{\frac{\gamma}{2}-1} e^{-x/2} & , x > 0 \\ 0 & , \text{o.w.} \end{cases}$$

we call the parameter (γ) the number of degrees of freedom of chi-square dist. $X \sim \chi^2_{(\gamma)}$

$$M_X = \gamma, \sigma_X^2 = 2\gamma, M_X(t) = (1-2t)^{-\frac{\gamma}{2}} = \frac{1}{(1-2t)^{\frac{\gamma}{2}}}$$

Weibull distribution (توزيع ويبل)

$$f(x, \alpha, \beta) = \begin{cases} \alpha \beta x^{\beta-1} e^{-\alpha x^\beta} & , 0 < x < \infty \\ 0 & , \text{o/w} \end{cases}$$

where α and β are parameter satisfy $\alpha, \beta > 0$.
This dist. used in reliability theory, for $\beta = 1$ the weibull dist. reduces to the exponential dist.

$$E(X) = \left(\frac{1}{\alpha}\right)^{1/\beta} \Gamma(1 + 1/\beta)$$

$$\sigma_x^2 = \left(\frac{1}{\alpha}\right)^{2/\beta} \left[\Gamma(1 + 2/\beta) - (\Gamma(1 + 1/\beta))^2 \right]$$

Ex₂ IS the weibull dist. pdf?

$$\int_{\text{all } x} f(x) dx = 1 \Rightarrow \int_0^\infty \alpha \beta x^{\beta-1} e^{-\alpha x^\beta} dx = 1$$

$$\text{Let } y = \alpha x^\beta \Rightarrow x = \left(\frac{y}{\alpha}\right)^{1/\beta}, \quad dx = \frac{1}{\beta} \left(\frac{y}{\alpha}\right)^{1/\beta-1} \frac{1}{\alpha} dy$$

$$\alpha \beta \int_0^\infty \left[\frac{y^{1/\beta}}{\alpha^{1/\beta}} \right]^{\beta-1} e^{-y} \frac{1}{\beta \alpha} \frac{y^{1/\beta-1}}{\alpha^{1/\beta-1}} dy = 1$$

$$\int_0^\infty y^{1-1/\beta} y^{1/\beta-1} e^{-y} dy = 1 \Rightarrow \int_0^\infty e^{-y} dy = 1$$

$$[e^{-y}]_0^\infty = 1 \Rightarrow 1 = 1 \text{ the weibull dist. pdf.}$$

Double exponential or Laplace distribution.

$$f(x, \alpha, \beta) = \begin{cases} \frac{1}{2\beta} e^{\frac{-|x-\alpha|}{\beta}} & , x > 0 \\ 0 & , \text{o.w} \end{cases}$$

where α, β parameter, $-\infty < \alpha < \infty, \beta > 0$

$$\mu_x = \alpha, \sigma_x^2 = 2\beta^2$$

Cauchy distribution (चौची वितरण)

$$f(x, \alpha, \beta) = \frac{1}{\pi\beta [1 + (\frac{x-\alpha}{\beta})^2]} , -\infty < x < \infty$$

where α, β parameter satisfy, $-\infty < \alpha < \infty, \beta > 0$.

Although the Cauchy dist. is symmetrical about the parameter α , its mean and higher moment do not exist.

Problems

Ex1) Given $X \sim b(n, \frac{1}{3})$. Find the Smallest value of (n) s.t. $P(X \geq 1) > 0.8$.

Sol

$$X \sim b(n, \frac{1}{3})$$

$$f(x) = \begin{cases} C_x^n p^x q^{n-x} & , x = 0, 1, 2, \dots, n \\ 0 & \text{o/w} \end{cases}$$

$$f(x) = \begin{cases} C_x^n \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{n-x} & , x = 0, 1, 2, \dots, n \\ 0 & \text{o/w} \end{cases}$$

$$P(X \geq 1) = 1 - P(X < 1)$$

$$= 1 - P(X = 0)$$

$$= 1 - f(0) \Rightarrow 1 - C_0^n \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{n-0}$$

$$= 1 - \left(\frac{2}{3}\right)^n > 0.8 \quad (\text{Since } P(X \geq 1) > 0.8)$$

$$\Rightarrow 1 - \left(\frac{2}{3}\right)^n > 0.8 \Rightarrow \left(\frac{2}{3}\right)^n < 0.2$$

