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**Fourier Series**

# Introduction:

For any periodic function with a fundamental frequency could be expanded to a series of trigonometric functions with a fundamental frequency and harmonic frequencies where.

The function is called *periodic* if it repeats itself after a constant interval of time, i.e.:

The Fourier series expansion of the function is as follows:

Here,.

The coefficients, and could be computed using trigonometric integration:

**Ex1:** Find the Fourier series expansion for the periodic function defined as:

**Answer:**

**Ex2:** Find the Fourier series expansion for the periodic function defined as:

**Answer:**

# Even and Odd Functions:

A periodic function is called *even* if it is symmetric over the -axis. This can be expressed as:

A periodic function is called *odd* if it is symmetric over the origin. This can be expressed as:

The function in Ex1 is even, while the function in Ex2 is odd.

For an even function:

,,

This expansion is called *half-range cosine expansion*.

For an odd function:

, ,

This expansion is called *half-range sine expansion*.

**Ex3:** Find the half-range sine expansion of

**Answer:**

**Ex4:** Find the half-range cosine expansion of

**Answer:**

# Parseval's Theorem:

For a periodic function with a period and a Fourier series expansion:

The Parseval's theorem states that:

**Ex5:** The Fourier series expansion of

Find the Parseval's value of this function.

**Answer:**

# Complex Form of Fourier series:

A periodic function with a period can be expanded into another form of Fourier series called *complex form* or *exponential form*. The complex form of the Fourier series is expressed as:

Here, the coefficient can be evaluated as:

The coefficients , and can be deduced directly from using the following equations:

From the complex form of the Fourier series, one can draw the amplitude spectrum of the function. This spectrum is drawn for versus.

**Ex6:** Find the complex Fourier series expansion for the periodic function, then find the coefficients , and . Find vs.

**Answer:**

, vs.

**Exercises:**

Calculate the Fourier trigonometric coefficients of the functions below:

1.
2. .
3. .
4. .
5. .
6. .
7. .
8. .

Calculate the Complex Fourier coefficient of the functions below, and then use it to calculate Fourier trigonometric coefficients:

1. .
2. .
3.
4. .

**Fourier Transform**

# Introduction:

For any periodic function with a period and fundamental frequency, the combined Fourier series integral of that function is formed as:

Recall that, the integral becomes:

When, and. The Fourier Integral will be:

From Fourier Integral, we can write two equations, which form the *two Fourier Transform* pair, which is a transformation from time domain to frequency domain.

The first equation is called the *Fourier Transform* (FT) of the function.

The second equation is called the *Inverse Fourier Transform* of the function.

**Ex1:** Find the FT for the function.

**Answer:** .

# Unit Impulse (Dirac) and Unit Step (Heaviside) Functions:

The unit impulse function (aka Dirac delta) is defined as:

The unit step function (aka Heaviside) is defined as:

The shifted Dirac and Heaviside functions are defined as:

The Dirac function is the derivative of the Heaviside function:

Some common properties of Dirac and Heaviside functions are listed below:

**Ex2:** Find the FT for the function and then for.

**Answer:** .

**Ex3:** Find the FT for the function.

**Answer:** .

**Ex4:** Find the FT for the function.

**Answer:** .

# Even and Odd Fourier Transforms:

Every function contains even and odd components and respectively.

If, then:

The real part of represents the FT of the even component of and the imaginary part of represents the FT of the odd component of. It can be concluded from that property that if is even, then is real; if is odd, then is pure imaginary.

# Amplitude and Phase Spectra of Fourier Transform:

Fourier Transform is a complex function that has an amplitude and phase for its values. The FT of a function can be split into two distinct functions; the first is the amplitude function and the second is the phase function.

Hence, the FT function could be represented using the amplitude and phase spectra as shown below:

The amplitude spectrum function is always an even function; the phase spectrum function is always an odd function.

**Ex5:** Find the real and imaginary components of the FT for the function, then find its magnitude and phase functions.

**Answer:** .

# Common Fourier Transform Pairs:

|  |  |  |
| --- | --- | --- |
| Function |  |  |
| Dirac |  |  |
| Delayed Dirac |  |  |
| Heaviside |  |  |
| Delayed Heaviside |  |  |
| Real Exponential |  |  |
| Imaginary Exponential |  |  |
| Cosine Wave |  |  |
| Sine Wave |  |  |

# Properties of Fourier Transform:

## Linearity:

If and, and are constants, then:

**Ex6:** Find the FT for the function.

**Answer:** .

## Time Shifting:

If, then:

**Ex7:** Find the FT for the function.

**Answer:** .

## Frequency Shifting:

If, then:

**Ex9:** Find the FT for the function.

**Answer:** .

## Time Scaling:

If, then:

**Ex8:** Find the FT for the function.

**Answer:** .

## Time Reversal:

If, then:

## Duality:

If, then:

**Ex10:** Find the FT for the function.

**Answer:** .

## Convolution:

If and, then:

## Time Domain Differentiation:

If, then:

**Ex11:** Find the FT for the function.Hint: .

**Answer:** .

## Frequency Domain Differentiation:

If, then:

**Ex12:** Find the FT for the function.

**Answer:** .

## Time Domain Integration:

If, then:

**Ex13:** Find the FT for function

**Answer:** .

# Fourier Transform of Periodic Functions:

For a periodic function that has a period and generated from an aperiodic function that has a FT, the FT of the periodic function will be:

**Ex14:** Find the FT for the periodic function

**Answer:** .

# Solving ODE using Fourier Transform:

For a LTI system defined by an ordinary differential equation (ODE) like:

The system could be solved using the FT as below:

Therefore, the system could be found:

If the input is given, the output could be found using inverse FT:

**Ex15:** For the LTI system, Find, draw the spectra, and find the system response when the input.

**Answer:** .

**Ex16:** Find when the LTI system is defined as:

**Answer:** .

# MATLAB Code for Fourier Transform:

Fourier analysis has been embedded in MATLAB and SIMULINK software. There are two MATLAB functions for Fourier analysis; the first is used for FT, and the second is used for inverse FT. Using these functions requires installing Symbolic Math Toolbox inside MATLAB. The syntax of these functions is as below:

In the first function, is the function in time domain that we want to find its FT, associated with the time variable. The frequency variable will be the associated variable of. In the second function, is the function in frequency domain that we want to find its inverse FT, associated with the frequency variable. The time variable will be the associated variable of. Before we can use these functions, the variables must be defined symbolically using the function. An example explains:

**Exercises:**

Find the FTs of the functions below, and draw the amplitude and phase spectra for each if requested:

1. Draw spectra.
2.
3. Draw spectra.
4. Draw Spectra.
5. .
6. .

**Laplace Transform**

# Introduction:

As in the previous chapter, the Fourier integral of a function is:

If we multiply the inside integral by the convergence factor and the outside integral by, we get:

If we denote by and replacing by the integral becomes:

This integral is called *Laplace Integral*. We can write two equations from that, which form the *two Laplace Transform* pair, which is a transformation from the time domain to the complex domain.

The first equation is called the *Laplace Transform* (LT) of the function.

The second equation is called the *Inverse Laplace Transform* of the function.

LT is the most important operational method in Engineering. By using it, ODEs are converted into algebraic equations.

# Graphic Representation of the Laplace Transform:

The LT complex plane is referred to as the -plane. The horizontal and vertical axes are referred to as the -axis and the -axis, respectively.

 is usually a rational function in , i.e.:

The orders and are positive integers. is called a *proper rational* function if, and an *improper rational* function if. The roots of the numerator polynomial are called the *zeros* of because for those values of. However, the roots of the denominator polynomial are called the *poles* of because is infinite for those values of.

The LT function is represented in the -plane by the locations of poles and zeros. An is used to indicate each pole location and an is used to indicate each zero location.

# Region of Convergence of Laplace Transform:

The range of values of the complex variable for which the LT converges is called the *region of convergence* (ROC). The poles of lie outside the ROC since does not converge at the poles. The zeros may lie inside or outside the ROC. Therefore, ROC lies in the right side of the rightest pole of . Furthermore, if ROC covers the positive part of -plane, then the function could be called *convergent*.

For, there is one zero in and two poles in and . The ROC begins at excluded rightward. Because of one pole is in the positive side of -plane, the function is divergent.

# Common Laplace Transform Pairs:

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|

|  |  |  |
| --- | --- | --- |
| Function |  |  |
| Dirac Delta |  |  |
| Delayed Dirac Delta |  |  |
| Heaviside Step |  |  |
| Delayed Heaviside Step |  |  |
| Ramp |  |  |
| Polynomial |  |  |
| Exponential |  |  |
| Ramp Exponential |  |  |
| Cosine Wave |  |  |
| Sine Wave |  |  |
| Damped Cosine Wave |  |  |
| Damped Sine Wave |  |  |

 |

# Properties of Laplace Transform:

If and, and are constants, then the following properties hold:

## Linearity:

**Ex1:** Find the LT of the functions and.

**Answer:** .

## Time Shifting:

If, then:

**Ex2:** Find the LT of the function.

**Answer:** .

## S-Domain Shifting:

**Ex3:** Find the LT of the function.

**Answer:** .

## Time Scaling:

**Ex4:** Find the LT of the function.

**Answer:** .

## Convolution:

If and, then:

**Ex5:** Find the inverse LT of the function.

**Answer:** .

## Time Domain Differentiation:

**Ex6:** Find the LT of the function.

**Answer:** .

## S-Domain Differentiation:

**Ex7:** Find the LT of the function.

**Answer:** .

## Initial Value Theorem:

The initial value of the function could be found using LT.

**Ex8:** Find the initial value of the function.

**Answer:** , .

## Final Value Theorem:

The final value of the function could also be found using LT.

**Ex9:** Find the final value of the function.

**Answer:** , .

# Laplace Transform of Periodic Functions:

For a periodic function that has a period and generated from an aperiodic function that has a LT, the LT of the periodic function will be:

**Ex10:** Find the LT of the periodic impulse train function.

**Answer:** .

# Inverse Laplace Transform:

The canonical equation of inverse LT is shown below:

Computation of inverse LT by using that equation is very complicated, then we could use the LT pairs as well as LT properties to find the time domain functions for its corresponding -domain functions.

There are many other ways to determine the inverse LT. The most widely used method is called Partial Fraction expansion that will be discussed here. Other ways are the convolution as in Ex6, and the Residue theorem that is used in complex integration. That method will be discussed later in the Complex Variable chapter.

# Partial Fraction Expansion:

If is a rational function of the form:

Here, and are the zeroes and the poles of, respectively, and. The inverse LT could be found using one of the following cases:

1. **Simple Pole case:** if all the poles of are simple, or distinct, i.e.:

Hence, the Heaviside Partial Fraction expansion theorem could be used. It states that:

Here, is the derivative of the denominator polynomial with respect to the complex variable, evaluated for the corresponding pole.

**Ex11:** Find the inverse LT of the function.

**Answer:** .

1. **Multiple Pole case:** if there are some poles of that are repeated, i.e.:

Hence, the partial fraction expansion of these poles is performed for the similar poles as:

Moreover, for the distinct poles, this is performed as:

**Ex12:** Find the inverse LT of the function.

**Answer:** .

1. **Complex Pole case:** if there are some poles of which are complex in the form:

Hence, it could be arranged to comply with the completing of square method:

Here,

The inverse LT will be as:

**Ex13:** Find the inverse LT of the function.

**Answer:** .

# Solving ODE using Laplace Transform:

Consider the ODE with initial conditions below:

Here, and are constants. is called the input which is applied to the system and is the output response due to the input. In LT, this is performed through the following steps:

* Convert the ODE into a LT style:
* Obtain the transfer function:

The output has two terms, the first is dependent on the input, and the second is dependent on the initial conditions of the system.

