Application of ordinary differential equations (Newton's law of cooling)

Example: A metal bar at a temperature of $100°C$ is placed in a room at a constant temperature of $0°C$. If after 20 minutes the temperature of the bar is $50°C$, then we have to find an equation for the temperature of the bar at any time.

To solve this problem, we have to use Newton's law of cooling.Which states that the rate of change of the temperature T of the body is proportional to the temperature difference between the body and its surrounding medium. If T_m is the temperature of the surrounding medium, then Newton's law of cooling implies

$$\frac{dT}{dt} = -k(T - T_m) \qquad \ldots\ldots \quad (1)$$

where k is a positive constant of proportionality, and the negative sign on the right hand side indicates the sign of cooling.

Thus in our problem, Eq.(1) becomes

$$\frac{dT}{dt} = -k(T - 0)$$
$$\Rightarrow \frac{dT}{dt} + kT = 0$$

and we have to solve this equation together with the boundary conditions when

$t = 0, T = 100$, and when $t = 20, T = 50$, to get an equation (expression) for the temperature of the bar at any time.

To solve this eq.

$\frac{dT}{dt} + kT = 0$
$\frac{1}{T}dT + kdt = 0$
$\text{Ln}|T| + kt = c$
$\text{Ln}|T| = -kt + c$
$T = e^{-kt+c}$
$T = e^{-kt} \cdot e^{c}$, let $e^{c} = C$
$T = Ce^{-kt} \qquad \ldots *$

when $t = 0, T = 100$ in *

$100 = Ce^{-k(0)} \Rightarrow 100 = C$ then * become $T = 100e^{-kt} \qquad \ldots\ldots **$

when $t = 20, T = 50$ in **

$$50 = 100e^{-k(20)} \Rightarrow e^{-20k} = \frac{50}{100}$$

$$-20k = \text{Ln}\left(\frac{1}{2}\right) \Rightarrow -20k = \text{Ln}1 - \text{Ln}2$$

$$-20k = 0 - \text{Ln}2$$

$$\Rightarrow -20k = \frac{-\text{Ln}2}{-20} \Rightarrow k = \frac{\text{Ln}2}{20}$$

$\therefore T = 100e^{\frac{-ln2t}{20}}$ this equation for the temperature of the bar at any time.

Where t=30 minutes

$$T = 100e^{\frac{-30ln2}{20}}$$

$$T = 100e^{\frac{-3\ ln2}{2}}$$