$$\text{IF } n=0 \Rightarrow \left(\frac{2}{3}\right)^0 = 1 \not< 0.2$$

$$\text{IF } n=1 \Rightarrow \left(\frac{2}{3}\right)^1 = .67 \not< 0.2$$

$$\text{IF } n=2 \Rightarrow \left(\frac{2}{3}\right)^2 = \frac{4}{9} = .44 \not< 0.2$$

$$\text{IF } n=3 \Rightarrow \left(\frac{2}{3}\right)^3 = \frac{8}{27} = .29 \not< 0.2$$

$$\text{IF } n=4 \Rightarrow \left(\frac{2}{3}\right)^4 = \frac{16}{81} = .19 < 0.2$$

$$\text{IF } n=5 \Rightarrow \left(\frac{2}{3}\right)^5 = \frac{32}{243} \approx .13 < .2$$

Then $n \geq 4$ s.t.

$\left(\frac{2}{3}\right)^n < .2 \Rightarrow \therefore$ the Smallest value of (n) is equal to (4) .

Q20) The number of failures occurring in a machine of a certain type in a year has a Poisson distribution with mean (0.4). In a factory there are ten of these machines what is:-

a) the expected total number of failures in the factory in a year?

b) the Probability that there are fewer than two failures in the factory in a year?

Sol: $X \sim P(\lambda) = \frac{e^{-\lambda} \lambda^x}{x!}$

a) $E(X) = \lambda = np = 10 \times 0.4 = \boxed{4}$

b) $P_r(X < 2) = P_r(X=0) + P_r(X=1)$

$$= e^{-4} + 4e^{-4}$$

$$= 5e^{-4} = 0.0916$$

Q 21) A factory uses tools of a particular type. From time to time failures in these tools occur and they need to be replaced. The number of such failures in a day has a Poisson dist. with mean (1.25).

At the beginning of a particular day there are five replacement tools in stock.

A new delivery of replacements will arrive after four days. If all five spares are used before the new delivery arrives then further replacements cannot be made until the delivery arrives. Find:

a) the prob. that three replacements are required over the next four days.

b) The expected number of replacements actually made over the next four days.

Sol = Let the number required over 4 days be X . Then,

$$\lambda = E(X) = 4 \times 1.25 = 5 \text{ and } X \sim P(5)$$

$$a) P_r(X=3) = \frac{e^{-\lambda} \lambda^x}{x!} = \frac{e^{-5} 5^3}{3!} = 0.1404$$

b) Let R be the number of replacements made.

$$E(R) = 0 \times P_r(X=0) + 1 \times P_r(X=1) + \dots + 5 \times P_r(X \geq 5)$$

The Multivariate distribution

1. The Trinomial dist.

The Binomial dist. can be generalized to the Trinomial dist., if n is a positive integer and p_1, p_2, p_3 are fixed constants, then

$$f(x, y) = \begin{cases} \frac{n!}{x! y! (n-x-y)!} p_1^x p_2^y p_3^{n-x-y} & , x, y = 0, 1, 2, \dots, n \\ 0 & \text{otherwise} \end{cases}$$

where x and y are nonnegative integers with $x+y \leq n$, and p_1, p_2, p_3 are positive proper fractions with $p_1 + p_2 + p_3 = 1$.

Accordingly, $f(x, y)$ satisfies the conditions of being a joint p.f. of two r.v.s x and y of the discrete type and said to have trinomial dist.

$$M_{x,y}(t_1, t_2) = (p_1 e^{t_1} + p_2 e^{t_2} + p_3)^n$$

For all real values of t_1 and t_2 the marginal M.g.f. of x and y are

$$M_x(t_1) = M_{x,y}(t_1, 0) = [(1-p_2) + p_1 e^{t_1}]^n$$

$$M_y(t_2) = M_{x,y}(0, t_2) = [(1-p_1) + p_2 e^{t_2}]^n$$

We see immediately, that x and y are stochastically independent, addition, x is $b(n, p_1)$ and y is $b(n, p_2)$

* 2) The bivariate normal distribution.