* Use Partial Fraction expansion or any other ways to get the inverse LT of to get:

**Ex14:** Solve the ODE: .

**Answer:** , .

# Solving LTI systems using Laplace Transform:

LTI systems are of high importance in control systems. Every LTI system has a transfer function, which represents the characteristics of that system. This system could also be represented as an ODE that could be modelled by LT to get the system transfer function as in the previous section.

The system could be represented as an impulse response function, which is the image of the transfer function in the time domain. In this case, the convolution process may be used if the time response of the input is present, but the convolution may take much effort especially if the system is complicated. Laplace method converts convolution to multiplication, therefore should be transformed by LT before being used in the Laplace method.

The output function could be found if the input is present by multiplying it by the system transfer function. At this point, the output function could be retransformed back to get the output time response.

**Ex15:** A system has the following impulse response function:

An input was applied to that system. Find the output time response of that system using Laplace method.

**Answer:** , ,

 , .

# Solving dynamic systems using Laplace Transform:

Dynamic system ODEs are the equations that consider the force inputs for computing the state variables. These types of ODE took wide fields in control system theory and design. In this case, the mechanical system is represented as an ODE with initial conditions. This ODE is converted by LT to a system similar to the transfer function in the previous section. The system output represent the element that is intended to be measured (distance, current, angle, rotation etc.)

The dynamic system is merged with the input as in the previous sections, and then the inverse LT is taken for the result to get the time response of the interested system variable.

**Ex16:** A pendulum is at rest (zero initial conditions) has the following ODE:

Here, is the pendulum mass, is the pendulum length, is the damping coefficient, and is the gravitational force. If a unit impulse input force was applied on the pendulum which is, find the output function and the time response of the pendulum angle, and respectively.

**Answer:** , .

# Analyzing Electrical circuits using Laplace Transform:

In the field of electrical circuit analysis, the use of LT made the analysis of electrical circuits simpler especially when the circuit has impedances (like inductance and capacitance). Rather than writing current differentiation for the inductance or current integration for the capacitance, the complex variable can handle this situation as in the following table:

|  |  |  |
| --- | --- | --- |
| Circuit Element | Voltage | Current |
| Supply |  |  |
| Resistance |  |  |
| Inductance |  |  |
| Capacitance |  |  |

**Ex17:** In a series RLC circuit, find the output function and the time response of the developed current in that circuit, and respectively, if the input voltage was. Let, and. Assume zero initial conditions.

**Answer:** , .

# MATLAB Code for Laplace Transform:

Laplace analysis has been embedded in MATLAB and SIMULINK software. There are two MATLAB functions for Laplace analysis; the first is used for LT, and the second is used for inverse LT. Using these functions requires installing Symbolic Math Toolbox inside MATLAB. The syntax of these functions is as below:

In the first function, is the function in time domain that we want to find its LT, associated with the time variable. The complex variable will be the associated variable of. In the second function, is the function in complex domain that we want to find its inverse LT, associated with the complex variable. The time variable will be the associated variable of. Before we can use these functions, the variables must be defined symbolically using the function. An example explains this:

**Exercises:**

1. Find LT for the given functions:
2. If , find .
3. Convert to LT:
4. Find Inverse LT:
5. Find Inverse LT:

What are the differences among the three results?

1. Solve ODE:
2. For a series circuit where, and. Find when.

**Z-Transform**

# Introduction:

Discrete sequence is a function that has a continuous set of samples (sequences) that are generated from convolving a continuous function by a train of unit impulses, where is the sampling time.

Taking LT of that function will yield:

Denoting by , the Z-Transform formula is obtained:

Therefore, Z-Transform (ZT) is the transformation of a discrete sequence from -domain to -domain ( is a complex number). ZT is the discrete counterpart of LT for continuous functions.

To determine ZT for a continuous function, three steps should be performed:

* Sampling to get by substituting every by;
* Taking LT for;
* Replacing every by.

# Graphic Representation of the Z-Transform:

The ZT complex plane is referred to as the -plane. The horizontal and vertical axes are referred to as the -axis and the -axis, respectively. The contour corresponding to is a circle of unit radius referred to as the *unit circle*. The ZT evaluated on the unit circle corresponds to the *Discrete Fourier Transform* DFT, which is the discrete counterpart of the Fourier Transform FT.

 is usually a rational function in , i.e.:

The orders and are positive integers. The roots of the numerator polynomial are called the *zeros* of. However, the roots of the denominator polynomial are called the *poles* of. The function is represented in the -plane by the locations of poles and zeros. An is used to indicate each pole location and an is used to indicate each zero location.

# Region of Convergence of Z-Transform:

The range of values of the complex variable for which the ZT converges is called the *region of convergence* (ROC). The poles of lie outside the ROC since does not converge at the poles. The zeros may lie inside or outside the ROC. Therefore, ROC lies in the exterior of the circle of the largest pole of from the origin. If is the largest pole of the function, then the ROC will be the exterior of the circle:

Furthermore, if ROC covers the unit circle, then the function could be called *convergent*. For, there is one zero in and two poles in and . The ROC begins at the circle excluded towards infinity. Because of the ROC does not contain the unit circle, the function is divergent.

# Common Z-Transform Pairs:

|  |  |  |
| --- | --- | --- |
| Sequence |  |  |
| Kronecker Delta |  |  |
| Delayed Kronecker Delta |  |  |
| Heaviside Step |  |  |
| Delayed Heaviside Step |  |  |
| Ramp |  |  |
| Power |  |  |
| Ramp Power |  |  |
| Cosine Wave |  |  |
| Sine Wave |  |  |
| Damped Cosine Wave |  |  |
| Damped Sine Wave |  |  |

# Properties of Z-Transform:

If and, and are constants, then the following properties hold:

## Linearity:

**Ex1:** Find the ZT of the sequence.

**Answer:** .

## Sample Shifting:

**Ex2:** Find the ZT of the sequence.

**Answer:** .

## Exponent Scaling:

**Ex3:** Find the ZT of the sequence.

**Answer:** .

## Discrete Convolution:

**Ex4:** Find the inverse ZT of the sequence.

**Answer:** .

## Z-Domain Differentiation:

**Ex5:** Find the ZT of the sequence.

**Answer:** .

## Initial Value Theorem:

The initial value of the sequence could be found using ZT.

