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<u>Solving Linear Recurrence Relations</u>

Linear recurrences

1. Linear homogeneous recurrences
2. linear non-homogeneous recurrences

<u>Definition:</u> A **linear homogenous recurrence relation of degree k** with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

Where $c_1, c_2, \dots, c_k$ are real numbers, and $c_k \neq 0$.

a_n is expressed in terms of the previous k terms of the sequence, so its degree is k. This recurrence includes k initial conditions

$a_0 = C_0 \qquad a_1 = C_1 \quad \dots \quad a_k = C_k.$

<u>Example:</u> Determine if the following recurrence relations are linear homogeneous recurrence relations with constant coefficients.

- $P_n = (1.11)P_{n-1}$

  a linear homogeneous recurrence relation of degree one.
- $a_n = a_{n-1} + a_{n-2}^2$

  not linear
- $f_n = f_{n-1} + f_{n-2}$

  a linear homogeneous recurrence relation of degree two
- $H_n = 2H_{n-1} + 1$

  not homogeneous because $f(x) = 1$.
- $a_n = a_{n-6}$

  a linear homogeneous recurrence relation of degree six
- $B_n = nB_{n-1}$

  does not have constant coefficien

<u>Theorem:</u> Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ be a linear homogeneous recurrence. Assume the sequence a_n satisfies the recurrence and the sequence g_n also satisfies the recurrence. So, $b_n = a_n + g_n$ and $d_n = \alpha a_n$ are also sequences that satisfy the recurrence. (α is any constant)

Note: Geometric sequences come up a lot when solving linear homogeneous recurrences. So, try to find any solution of the form $a_n = r^n$ that satisfies the recurrence relation.

Recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

Try to find a solution of form r^n

$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$

$r^n - c_1 r^{n-1} - c_2 r^{n-2} - \cdots - c_k r^{n-k} = 0$

$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$ (dividing both sides by r^{n-k})

This equation is called the **characteristic equation**.

Example: The Fibonacci recurrence is $F_n = F_{n-1} + F_{n-2}$. Its characteristic equation is $r^2 - r - 1 = 0$.

Theorem: r is a solution of $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$ if and only if r^n is a solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$.

Example: Consider the characteristic equation $r^2 - 4r + 4 = 0$.

$r^2 - 4r + 4 = (r-2)^2 = 0$

So, $r = 2$, then 2^n satisfies the recurrence $F_n = 4F_{n-1} - 4F_{n-2}$

$2^n = 4.2^{n-1} - 42^{n-2}$

$2^n - 4.2^{n-1} + 42^{n-2} = 0$

$2^{n-2}(2^2 - 4.2 + 4) = 0$

$2^{n-2}(4 - 8 + 4) = 0$

Theorem: Consider the characteristic equation $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$ and the recurrence $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$. Assume $r_1, r_2, \ldots, r_m$ all satisfy the equation. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ by any constants. So, $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_m r_m^n$ satisfies the recurrence.

Example: What is the solution of the recurrence relation $f_n = f_{n-1} + f_{n-2}$ with $f_0 = 0 \ and \ f_1 = 1$?

Since it is linear homogeneous recurrence, first find its characteristic equation

$r^2 - r - 1 = 0, r_1 = \frac{1+\sqrt{5}}{2} \ and \ r_2 = \frac{1-\sqrt{5}}{2}$

So, by theorem $f_n = \alpha_1(\frac{1+\sqrt{5}}{2})^n + \alpha_2(\frac{1-\sqrt{5}}{2})^n$ is a solution. Now, we should find α_1 and α_2 using initial conditions

$f_0 = \alpha_1 + \alpha_2 = 0$

$$f_1 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

So, $\alpha_1 = \frac{1}{\sqrt{5}}$ and $\alpha_2 = -\frac{1}{\sqrt{5}}$.

$a_n = \frac{1}{\sqrt{5}}.(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}.(\frac{1-\sqrt{5}}{2})^n$ is a solution.

Example: What is the solution of the recurrence relation $a_n = -a_{n-1} + 4a_{n-2} + 4a_{n-3}$ with $a_0 = 8, a_1 = 6$ and $a_2 = 26$?

Since it is linear homogeneous recurrence, first find its characteristic equation

$r^3 + r^2 - 4r - 4 = 0$

$(r+1)(r+2)(r-2) = 0$ $r_1 = -1, r_2 = -2$ and $r_3 = 2$

So, by theorem $a_n = \alpha_1(-1)^n + \alpha_2(-2)^n + \alpha_3(2)^n$ is a solution. Now, we should find α_1, α_2 and α_3 using initial conditions.

$a_0 = \alpha_1 + \alpha_2 + \alpha_3 = 8$

$a_1 = -\alpha_1 - 2\alpha_2 + 2\alpha_3 = 6$

$a_2 = \alpha_1 + 4\alpha_2 + 4\alpha_3 = 26$

So, $\alpha_1 = 2, \alpha_2 = 1$ and $\alpha_3 = 5$.

