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Computer Mathematics

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Problems:

1. Use mathematical induction to prove $\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1}-a}{r-1}$ where $r \neq 1$
2. Prove that $L_n = F_{n-1} + F_{n+1}$ by using the Lucas sequence $L_{n+1} = L_n + L_{n-1}$ and this relation $f_{n+2} = F_n p + F_{n+1} q$ with values $p = 1 \; and \; q = 3$?
3. Determine values of the constants A and B such that $a_n = An + B$ is a solution of recurrence relation $a_n = 2a_{n-1} + n + 5$?
4. Find the solution to the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with the initial conditions $a_0 = 2, a_1 = 5, and \; a_2 = 15$?
5. What is the solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?
6. What is the solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2} + 2^n \; for \; n \geq 2$ with $a_0 = 1 \; and \; a_1 = 2$?
7. Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?
8. What is the solution of the recurrence relation $a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$ with $a_0 = 1$ and $a_1 = 2$?

$a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$

Since it is linear non-homogeneous recurrence, b_n is similar to $f(n)$

Guess: $b_n = cn^2 + en + d$

$b_n = b_{n-1} + b_{n-2} + n^2 + n + 1$

$$cn^2 + en + d = c(n-1)^2 + e(n-1) + d + c(n-2)^2 + e(n-2) + d + n^2 + n + 1$$

$$= cn^2 - 2cn + c + en - e + d + cn^2 - 4cn + 4c + en - 2e + d + n^2 + n + 1$$

$$= (cn^2 + cn^2 + n^2) + (-2cn + en - 4cn + en + n) + (-e + d + c + 4c - 2e + d + 1)$$

$c = 2c + 1 \rightarrow c = -1$

$e = -6c + 2e + 1 \rightarrow e = -7$

$d = -3e + 5c + 2d + 1 \rightarrow d = -17$

$b_n = -n^2 - 7n - 17$

$a_n = b_n + h_n \rightarrow h_n = h_{n-1} + h_{n-2}$

$r^2 - r - 1 = 0 \rightarrow r = \frac{1 \pm \sqrt{5}}{2}$

$h_n = \alpha_1 (\frac{1+\sqrt{5}}{2})^n + \alpha_2 (\frac{1-\sqrt{5}}{2})^n$

$a_n = -n^2 - 7n - 17 + \alpha_1 (\frac{1+\sqrt{5}}{2})^n + \alpha_2 (\frac{1-\sqrt{5}}{2})^n$

$a_0 = -17 + \alpha_1 + \alpha_2 = 1$

$a_1 = -1 - 7 - 17 + \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) \quad + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) \quad = 2$

$= -25 + \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) \quad + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) \quad = 2$

$\alpha_1 = 9 + \frac{17}{\sqrt{5}}, \;\; \alpha_2 = 9 - \frac{17}{\sqrt{5}}$

$\therefore a_n = -n^2 - 7n - 17 + (9 + \frac{17}{\sqrt{5}})(\frac{1+\sqrt{5}}{2})^n + (9 - \frac{17}{\sqrt{5}})(\frac{1-\sqrt{5}}{2})^n$

Counting

Counting is the task of finding the number of elements (also called the **cardinality**) of a given set.

When the set is small, we can count its elements "by hand". When sets are larger we need a more systematic way to count.

Say you have a six-sided die and a two-sided coin it comes out heads (H) or tails (T). What is the number of possible outcomes when both the die and the coin are tossed? There are 6 possible outcomes for the die and 2 for the coin, so the total number of outcomes is 6 . 2 = 12.

It will be useful to describe this kind of problem using the language of sets. The set **S** of possible outcomes of the die is **S** = {1, 2, 3, 4, 5, 6}, so **|S|** = 6. The set **T** of possible outcomes of the coin is **T** = {H, T}, so **|T|** = 2. The set of possible outcomes of the die and the coin is the product set $\boldsymbol{S} \times \boldsymbol{T}$:

$$\boldsymbol{S} \times \boldsymbol{T} = \{(1,\mathrm{H}),(1,\mathrm{T}),(2,\mathrm{H}),(2,\mathrm{T}),\ldots,(6,\mathrm{H}),(6,\mathrm{T})\}.$$

The number of elements of $\boldsymbol{S} \times \boldsymbol{T}$ is **|S|**.**|T|** =6 . 2 = 12.