**Ex6:** Find the initial value of the sequence.

**Answer:** .

## Final Value Theorem:

The final value of the sequence could also be found using ZT.

**Ex7:** Find the final value of the sequence.

**Answer:** .

# Z-Transform of Periodic Sequences:

For a periodic sequence, that has a period and generated from a sequence that has a ZT, the ZT of the periodic sequence will be:

**Ex8:** Find the ZT of the periodic sequence, and then find that periodic sequence.

**Answer:** , .

# Inverse Z-Transform:

The canonical equation of inverse ZT is shown below:

Here, the integration refers to the counterclockwise contour of integration around the origin. This type of integration is related to the theory of Complex Variables.

Computation of inverse ZT by using that equation is very complicated, then we could use the ZT pairs as well as ZT properties to find the -domain sequences for their corresponding -domain functions.

There are many other ways to determine the inverse ZT. The most used methods are called Partial Fraction expansion and Power Series expansion that will be discussed here. Other ways are the discrete convolution as in Ex4, and the Residue theorem used in complex integration which will be discussed later in the Complex Variable chapter.

# Partial Fraction Expansion:

If is a rational function of the form:

Here, is the numerator polynomial of and are the poles of.

Before computing the poles, should be divided by to make it solvable:

Then, the inverse ZT could be found using one of the following cases:

1. **Simple Pole case:** if all the poles of are simple, or distinct, i.e.:

Hence, the Partial Fraction expansion theorem could be used. It states that:

Here, are the coefficients of the power series:

Therefore, the discrete sequence could be determined:

**Ex9:** Find the inverse ZT of the function.

**Answer:** .

1. **Multiple Pole case:** if there are some poles of that are repeated, i.e.:

Hence, the partial fraction expansion of these poles is performed for the similar poles as:

Moreover, for the distinct poles, this is performed as:

**Ex10:** Find the inverse ZT of the function.

**Answer:** .

1. **Complex Pole case:** if there are some poles of which are complex in the form:

Hence, it could be arranged to be:

**Ex11:** Find the inverse ZT of the function.

**Answer:** .

# Power Series Expansion:

Another simple method that is used to find inverse ZT is called Power Series expansion. This method is came from the fact that sequence value is the coefficient of its corresponding power of:

This method is very effective when it is desired to represent as a power series with some primary coefficients of the polynomial. If is a rational function, then it could be determined by long division:

**Ex12:** Find the Power Series expansion of the function, and then use Partial Fraction expansion to find.

**Answer:**

.

# Solving System of Difference Equations using Z-Transform:

The standard representation of difference equation is below:

If the inputis introduced as a sequence form, then the values of the previous input sequences should be given, or else it will be considered as zeroes. If the input is introduced as a ZT form, then it should be solved using ZT. The previous sequences are added as below:

Then, after manipulation, the transfer function is appeared:

Thus, the output function could be determined if is given. The sequence could be computed from using inverse ZT.

**Ex13:** A system that has the following difference equation:

Find and determine when.

**Answer:** , .

**Ex14:** An input sequence was introduced to a linear system which generates an output sequence . Find and determine when.

**Answer:** , .

# MATLAB Code for Z-Transform:

Z-analysis has been embedded in MATLAB and SIMULINK software. There are two MATLAB functions for Z-analysis; the first is used for ZT, and the second is used for inverse ZT. Using these functions requires installing Symbolic Math Toolbox inside MATLAB. The syntax of these functions is as below:

In the first function, is the sequence in -domain that we want to find its ZT, associated with the sequence variable. The complex variable will be the associated variable of. In the second function, is the function in complex domain that we want to find its inverse ZT, associated with the complex variable. The sequence variable will be the associated variable of. Before we can use these functions, the variables must be defined symbolically using the function. An example explain this:

**Exercises:**

1. For the sequence . Draw ROC.
2. Find the inverse Z-Transform for the following:
3. A system is described by the difference equation. Find when , .
4. A system is described by the difference equation. Find and. Find for the corresponding :
5. Fibonacci Polynomial: . Find .

**Complex Variables**

# Introduction on Complex Numbers:

A *complex number* is an ordered pair of real numbers and. is called the *real part* of and is called the *imaginary* *part* of . This number system extension leads to the Complex Number system , which is defined as:

# Complex Plane:

The complex numbers could be represented as points in the complex plane, which its -axis represents the real part and -axis represents the imaginary part. This plane is usually called *Argand plane*.

# Complex Number Representation:

There are three forms for representing complex numbers:

1. Cartesian Form:
2. Polar Form:
3. Exponential Form:

Polar form of the complex numbers contains two real elements: the first is the *magnitude* which is the absolute value of , and the second is the *argument* which is the phase of .

To convert one form to another:

The magnitude of could be any positive real number, whereas the argument of could be any real number, but its principal representation should belong to the interval (i.e.). The argument is measured in radians, not degrees. The argument is undefined when .

Euler formula states that:

The magnitude of is unity: .

**Ex1:** Convert to the polar form.

**Answer:** .

**Ex2:** Convert to the Cartesian form.

**Answer:** .

# Complex Arithmetic:

Let

Let

## Addition and Subtraction:

## Multiplication in Cartesian Form:

## Multiplication in Polar Form:

## Division in Cartesian Form:

## Division in Polar Form:

**Ex3:** If and find, and .

**Answer:** , , .

**Ex4:** Express in the form of .

**Answer:** .

**Ex5:** Express in the form of .

**Answer:** .

**Ex6:** Express in the exponential form.

**Answer:** .

# Complex Conjugate:

The number is the *complex conjugate* of the number . It is generated by reflecting through the real axis. It is also expressed as in polar form and in the exponential form. The properties of the complex conjugate number is illustrated as follows:

# Integer Powers and Roots, De Moivre’s Formula:

If we use the polar multiplication to square a complex number , we find:

If we cube , we find:

This leads to the general formula of integer powers of :

For , the formula becomes the De Moivre’s Formula:

This formula is valid for any integer .