$a_n = 2(-1)^n + (-2)^n + 5(2)^n$ is a solution.

Theorem: Consider the characteristic equation $r^k - c_1r^{k-1} - c_2r^{k-2} - \cdots - c_k = 0$ and the recurrence $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$. Assume the characteristic equation has $t \leq k$ distinct solutions. Let $\forall i(1 \leq i \leq t)r_i$ with multiplicity m_i be a solution of the equation and let $\forall i,j(1 \leq i \leq t$ and $0 \leq j \leq m_i - 1)\alpha_{ij}$ be a constant. So,

$$a_n = (\alpha_{10} + \alpha_{11}n + \cdots + \alpha_{1m_1-1}n^{m_1-1})\, r_1^n + (\alpha_{20} + \alpha_{21}n + \cdots + \alpha_{2m_2-1}n^{m_2-1})\, r_2^n + \ldots + (\alpha_{t0} + \alpha_{t1}n + \cdots + \alpha_{tm_t-1}n^{m_t-1})\, r_t^n$$

satisfies the recurrence.

Example: What is the solution of the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$with $a_0 = 1$ and $a_1 = 6$?

First find its characteristic equation

$r^2 - 6r + 9 = 0 \quad \rightarrow \quad (r-3)^2 \quad \rightarrow \quad r_1 = 3$ (Its multiplicity is 2)

So, by theorem $a_n = (\alpha_{10} + \alpha_{11} n)(3)^n$ is a solution. Now, we should find α_{10} and α_{11} using initial conditions.

$a_0 = \alpha_{10} = 1$

$a_1 = 3\alpha_{10} + 3\alpha_{11} = 6$

Hence, $\alpha_{10} = 1 and\ \alpha_{11} = 1$.

$a_n = (3)^n + n(3)^n$ is a solution.

Linear non-homogeneous recurrences

Definition: A **linear non-homogenous recurrence relation of degree k** with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n)$$

Where $c_1, c_2, \dots, c_k$ are real numbers, and $f(n)$ is a function depending only on n.

The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

is called the associated homogeneous recurrence relation.

This recurrence includes k initial conditions

$a_0 = C_0 \qquad a_1 = C_1 \quad \dots \quad a_k = C_k.$

Example: The following recurrence relations are linear nonhomogeneous recurrence relations.

- $a_n = a_{n-1} + 2^n$
- $a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$
- $a_n = a_{n-1} + a_{n-2} + n!$
- $a_n = a_{n-6} + n2^n$

Theorem: Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n)$ be a linear nonhomogeneous recurrence. Assume the sequence b_n satisfies the recurrence and another sequence a_n also satisfies the non-homogeneous recurrence if and only if $h_n = a_n - b_n$ is also sequences that satisfies the associated homogeneous recurrence.

Example: What is the solution of the recurrence relation $a_n = a_{n-1} + a_{n-2} + 3n + 1$ for $n \geq 2$ with $a_0 = 2\ and\ a_1 = 3$?

Since it is linear non-homogeneous recurrence, b_n is similar to $f(n)$

Guess: $b_n = cn + d$

$b_n = b_{n-1} + b_{n-2} + 3n + 1$

$cn + d = c(n-1) + d + c(n-2) + d + 3n + 1$

$cn + d = cn - c + d + cn - 2c + d + 3n + 1$

$(c - 2c)n + (d - 2d) = -3c + 3n + 1$

$-cn - d = 3n - 3c + 1$

$c = -3 \qquad d = -10$

So, $b_n = -3n - 10$

(b_n only satisfies the recurrence, it does not satisfy the initial conditions.)

We are looking for an that satisfies both recurrence and initial conditions. $a_n = b_n + h_n$ where h_n is a solution for the associated homogeneous recurrence: $h_n = h_{n-1} + h_{n-2}$

By previous example, we know $h_n = \alpha_1(\frac{1+\sqrt{5}}{2})^n + \alpha_2(\frac{1-\sqrt{5}}{2})^n$

$a_n = b_n + h_n$

$\quad = -3n - 10 + \alpha_1(\frac{1+\sqrt{5}}{2})^n + \alpha_2(\frac{1-\sqrt{5}}{2})^n$

Now we should find constants using initial conditions

$a_0 = -10 + \alpha_1 + \alpha_2 = 2$

$a_1 = -13 + \alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right) = 3$

Hence, $\alpha_1 = 6 + 2\sqrt{5}\ and\ \alpha_2 = 6 - 2\sqrt{5}$.

So, $a_n = -3n - 10 + (6 + 2\sqrt{5})\left(\frac{1+\sqrt{5}}{2}\right)^n + (6 - 2\sqrt{5})\left(\frac{1-\sqrt{5}}{2}\right)^n$.