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]$$

where $\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho$ are constants and parameters satisfies $-\infty < x < \infty, -\infty < y < \infty, -\infty < \mu_x < \infty, -\infty < \mu_y < \infty$

$$\sigma_x > 0, \sigma_y > 0, -1 \leq \rho \leq 1$$

$$(x, y) \sim N(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$$

The moment generating function of a bivariate normal dist. can be determined as follows:

$$\sqrt{\frac{1}{2}} = \sqrt{2} * \frac{1}{2} \sqrt{2\pi} = \sqrt{\pi}$$

Chapter Four Distributions of Functions of Random Variables

As the title of this chapter indicates, we are interested in finding the distributions of function of random variables. More precisely (أي)، for given r.v.s, say $X_1, X_2, \dots, X_n$ denote (n) r.v.s having the joint pdf $P(X_1, X_2, \dots, X_n)$, and that $Y_1 = U_1(X_1, X_2, \dots, X_n)$
 $Y_2 = U_2(X_1, X_2, \dots, X_n), \dots, Y_k = U_k(X_1, X_2, \dots, X_n)$
we want in general to find the joint dist of $Y_1, \dots, Y_k$ where $k \leq n$ as well as the marginal dist of any combination of $Y_1, Y_2, \dots, Y_k$.

In this chapter three techniques for finding the dist of functions of r.v.s will be presented. These three techniques are called:

- 1) The Transformation techniques.
- 2) The Cumulative distribution function techniques.
- 3) The Moment generating function techniques.

(أيضا، تحويلات لاول المتغيرات، و...)

$$\begin{matrix} (x_1, x_2) & & (y_1, y_2) \\ A & \xrightarrow{\quad} & B \end{matrix}$$

1) Transformation techniques of (2) random var.
a) Transformation of two discrete r.v's.

Let X_1 & X_2 are two d.r.v.s with J.P.M.F.
 $P(X_1, X_2) > 0$, such that A is space of (X_1, X_2)
 Let

$$\left. \begin{aligned} Y_1 &= U_1(X_1, X_2) \\ Y_2 &= U_2(X_1, X_2) \end{aligned} \right\} \text{ be (1-1) transformation of } A \text{ onto } B \text{ (the space of } (y_1, y_2))$$

$$\left. \begin{aligned} X_1 &= U_1^{-1}(y_1, y_2) \\ X_2 &= U_2^{-1}(y_1, y_2) \end{aligned} \right\} \text{ be (1-1) transformation of } B \text{ to } A$$

Then the J.P.M.F. of (y_1, y_2) is -

$$g(y_1, y_2) = P[U_1^{-1}(y_1, y_2), U_2^{-1}(y_1, y_2)]$$

Q5) b)

Example: Given a J.P.M.F. of X_1 & X_2

$$f(x_1, x_2) = \begin{cases} \left(\frac{2}{3}\right)^{x_1+x_2} & \left(\frac{1}{3}\right)^{2-x_1-x_2} \\ 0 & \text{otherwise} \end{cases}, (x_1, x_2) = (0,0), (0,1), (1,0), (1,1)$$

Find the J.P.M.F. of $\left. \begin{aligned} Y_1 &= X_1 - X_2 \\ Y_2 &= X_1 + X_2 \end{aligned} \right\} g(y_1, y_2)$

Sol: & Marginal of y_1

$$U_1 = Y_1 = X_1 - X_2$$

$$U_2 = Y_2 = X_1 + X_2$$

وإذا كان X_2 و X_1 قيم $1, 0, 1$
 y_2 و y_1

b) Transformation of Continuous type

In this section we shall examine the same problem when the r.v.s are the Continuous type. To find the dist. of y , we apply the Law.

$$g(y) = f(x) \cdot \left| \frac{dx}{dy} \right| = f(x) \cdot |J|$$

where $|J|$ is Jacobian

Ex) Assume that X be a r.v. having p.d.f

$$f(x) = \begin{cases} 2x & , 0 < x < 1 \\ 0 & , \text{otherwise} \end{cases}$$

Find the p.d.f. of $Y = 8X^3$?

Sol

x	$y = 8x^3$
0	0
1	8

$$A = \{x; 0 < x < 1\}, f(x) > 0$$

$$B = \{y; 0 < y < 8\}, g(y) > 0$$

$$y = 8x^3 \quad (1)$$

$$y = 8x^3$$

$$x^3 = \frac{y}{8} \Rightarrow x = \sqrt[3]{\frac{y}{8}} \Rightarrow x = \frac{\sqrt[3]{y}}{2}$$

$$g(y) = f(x) \cdot |J|$$

$$|J| = \left| \frac{dx}{dy} \right| = \frac{d(\sqrt[3]{y}/2)}{dy} = \left(\frac{\sqrt[3]{y}}{2} \right)'$$

$$= \frac{1}{2} (y^{\frac{1}{3}})' = \frac{1}{2} \cdot \frac{1}{3} y^{\frac{1}{3}-1} = \frac{1}{6} y^{-\frac{2}{3}}$$