**Ex7:** Find .

**Answer:** .

If we calculate the square root of a complex number , we find:

For any integer , the th root of a complex number is defined as:

There are distinct roots for any complex number . The general formula of -rooting of is as follows:

In that formula, could be any integer between and .

**Ex8:** Find all the cubic roots of unity.

**Answer:** .

# Complex Functions:

A transformation, where is an independent complex variable and is the dependent complex value, is called a *complex function* of. Every value of corresponds to a single value of, therefore this function is called a *single-valued* complex function.

Every complex point in the -plane is transformed to a complex point in the -plane by means of the complex function which is actually a pair of two real functions and each depending on two real variables and . Therefore, the point is a function of real variables of the point .

**Ex9:** If , find and then compute for

**Answer:** .

The complex variables are located in some region of the complex plane; this region could be part of the complex plane or the entire plane. There are many examples for regions, some of which are:

The region is called a *circle* of radius and located on which holds all the complex numbers which have a distance from equal to radius .

The region is called a *closed disk* of radius and located on which holds all the complex numbers which have a distance from in less than or equal to radius .

The region is called an *open disk* of radius and located on which holds all the complex numbers which have a distance from in less than radius.

Half Planes: Half planes are the regions of all complex numbers that are located in the upper, lower, right or left sides of the complex plane, which are defined by their identifier. For example, is called the *closed left half plane*, and is called the *open upper half plane*.

# Limits and Continuity:

A function is said to have a *limit* as approaches to a point :

The function has its limit if it is defined in a neighborhood of and the values of are close to for all close to .

If there is a positive real number such that all in the disk , then there is a positive real number such that . In the complex calculus, is approaching from any direction and the limit exists wherever is approaching from. In the single-valued complex function: if the limit exist, it is unique.

A function is said to be *continuous* at if it is defined at and have a limit around equals the function value :

# Derivatives:

For a single-valued complex function, the *derivative* of at a point, namely, is defined as:

The derivative exists if the limit exists, then is said to be differentiable at. The derivative exists and it is unique wherever is approaching from. All the differentiation rules of the real-valued functions hold for the complex-valued functions.

**Ex10:** Show that is differentiable, but is not.

# Analytic Complex Functions:

The function, which is defined over a region, is called *analytic* in if is differentiable at all the complex points which belong to. The function is called *analytic at a point* if there is a neighborhood such that is analytic ( exists) in that neighborhood. If is analytic in the entire -plane, then it is called an *entire function*.

# Cauchy-Riemann Equations:

Recall that , we find:

As it stated earlier, the derivative exists wherever the path of the neighborhood is directed, therefore we will choose two paths. The first when (i.e. along) and the second when (i.e. along)

For the first path:

For the second path:

Comparing the real and imaginary parts in results above, the important Cauchy-Riemann equations are obtained:

It can be concluded that if and only if the partial derivatives of and in and satisfy the Cauchy-Riemann equations and are continuous in a region, then the function is analytic in .

**Ex11:** Show that is analytic, but is not.

# Laplace’s Equations and Harmonic Functions:

If we differentiate the first Cauchy-Riemann equation with respect to and the second with respect to , we get:

Substituting both equations into each other yields:

This equation is called *Laplace’s Equation* for .

In the same way, one can obtain the Laplace’s equation for :

Functions that satisfy Laplace’s equation are called *harmonic functions*. The real and imaginary functions and which satisfy Cauchy-Riemann and Laplace’s equations are called *conjugate harmonic functions*.

**Ex12:** Verify that is harmonic in the entire complex plane, then find its harmonic conjugate function.

**Answer:** .

# Elementary Complex Functions:

In this section, the most important elementary complex functions will be discussed.

## Complex Polynomial Functions:

The complex polynomial function is defined as:

The constants are complex, provided that . The positive integer is called the degree of the polynomial. This function is entire.

## Complex Rational Functions:

The complex rational function is defined as:

The functions and are complex polynomials. This function is entire except at the points which make .

## Complex Exponential Functions:

The complex exponential function is defined as:

This function is entire. The magnitude and phase of are illustrated as:

If the complex exponential function is defined as:

Hence, this function could be converted into the standard form by:

**Ex13:** Compute.

**Answer:** .

## Complex Trigonometric Functions:

The complex trigonometric functions are defined as:

The functions and are entire; and are entire except when ; and are entire except when .

In terms of and , the functions and can be written as:

**Ex14:** Compute.

**Answer:** .

## Complex Hyperbolic Functions:

The complex hyperbolic functions are defined as:

The functions and are entire. In terms of and , the functions and can be written as:

Relation between the trigonometric and hyperbolic functions:

**Ex15:** Compute.

**Answer:** .

## Complex Logarithmic Functions:

The complex logarithmic functions are defined as:

This function is defined in the entire complex plane except at , and it’s entire except at and the negative real axis.

**Ex16:** Compute.

**Answer:** .

## Complex Power Functions:

The complex power function is defined as:

This function could be converted to:

This function is analytic in the entire complex plane except at .

**Ex17:** Compute.

**Answer:** .

## Inverse Trigonometric and Hyperbolic Functions:

**Ex18:** Find if .

**Answer:** .

# Complex Line Integrations:

The complex definite integrals are called *complex line integrals* written as:

 The integrand is integrated over a given curve, which is called *the path of integration*. The curve may be represented as:

Here, is called as the *positive sense* on.