In general, given any two finite sets **S** and **T** the product set $\boldsymbol{S} \times \boldsymbol{T}$ consists of all ordered pairs of elements (**s**, **t**) such that **s** is in **S** and **t** is in **T**:

$$\boldsymbol{S} \times \boldsymbol{T} = \{(s,t); s \in \boldsymbol{S} \; and \; t \in \boldsymbol{T}\}$$

Example

Let R and B be the sets of outcomes of a toss of a red and a blue six-sided die, respectively. Then $R = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 2, 3, 4, 5, 6\}$. When both dies are tossed, the set of outcomes is $R \times B = \{(1, 1), (1, 2), \dots, (6, 6)\}$

and the number of outcomes is $\boldsymbol{R \times B = |R|.|B| = 36}$

In cases like this when S = T we can denote the set $S \times T$ by $S^{\mathbf{2}}$**.** (This is the square of a set, not the square of a number). The set $S^{\mathbf{2}}$ has $|S|^2$ elements.

<u>Problems</u>

1. How many elements of the set of outcomes when 9 different six-sided dies are tossed?
2. How many elements of the product set of $S \times T \times R$ are there? Where $S = \{1,1,2\}$,$T = \{2, 1,1\}$, and $R = \{1, 2, 2\}$.

<u>The sum rule</u>

The sum rule says that if $A_1,\ A_2, \dots,\ A_n$ are disjoint sets then

$$|A_1\ \cup\ A_2 \cup \dots \cup\ A_n| = |A_1| + |A_2| + \cdots + |A_n|$$

<u>Example</u>

If you have 10 white balls, 7 blue balls, and 4 red balls, then the total number of balls you have is $10\ +\ 7\ +\ 4\ =\ 21$.

<u>Example</u>

Bart allocates his little sister Lisa a quota of 20 bad days, 40 irritable days, and 60 generally surly days. On how many days can Lisa be out-of-sorts one way or another?

Let set B be her bad days, I be her irritable days, and S be the generally surly. In these terms, the answer to the question is $|B\ \cup I \cup S|$. Now assuming that she is permitted at most one bad quality each day, the size of this union of sets is given by

$|B\ \cup I \cup S| = |B| + |I| + |S| = 20 + 40 + 60 = 120$ days.

Few counting problems can be solved with a single rule. More often, a solution is a flurry of sums, products, and other methods.

<u>Example</u>

For solving problems involving passwords, telephone numbers, and license plates, the sum and product rules are useful together. For example, on a certain computer system, a valid password is a sequence of between six and eight symbols. The first symbol must be a letter (which can be lowercase or uppercase), and the remaining symbols must be either letters or digits. How many different passwords are possible?

Let's define two sets, corresponding to valid symbols in the first and subsequent positions in the password.

$$F = \{a, b, \dots, z, A, B, \dots, Z\}$$

$$S = \{a, b, \dots, z, A, B, \dots, Z, 0,1, 2, \dots ,9\}$$

In these terms, the set of all possible passwords is:

$$(F \times S^5) \cup (F \times S^6) \cup (F \times S^7)$$

Thus, the length-six passwords are in the set $F \times S^5$, the length-seven passwords are in $F \times S^6$, and the length-eight passwords are in $F \times S^7$. Since these sets are disjoint, we can apply the sum rule and count the total number of possible passwords as follows:

$$|(F \times S^5) \cup (F \times S^6) \cup (F \times S^7)| = |F \times S^5| + |F \times S^6| + |F \times S^7|$$

$$= |F|.|s|^5 + |F|.|s|^6 + |F|.|s|^7$$

$$= 52\ .\ 62^5 + 52\ .\ 62^6 + 52\ .\ 62^7$$

$$\approx 1.8.10^{14} \text{ different passwords}$$

<u>Product Rule</u>

Let S be a set of length-k sequences. If there are:

- n_1 possible first entries,
- n_2 possible second entries for each first entry,
- n_3 possible third entries for each first entry,

  ⋮
- n_k possible kth entries for each sequence of first $k-1$ entries,

Then $|S| = n_1.n_2.n_3 \dots n_k$

<u>Example</u>

In how many different ways can we place a pawn (P), a knight (N), and a bishop (B) on a chessboard so that no two pieces share a row or a column?

The position of the three pieces is specified by a six numbers $(r_P\ , C_P\ , r_N\ , C_N, r_B\ , C_B)$ where, r_P, r_N and r_B are distinct rows and C_P, C_N and C_B are distinct columns. In particular, r_P

is the pawn's row C_P is the pawn's column r_N is the knight's row, etc. Now we can count the number of such sequences using the product rule:

- r_P is one of 8 rows
- C_P is one of 8 columns
- r_N is one of 7 rows (anyone but r_P)
- C_N is one of 7 columns (anyone but C_P)
- r_B is one of 6 rows (anyone but r_P or r_N)
- C_B is one of 6 columns (anyone but C_P or C_N)

Thus, the total number of configurations is $8.8.7.7.6.6 = (8.7.6)^2$

Example

A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Example

In how many ways can we select three students from a group of five students to stand in line for a picture? In how many ways can we arrange all five of these students in a line for a picture? First, note that the order in which we select the student's matters. There are five ways to select the first student to stand at the start of the line. Once this student has been selected, there are four ways to select the second student in the line. After the first and second students have been selected, there are three ways to select the third student in the line. By the product rule, there are $5.4.3 = 60$ ways to select three students from a group of five students to stand in line for a picture.

To arrange all five students in a line for a picture, we select the first student in five ways, the second in four ways, the third in three ways, the fourth in two ways, and the fifth in one way. Consequently, there are $5.4.3.2.1 = 120$ ways to arrange all five students in a line for a picture.

Permutations

A **permutation** of a set of distinct objects is an ordered arrangement of these objects. We also are interested in ordered arrangements of some of the elements of a set. An ordered arrangement of *r* elements of a set is called an ***r*-permutation**.

Example

Let $S = \{1, 2, 3\}$.The ordered arrangement 3, 1, 2 is a permutation of S. The ordered arrangement 3, 2 is a 2-permutation of S.

The number of r-permutation of a set with n elements is denoted by $p(n, r)$ We can find $P(n, r)$ using the product rule.

Theorem

If n and r are integers with $0 \leq r \leq n$, then $P(n, r) = \frac{n!}{(n-r)!}$

Example

Let $S = \{a, b, c\}$.The 2-permutation of S are the ordered arrangements a, b; a, c; b, a; b, c; c, a; and c, b. Consequently, there are six 2-permutation of this set with three elements, it follows that $P(n, r) = \frac{n!}{(n-r)!} = \frac{3!}{(3-2)!} = 3 \cdot 2 = 6$.

Example

How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

Because it matters which person wins which prize, the number of ways to pick the three prize winners is the number of ordered selections of three elements from a set of 100 elements, that is, the number of 3-permutations of a set of 100 elements. Consequently, the answer is $P(n, r) = \frac{n!}{(n-r)!} = \frac{100!}{(100-3)!} = 100 \cdot 99 \cdot 98 = 970{,}200$.

Example

Suppose that there are eight runners in a race. The winner receives a gold medal, the second-place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties?

The number of different ways to award the medals is the number of 3-permutations of a set with eight elements. Hence, there are $P(n, r) = \frac{n!}{(n-r)!} = \frac{8!}{(8-3)!} = 8 \cdot 7 \cdot 6 = 336$ possible ways to award the medals.