2) The Cumulative distribution function technique

The Joint dist. of r.v's $X_1, X_2, \dots, X_n$ is given as $P(X_1, \dots, X_n)$, then the Joint dist. of r.v's of $Y_1, Y_2, \dots, Y_k$ can be determined, where $U_j(X_1, \dots, X_n)$, $j = 1, 2, \dots, k$, $k \leq n$, by using C.d.F technique, then

$$F(y_1, y_2, \dots, y_k) = P_r(Y_1 \leq y_1, Y_2 \leq y_2, \dots, Y_k \leq y_k)$$

$$= P_r[U_1(X_1, \dots, X_n) \leq y_1, \dots, U_k(X_1, \dots, X_n) \leq y_k]$$

If Y is univariate r.v., then:-

$$F(y) = P_r(Y \leq y)$$

$$F(y) = P_r[U(X_1, X_2, \dots, X_n) \leq y], \text{ then}$$

$$f(y) = F'(y), \text{ where } f(y) \text{ is P.d.f. of } y$$

Ex) Let $f(x) = \begin{cases} \frac{1}{2}, & -1 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$

Find the distribution of $Y = X^2$ by using C.d.F?

Sol:-

$$F(y) = P_r(Y \leq y) = P_r(X^2 \leq y) \quad \text{--- (1)}$$

$$= P_r(|X| \leq \sqrt{y})$$

$$= P_r(-\sqrt{y} \leq X \leq \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2} dx \quad F(y)$$

$$= \frac{1}{2} x \Big|_{-\sqrt{y}}^{\sqrt{y}} = \frac{1}{2} (\sqrt{y} + \sqrt{y}) = \frac{1}{2} \cdot 2\sqrt{y} = \sqrt{y}$$

$$\therefore F(y) = \sqrt{y}$$

③ استخراج $f(y)$ من $F(y)$!

Now,

$$f(y) = F'(y)$$

بالاشتقاق

$$f(y) = \sqrt{y}' = \left(y^{\frac{1}{2}}\right)' = \frac{1}{2} y^{-\frac{1}{2}} = \frac{1}{2\sqrt{y}}$$

لأن الحد y من الحدود لا يتغير

X

$$Y = X^2$$

④ استخراج $f(y)$ من $F(y)$

-1
1

$$(-1)^2 = 1 \approx 0$$

$$1^2 = 1$$

$$f(y) = \begin{cases} \frac{1}{2\sqrt{y}} & , 0 < y < 1 \\ 0 & , \text{otherwise} \end{cases}$$

$$\text{Ex) Let } f(x) = \begin{cases} \frac{x}{6} & , x = 1, 2, 3 \\ 0 & , \text{otherwise} \end{cases}$$

Find the dist. function of $Y = X^2$ by using C.d.F. technique.

Sol

$$F(y) = P_r(Y \leq y) = P_r(X^2 \leq y) = P_r(|X| \leq \sqrt{y})$$

$$= P_r(-\sqrt{y} \leq X \leq \sqrt{y})$$

$$= P_r(X \leq \sqrt{y})$$

لأن X متغير عشوائي

$$= \sum_{x=1}^{\sqrt{y}} \frac{x}{6} = \frac{1}{6} \sum_{x=1}^{\sqrt{y}} x$$

$$= \frac{1}{6} \cdot \frac{\sqrt{y}(\sqrt{y}+1)}{2} = \frac{\sqrt{y}(\sqrt{y}+1)}{12}$$

$$\frac{N(N+1)}{2}$$

$$N(N+1)$$

3. The Moment generating function techniques

The another method for determining the dist. of function of r.v's which we shall find to be particularly useful in certain instance for given r.v's $X_1, \dots, X_n$ with given density $f(X_1, \dots, X_n)$ and given functions $g_1(X_1, \dots, X_n), \dots, g_k(X_1, \dots, X_n)$ find the joint dist. of $Y_1 = g_1(X_1, \dots, X_n), \dots, Y_k = g_k(X_1, \dots, X_n)$.

Now, the joint M.g.f. of $Y_1, \dots, Y_k$, if it exists is $M_{Y_1, Y_2, \dots, Y_k}(t_1, \dots, t_k) = E[e^{t_1 Y_1 + t_2 Y_2 + \dots + t_k Y_k}]$

$$= \int \dots \int_{\text{all } y_1, \text{ all } y_k} e^{t_1 Y_1 + \dots + t_k Y_k} f(X_1, \dots, X_n) \prod_{i=1}^n dx_i$$

$$= \int \dots \int_{\text{all } y_1, \text{ all } y_k} e^{t_1 g_1(X_1, \dots, X_n) + \dots + t_k g_k(X_1, \dots, X_n)} f(X_1, \dots, X_n) \prod_{i=1}^n dx_i$$

Ex) Suppose X has standard normal dist. and let $Y = X^2$, find the dist. of Y by Moment generating fun. technique?