It is assumed that is a smooth curve, i.e. it has a continuous and nonzero derivative:

If the initial and final points of the line integration coincide, then will be called as *closed path* like a circle etc. The notation of complex closed integration will be as below:

The line integral of a complex function is formulated as:

If is analytic inside then the integration above exists and its value is independent of the choice of integration paths for the same initial and final points.

# Evaluation of Line Integrations:

There are two methods for evaluating line integrations. The first one uses the common approach by substituting the lower and upper limits of the curve in the indefinite integral of the function . This method is simpler; however, it is suitable only for the analytic functions inside . If is the indefinite integral of the function which is analytic in a simply connected domain , then for all the paths joining two points and in , we have:

**Ex19:** Evaluate .

**Answer:** .

The second method is not restricted to analytic functions but it applies to any continuous complex functions, which is performed using the positive sense variable. If we represent the integral curve by, then:

**Ex20:** Evaluate around the unit circle.

**Answer:** .

# Cauchy’s Integral Theorem:

Let be a simple closed curve (which is also called as *contour*) inside a simply connected domain . If is analytic within the region bounded by as well as on , then the Cauchy integral theorem states that:

The integral above is called *contour integral*.

The Cauchy’s integral theorem holds for the functions that are entire, and for the functions that have singularities outside the contour .

**Ex21:** Evaluate around the unit circle.

**Answer:** .

# Cauchy’s Integral Formula:

Let be analytic inside a simply connected domain . Then, for any contour inside and a point inside :

**Ex22:** Evaluate around the unit circle.

**Answer:** .

**Ex23:** Evaluate around the circles and .

**Answer:** .

# Derivatives of Analytic Functions:

By using Cauchy’s integral formula, it can be shown that complex analytic functions have derivatives of all orders. The existence of those derivatives will result from the general integration formula below:

**Ex24:** Evaluate around the circle.

**Answer:** .

# Taylor’s Series:

Every analytic function can be represented as power series like Taylor’s series. The power series of is of the form:

The Taylor’s series converges for all inside the open disk , where is the distance from to the nearest singularity of . If is entire (i.e. no singularity), then it converges for all .

# Laurent’s Series:

If it is desired to develop a function representation for in powers of , where has a singularity on , Taylor’s series could not be used. A new kind of series, *Laurent’s series*, may be used. This series has negative integer powers of , as well as the positive powers as in Taylor’s series. Laurent’s series representation is as below:

The series positive and negative parts are called the *analytic* and *principal* parts.

 **Ex25:** Find the Laurent’s Series expansion of around .

**Answer:** .

# Singularities:

The function is said to be *singular* or has a *singularity* at a point if it is not analytic (or may not be defined) at ; however, every point in the neighborhood of contains points of which is analytic. The point is called a *singular point* of . If there is no other singularities in the neighborhood of , then it is called *isolated singularity*.

# Poles:

If the principal part of Laurent’s series of has finite terms of the form:

The singularity of at is called a *pole*, and is called its *order*. The first order poles are known as *simple poles*, the second order ones as *double poles*, and the third order ones as *triple poles* and so on.

If has a pole at , then as in any manner.

# Zeros:

The point that makes an analytic function is called a *zero* in . A zero of order has and at . The first order zeros are known as *simple zeros*, the second order ones as *double zeros*, and the third order ones as *triple zeros* and so on.

The zeros of an analytic function are always isolated i.e. each of them has a neighborhood that contains no further zeros of .

If is analytic and has a zero of th order at , then has a pole of th order at provided that is analytic at and .

# Residues of Analytic Functions:

If is analytic everywhere in and at a contour except at a point , then the function has a Laurent’s series expansion that converges for all points near . The coefficient of the first negative power of Laurent’s series, namely, is given by:

The coefficient is called the *residue* of at , and it is denoted as:

# Evaluation of Residues:

To evaluate the residues of a function at a pole of order , one can use the following general formula:

This formula could be reduced to a simple one for simple poles:

If the function is formulated as rational polynomials , where has a simple zero at and , then the residue formula could be formulated as:

**Ex26:** Find all the residues of .

**Answer:** .

**Ex27:** Find all the residues of . There is a double pole at .

**Answer:** .

# Residue Theorem:

 If is analytic everywhere inside and on a contour except at a point , then the residue of at could be used to evaluate the contour integral of that function, provided the nonexistence of any other singularities of inside or on :

From the above, we found that only the coefficient of the first negative term of Laurent’s series (the residue) is required for evaluating the contour integral. The contour integral of around a singular point equals times the residue of at .

If is analytic everywhere inside and on a contour except at a finite many singular points , then the contour integral of around them equals times the sum of the residues of at .

**Ex28:** Evaluate around any contour such that: A: 0 and 1 inside; B: 0 inside, 1 outside; C: 0 outside, 1 inside; D: 0 and 1 outside.

**Answer:** A: ; B: ; C: ; D: .

**Ex29:** Evaluate around the contour .

**Answer:** *.*

# Evaluation of Real Integrals using Residue Theorem:

One of the many usefulness of the residue integrals is that the complicated real integrals can be integrated by using the residue theorem. There are three types of real definite integrals, which are:

**First Type:** , is a rational function: In this type, we consider the contour integral along a line belongs the -axis from to inside a semicircle above the -axis having that line as its diameter, then we let .

In the above integral, only the poles that reside on and above the real axis are considered in the summation. If is even, then this can be used to evaluate .

**Ex30:** Evaluate .

**Answer:** *.*

**Second Type:** , is a rational function of and : In this type, let , then , , and let , then . After substitution in the integral above, it is converted to the standard contour integral where the contour is the unit circle.

**Ex31:** Evaluate .

**Answer:**.

**Third Type:** , is a rational function: In this type, we consider the contour integral along the same contour of the first type.

