University of Baghdad

College of Science for Women

Department of Computer Science

Computer Mathematics

Ahmed J. Kadhim, M. Sc.

2023 - 2024

Recurrences

A recurrence describes a sequence of numbers. Early terms are specified explicitly and later terms are expressed as a function of their predecessors. As a trivial example, this recurrence describes the sequence 1, 2, 3, etc.:

$$T_1 = 1$$

$$T_n = T_{n-1} + 1, \qquad (for\ n \geq 2)$$

Here, the first term is defined to be 1 and each subsequent term is one more than its predecessor.

Example: Use mathematical induction to show that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for all nonnegative integers n.

Solution: Let $P(n)$ be the proposition that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for the integer n.

$P(0)$ is true because $2^0 = 1 = 2^1 - 1$

For the inductive hypothesis, we assume that $P(k)$is true for an arbitrary nonnegative integer k. That is, we assume that

$1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$ ……(*)

then P(k+ 1) is also true. That is, we must show that

$1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 1 = 2^{k+2} - 1$?

From equation (*)$1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = (1 + 2 + 2^2 + \cdots + 2^k) + 2^{k+1}$

$$= 2^{k+1} - 1 + 2^{k+1} = 2.2^{k+1} - 1$$

$$= 2^{k+2} - 1$$

By mathematical induction we knowthat $P(n)$ is true for all nonnegative integers n.

Example: Show that if n is a positive integer, then

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Solution: Let $P(n)$ be the proposition that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for the positive integer n

P(1) is true, because $\frac{1.(1+1)}{2} = 1$

we assume that $P(k)$ holds for an arbitrary positive integer k. That is, we assume that

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2} \qquad \text{……(*)}$$

It must be shown that $P($k+ 1$)$ is true

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

This last equation shows that $P(k+1)$ is true under the assumption that $P(k)$ is true

Example: Use mathematical induction to show that $1 + 3 + 5 + \cdots + (2n-1) = n^2$ is true for all positive integers n. (H.W)

Applications of Recurrence Relations

The Fibonacci sequence

Fibonacci published in the year 1202 is now famous rabbit puzzle:

A man put a male-female pair of newly born rabbits in a field. Rabbits take a month to mature before mating. One month after mating, females give birth to one male-female pair and then mate again. No rabbits die. How many rabbit pairs are there after one year?

To solve, we construct Table 1.1. At the start of each month, the number of juvenile pairs, adult pairs, and total number of pairs are shown. At the start of January, one pair of juvenile rabbits is introduced into the population. At the start of February, this pair of rabbits has matured. At the start of March, this pair has given birth to a new pair of juvenile rabbits. And so on.

Month	J	F	M	A	M	J	J	A	S	O	N	D	J
Juvenile	1	0	1	1	2	3	5	8	13	21	34	55	89
Adult	0	1	1	2	3	5	8	13	21	34	55	89	144
Total	1	1	2	3	5	8	13	21	34	55	89	144	233

Table 1.1: Fibonacci's rabbit population

We define the Fibonacci numbers F_n to be the total number of rabbit pairs at the start of the nth month. The number of rabbits pairs at the start of the 13th month, $F_{13} = 233$, can be taken as the solution to Fibonacci's puzzle.

Further examination of the Fibonacci numbers listed in Table 1.1, reveals that these numbers satisfy the recursion relation

$$F_{n+1} = F_n + F_{n-1}. \qquad (1.1)$$

This recursion relation gives the next Fibonacci number as the sum of the preceding two numbers. To start the recursion, we need to specify F_1 *and* F_2. In Fibonacci's rabbit problem, the initial month starts with only one rabbit pair so that $F_1 = 1$. And this initial rabbit pair is newborn and takes one month to mature before mating so $F_2 = 1$. The first few Fibonacci numbers, read from the table, are given by

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$$

and has become one of the most famous sequences in mathematics.

Example: The Lucas numbers are closely related to the Fibonacci numbers and satisfy the same recursion relation $L_{n+1} = L_n + L_{n-1}$, but with starting values $L_1 = 1$ *and* $L_2 = 3$. Determine the first 12 Lucas numbers.

$if\ n = 2 \rightarrow L_{2+1} = L_2 + L_{2-1} = 3 + 1 = 4$

$if\ n = 3 \rightarrow L_{3+1} = L_3 + L_{3-1} = 4 + 3 = 7$

By the same way, we found the first 12 Lucas numbers.

$$1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322$$

Example: The generalized Fibonacci sequence satisfies $f_{n+1} = f_n + f_{n-1}$ with starting values $f_1 = p$ *and* $f_2 = q$. Using mathematical induction, prove that

$$f_{n+2} = F_n p + F_{n+1} q. \qquad (2)$$

We now prove (2) by mathematical induction

Base case: To prove that (2) is true for $n = 1$, we write $F_1 p + F_2 q = p + q = f_3$. To prove that (2) is true for $n = 2$, we write $F_2 p + F_3 q = p + 2q = f_3 + f_2 = f_4$.

Induction step: Suppose that (2) is true for positive integers $n = k - 1$ *and* $n = k$. Then we have

$f_{n+2} = f_{n+1} + f_n$

If $n = k + 1$, then

$$\begin{aligned} f_{k+3} &= f_{k+2} + f_{k+1} \\ &= (F_k p + F_{k+1} q) + (F_{k-1} p + F_k q) \\ &= (F_k + F_{k-1}) p + (F_{k+1} + F_k) q \\ &= F_{k+1} p + F_{k+2} q \end{aligned}$$

so that (2) is true for $n = k + 1$. By the principle of induction, (2) is therefore true for all positive integers.