Sol

$$M_Y(t) = E[e^{tY}] = E[e^{tX^2}]$$

$$= \int_{-\infty}^{\infty} e^{tx^2} f(x) dx = \int_{-\infty}^{\infty} e^{tx^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2(1-2t)} dx$$

Order Statistics (ترتیبی آمارها)

Def: Let $Y_1, Y_2, \dots, Y_n$ be a random sample from continuous r.v's with p.d.f $f(y)$ and distribution function $F(y)$.

[ترتیبی آمارها، ترتیبی آمارها، ترتیبی آمارها]

$Y_{(1)}, Y_{(2)}, \dots, Y_{(n)}$ where $Y_{(1)} \leq Y_{(2)} \leq \dots \leq Y_{(n)}$
where $Y_{(1)} = \min(Y_1, Y_2, \dots, Y_n)$

$$Y_{(1)} = \min(Y_1, Y_2, \dots, Y_n)$$

$$Y_{(n)} = \max(Y_1, Y_2, \dots, Y_n)$$

The Aim: To find the distribution function and probability function of $Y_{(1)}$ & $Y_{(n)}$.

Here, we have four rules.

① For $Y_{(n)} = \max(Y_1, \dots, Y_n)$

* The distribution function C.d.F.

$$F_{Y_{(n)}}(y_n) = (F_Y(y_{n-1}))^n$$

* The p.d.f of $Y_{(n)}$ is

$$f_{Y_{(n)}}(y_n) = n [F_Y(y_n)]^{n-1} f_Y(y_n)$$

② For $Y_{(n)} = \min(Y_{(1)}, Y_{(2)}, \dots, Y_{(n)})$

* The C.d.f is:-

$$F_{Y_{(n)}}(y_1) = 1 - (F_Y(y_1))^n$$

* The p.d.f. is:-

$$f_{Y_{(n)}}(y_1) = n [1 - F_Y(y_1)]^{n-1} \cdot f_Y(y_1)$$

X_1	X_2	X_3	X_4
5	1	3	2

y_1

y_2

y_3

y_4

$$Y_{(1)} < Y_{(2)} < Y_{(3)} < Y_{(4)}$$

i.e

$$Y_{(1)} < Y_{(2)} < \dots < Y_{(n)} \quad , \quad n = \text{Sample Size}$$

① $Y_{(1)} = \min\{X_1, \dots, X_n\}$
 ② $Y_{(2)} = \text{Second}\{X_1, \dots, X_n\}$
 ③ $Y_{(n)} = \text{Max}\{X_1, \dots, X_n\}$

$$Y_1 = \min\{X_1, \dots, X_n\}$$

$$Y_2 = \text{Second}\{X_1, \dots, X_n\}$$

$$Y_n = \text{Max}\{X_1, \dots, X_n\}$$

$$g(y_k) = \frac{n!}{(k-1)!(n-k)!} [F(y_k)]^{k-1} [1-F(y_k)]^{n-k} p(y_k)$$

where

$$F(y_k) = \int_{-\infty}^{y_k} f(x) dx$$

and

min value of x

$$* g(y_1) = n [1-F(y_1)]^{n-1} p(y_1) \quad (\text{min})$$

$$* g(y_n) = n [F(y_n)]^{n-1} p(y_n) \quad (\text{max})$$

$$* y_n - y_1 \Rightarrow \text{range of r.s.}$$

$$* \frac{y_n + y_1}{2} \Rightarrow \text{mid-range}$$

$$* \frac{y_{n+1}}{2} \Rightarrow \text{median [is case of odd observation]}$$

$$* \text{The Joint of } y_1, y_2, \dots, y_n \text{ is}$$

$$g(y_1, y_2, \dots, y_n) = n! p(y_1) p(y_2) \dots p(y_n), a < y_1 < y_2 < \dots < y_n$$

$$= n! \prod_{i=1}^n p(y_i)$$

$$* \text{The Joint of } i \& j \text{ is}$$

$$g(y_i, y_j) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} [F(y_i)]^{i-1} [F(y_j) - F(y_i)]^{j-i-1} [1-F(y_j)]^{n-j} p(y_i) p(y_j)$$

$$p(i) < p(j),$$

$$a < i < j < b$$