**Ex32:** Evaluate .

**Answer:**.

**Exercises:**

1. Convert the following complex numbers from the Cartesian form to the Polar form:
2. Convert the following complex numbers from the Exponential form to the Cartesian form:
3. Find all the roots of the following complex numbers:
4. A formula of the square root of a complex number states that: . Find square root of:
5. Graph the regions:
6. Find and for the following functions:
7. Are the following functions analytic? Specify the region of analyticity:
8. Are the following functions harmonic? If yes, find their harmonic conjugates:
9. Compute for the following in the form of :
10. Compute the following in the form of :
11. Compute for the following in the form of :
12. Compute the following in the form of :
13. Evaluate the following complex line integrations:
14. Check if the Cauchy’s integral theorem hold for the following integrals around the unit circle:
15. Evaluate the following integrals using Cauchy’s integral formula:
16. Evaluate the following integrals around the circle :
17. Evaluate the following integrals using Residues:
18. Evaluate the following real integrals:

**Numerical Methods**

# Introduction on Numerical Methods:

*Numerical methods* are mathematical methods used to solve problems on computers or calculators by means of numeric calculations, resulting in a table of numbers and/or graphical representations. These methods are used when the conventional analytic methods fail to find a solution. The steps of the numerical computations from a given situation in engineering to the final answer usually includes modeling, choosing a numeric method, programming computation and result interpreting.

# Errors in Numerical Methods:

Because of the numerical methods are approximations for the exact methods, the results of the numerical methods contain errors. These errors arise from round-off errors resulting from number rounding in computer, and from truncating errors resulting from using approximate representations for some complex formulae like Taylor series or Difference Equations.

The *error*, resulted from a certain numerical method, is the difference between the actual and the approximate value, and, respectively:

The error is always smaller than a value called *error bound*, i.e:

# Solving Nonlinear Equations:

For a given function, it is desired to find the values of that make. These values of are called the *solutions* or *roots* of. Generally, finding function solutions in analytical methods is almost impossible except for some cases. Solving such equations is important in engineering applications, which uses an approximation method called *iteration method*. This method starts from an initial guess of the solution and compute step-by-step approximations of unknown solution for a given accuracy. This method is easy to program in software packages because the computational operations are the same in each step.

# Newton’s Method:

This method is also called Newton-Raphson’s method used to find a solution for a function . It is assumed that this method has a continuous derivative . This method is commonly used because of simplicity and great speed. The derivative can be expressed as:

If we let and simplify for , we can obtain the general formula of Newton’s method:

Generally, the iteration stops when the estimated relative error is smaller than a given small value, or the absolute value of the function is smaller than a given small value.

The convergence of Newton method could be determined as in the following formula:

This implies that Newton method converges rapidly if and are nonzero. Hence, this method is called *second order convergent*.

**Ex1:** Find the solution of the function using Newton’s method for three iterations starting from .

**Answer:** .

# Secant Method:

Secant method is a powerful method for solving nonlinear equations. It is similar to Newton’s method. However, the derivative in Newton’s method is replaced by a difference quotient in the secant method:

The general formula of the secant method is illustrated below:

This method requires two initial values and , compared with Newton’s method, which require one initial condition .

Secant method can use the same stoppage condition that the Newton’s method uses.

**Ex2:** Find the solution of the function using secant method for three iterations starting from .

**Answer:** .

# Numerical Integration:

Numerical integration means the numerical evaluation of integrals:

In the above integration, and are the bounds of the integration interval and is a function given analytically or empirically by a table of values. represents the area under the curve of between and .

Many engineering applications lead to functions that have very hard or maybe impossible integrals or have empirical functions given by recorded numerical values. Then we should use numerical integration methods to evaluate approximate values of such integrals. Numerical methods approximate the integrand by functions that can easily be integrated.

# Trapezoidal Rule:

Trapezoidal rule of numerical integration divides the integration interval into subintervals of equal length :

In each subinterval, the function is approximated by the constant :

Then, the area under the curve of between and is approximated by the sum of trapezoids of areas:

The truncation error of the trapezoidal rule could be estimated using the following formula:

The unknown value is between and .

**Ex3:** Integrate inside the interval . Use .

**Answer:** .

# Simpson’s Rule:

Simpson’s rule is one of the widely used methods of numerical integration; its simplicity and high accuracy made it the favorite method in numerical integration. It is a quadratic approximation method, compared with the trapezoidal rule, which is a linear approximation method.

Simpson’s rule divides the interval into even subintervals of equal length , and we do as we did in trapezoidal rule in function approximation. Then, the area under the curve of between and is approximated by the sum of areas:

The truncation error of the Simpson’s rule could be estimated using the following formula:

The unknown value is between and .

**Ex4:** Integrate inside the interval . Use .

**Answer:** .

# Numerical Differentiation:

Numerical differentiation methods are used in computer programs to calculate the values of the derivative, double derivative, or higher derivatives using the function evaluation or tabulated data. Because the computer program cannot deal with the concept of “limit” in the evaluation of differentials, the numerical differentiation methods simulate the behavior of limit to give the approximation of the differentials.

Three formulae can be used to evaluate first derivative: they are the *forward*, *backward* and *center difference* equations, respectively:

One formula can be used to evaluate second derivative, which is called the *center difference* equation:

**Ex5:** For, find the forward, backward, central first difference and central second difference at . Use the points and

**Answer:** .

# Solving First-Order Ordinary Differential Equations:

Practical engineering problems may lead to ODE’s that cannot be solved analytically or have a complicated solution. Therefore, numerical methods for solving engineering ODE’s are of great practical importance.

The ODE of the first order can be written as:

Here, is the derivative of the dependent variable and is a function of the independent variable and the dependent variable . The goal of the numerical method is to find a solution for every input , starting from an initial value for an initial input . This is called *initial value problem*.

In this section, the methods of computing approximate numerical values the solution at equidistant values of in a systematic approach will be discussed:

Here, is a fixed number called *step size* could be chosen using some formulae.

# Euler Method:

If we use Taylor’s series to approximate we find:

For small , the terms which contain , and higher could be neglected to get the approximation:

This method is called *Euler method*. This method is rarely used in practice because of high truncation error; it is called *first-order method*.

**Ex6:** Find the output values , for the following initial value problem using Euler method and choosing :

**Answer:** .

# Improved Euler Method:

In each step of the improved Euler method, a predicted value of is computed:

Then, the predicted value is used to compute the corrected value of :

This method is called *predictor-corrector method*. It is a *second-order method* since its truncation error is proportional to :

**Ex7:** Find the output values , for the following initial value problem using improved Euler method and choosing :

**Answer:** .

# Runge-Kutta Method:

Runge-Kutta method is one of the most widely used methods for solving first-order ODE’s because of its high accuracy, numerical stability and fitness for computer programming. It does not need special starting procedure, makes light demand on storage, and use the same straightforward procedure repeatedly. It is a *fourth-order method* since its truncation error is proportional to .

Runge-Kutta method begins by computing four predicted values . The new output value is computed from these values:

**Ex8:** Find the output values , for the following initial value problem using Runge-Kutta method and choosing :

**Answer:** .

**Exercises:**

1. Find the solutions of the following nonlinear equations using Newton and Secant methods. Perform the first four iterations:
2. Evaluate the following integrals using trapezoidal and Simpson’s rules. Solve for :
3. Solve the following ODE’s using modified Euler and Runge-Kutta methods. Solve for and :

**Matrix Algebra**

# Introduction on Matrices and Vectors:

*Matrices* are rectangular patterns or arrangements of numbers (or functions) enclosed in brackets []. These numbers (or functions) are called *entries* of the matrix. For example, the following are matrices:

The first matrix has two rows and three columns, the second and third matrices are *square* matrices (3 and 2 rows and columns, respectively). The fourth and fifth matrices are called *vectors*, because it has just one row and one column, respectively.

The matrices are denoted by capital bold letters or by the general entry in brackets: . An matrix has rows and columns:

Therefore, the first matrix is matrix, the second is matrix, the third is matrix, the fourth is matrix, and the fifth is matrix.

Each entry has two subscripts, the first is called *row number* and the second is called *column number*. If , then is called an *square matrix* and the entries are called the *main diagonal* of . If , then is called a *rectangular matrix*.

Vectors are matrices with one row or column and their entries are called the components of the vector. The vectors are denoted by small bold letters or by the general entry in brackets: . The row vector and the column vector are of the form:

# Matrix Addition and Scalar Multiplication

Two matrices and are equal if and only if they have the same size and the corresponding entries are equal: .

The sum of two matrices and of the same size is written and has the entries**:** . Matrices of different sizes cannot be added.

The product of any matrix and any scalar is written , and the entries are obtained by multiplying each entry of by . Matrix addition and scalar multiplication are commutative and associative. The zero matrix is a matrix that has all of its entries are zeros.

# Matrix Multiplication:

The product of an matrix times matrix is defined if and only if , then the matrix has the entries:

The condition means that the second matrix must have as many rows as the first matrix has columns, namely.

**Ex1:** Multiply by.

**Answer:** .

Matrix multiplication is not commutative; i.e. , so that the order of the matrices must be observed carefully. However, matrix multiplication is associative.

# Matrix Transposition:

The *transpose* of an matrix is the an matrix that has the first row of as its first column, the second row of as its second column and so on.

Transposition converts row vectors to column vectors and conversely.

Some important rules of transpositions are as follows:

# Special Matrices:

**Diagonal Matrix**: Diagonal matrices are square matrices that have nonzero entries in their diagonals, i.e. .

**Identity Matrix**: Identity matrices are diagonal matrices that all of its diagonal entries are unity. The Identity matrix is denoted as .

**Symmetric Matrix**: Symmetric matrices are square matrices that are equal to its transpose, i.e. .

**Skew-Symmetric Matrix**: Skew-Symmetric matrices are square matrices that are equal to its negative transpose, i.e. .

**Ex2:** Specify these matrices: .

**Answer:** Skew-symmetric, diagonal, symmetric, identity.

# Determinant of the Matrix:

A *determinant* of order is a scalar associated with an square matrix which is written as:

The factor is called the minor of in , which is the determinant of order , namely, the determinant of the submatrix of obtained from by omitting the row and column of the entry , that is, the th row and the th column.

For

For

**Ex3:** Find the determinant of the matrix.

**Answer:** .

# Cramer’s Rule for solving linear systems:

If a linear system of equations in the same number of unknowns :

If the determinant is nonzero, the system has one solution. This solution is given by the formulas:

The determinant is the determinant obtained from by replacing the th column in by the column with the entries .

**Ex4:** Given the linear system , Find the unknown vector using Cramer’s rule.

**Answer:** .

# Inverse of a matrix:

The inverse of an square matrix is denoted by and is an square matrix such that:

If has an inverse, then is called *nonsingular matrix* and the inverse is unique; otherwise it is called *singular*.

For

Generally

The factor is called the cofactor of in .

To obtain the inverse of an square matrix , we should do the following:

Find the determinant of the matrix ;

Transpose to get ;

Calculate the cofactor for each element in to get ;

Divide over to get .

**Ex5:** Given the matrix . Find .

**Answer:** .

If is a diagonal matrix, then its invers is a diagonal matrix with entries which are the reciprocal of their counterpart in **.**

**Ex6:** Given the matrix . Find .

**Answer:** .

# Eigenvalues and Eigenvectors:

Let be an square matrix and consider the vector equation:

Here, is an unknown vector and is an unknown scalar. We look for vectors for which the multiplication by has the same effect as the multiplication by a scalar . A value of for which the above equation has a solution is called *eigenvalue*, and the corresponding is called the *eigenvector* of corresponding to that eigenvalue .

The eigenvalues of a square matrix are the roots of the characteristic equation:

**Ex6:** Given the matrix . Find the eigenvalue and eigenvectors of if one of the eigenvalues is .

**Answer:**

**Exercises:**

1. Find the inverse of the following matrices:
2. Solve the linear system using matrix inversion:
3. Find the eigenvalues and eigenvectors of the following matrices: