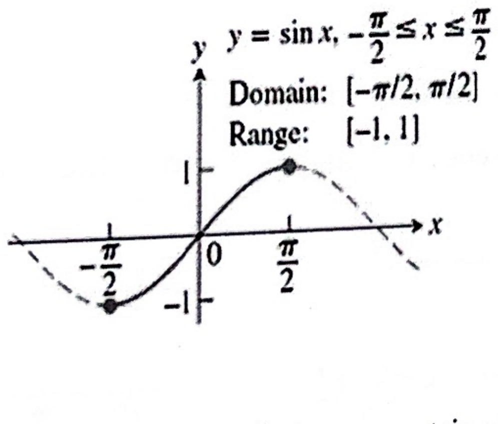
Inverse of Trigonometric functions

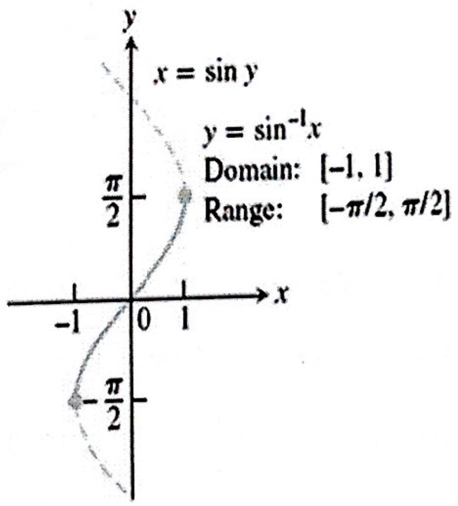
1- The **arcsine of x ($\sin^{-1}x$)** is the angle in $[-\pi/2, \pi/2]$ whose sine is x. The function $y=\sin x$ is one-to-one, if we restrict its domain to the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. It has an inverse which is denoted by:

$$\boxed{y = \sin^{-1} x}$$

and is sometimes written as $y=\textbf{arcsin}x$ and for the function $y=\sin^{-1}x$

$$D_f = [-1,1] \text{ and } R_f = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$





Note: The graph of $\sin^{-1}x$ is symmetric about the origin because that the graph of $\sin x$ is symmetric about the origin this means that

$$\sin^{-1}(-x) = -\sin^{-1}x$$

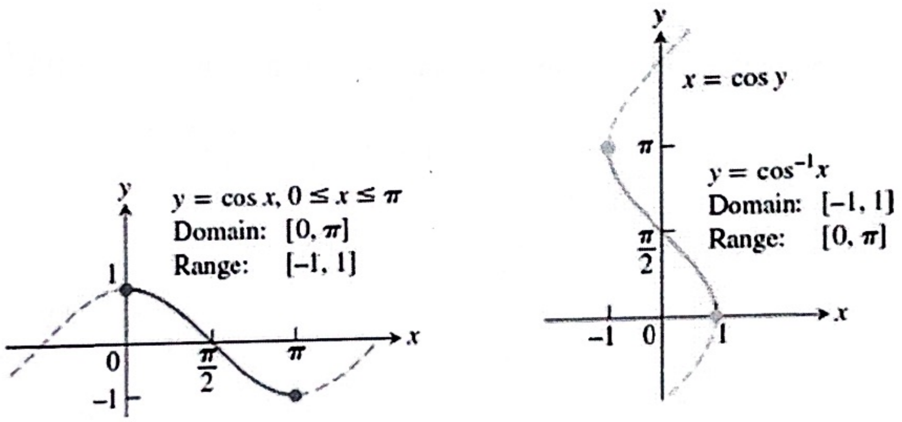
2. The **arccosine of** x **($\cos^{-1}x$)** is the angle in $[0,\pi]$ whose cosine is x. The function $y=\cos x$ is one-to-one, if we restrict its domain to the interval $0 \le x \le \pi$. It has an inverse which is denoted by:

$$\boxed{y=\cos^{-1} x}$$

and is sometimes written as $y=$**arc** $\cos x$ and for the function $y=\cos^{-1}x$

$$D_f=[-1,1] \text{ and } R_f=[0,\pi]$$

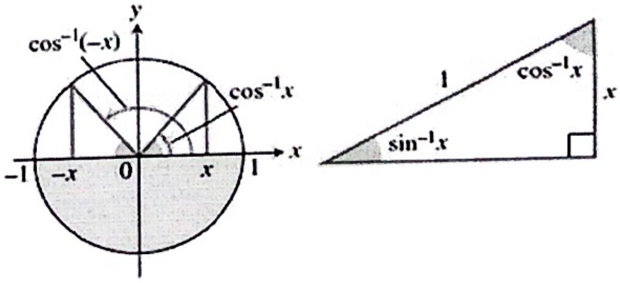
Note: The graph of $y=\cos^{-1}x$ has no such symmetry



Note: We can see from the figures below the following identities:

$\cos^{-1}x+\cos^{-1}(-x)=\pi$ $\quad \therefore \cos^{-1}(-x)=\pi-\cos^{-1}x$ and form the triangle

$\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$ for $x>0$



3. The **arctangent of *x* ($\tan^{-1}x$)** is the angle in $(-\pi/2, \pi/2)$ whose tangent is *x*. The function $y=\tan x$ is one-to-one, if we restrict its domain to the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$. It has an inverse which is denoted by:

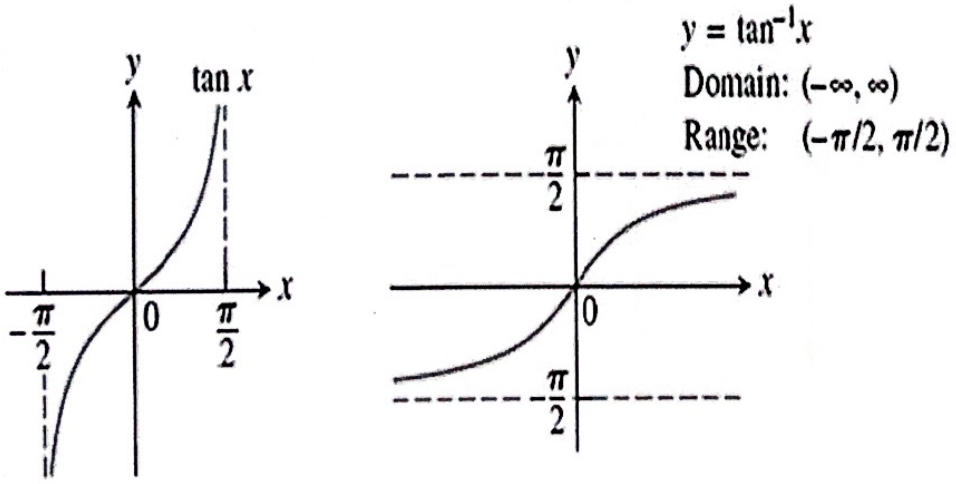
$$\boxed{y = \tan^{-1} x}$$

and is sometimes written as $y=$**arc**$\tan x$ and for the function $y=\tan^{-1}x$

$$D_f = (-\infty, \infty) \text{ and } \qquad R_f = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Note: The graph of $\tan^{-1}x$ is symmetric about the origin because that the graph of $\tan x$ is symmetric about the origin, this means that

$$\tan^{-1}(-x) = -\tan^{-1}x$$



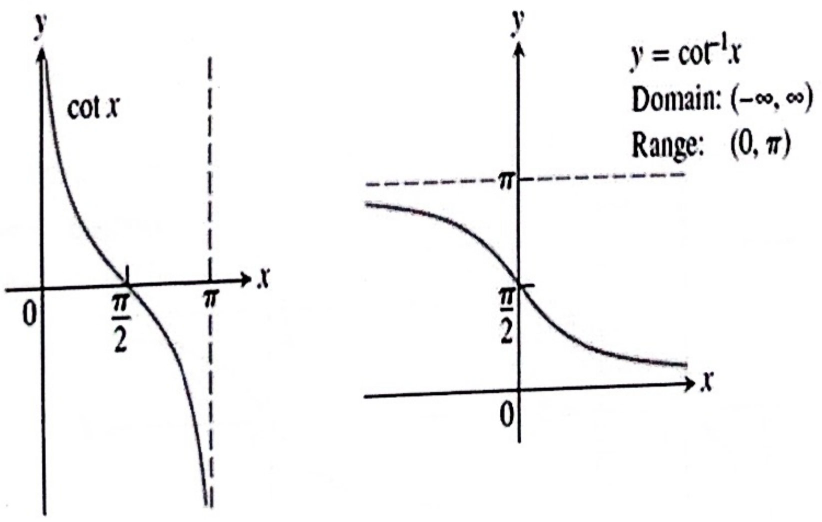
4. The **arctcoangent of *x* (cot^{-1}x)** is the angle in $(0,\pi)$ whose cotangent is *x*.

The function y=cot x is one-to-one, if we restrict its domain to the interval $0<x<\pi$. It has an inverse which is denoted by:

$$\boxed{y=\cot^{-1} x}$$

and is sometimes written as y=**arc**cot x and for the function y=cot^{-1}x

$D_f=(-\infty,\infty)$ and $R_f=(0,\pi)$.

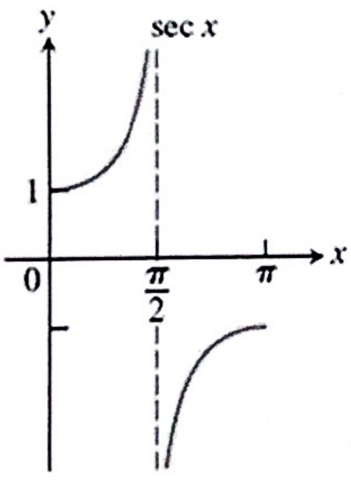


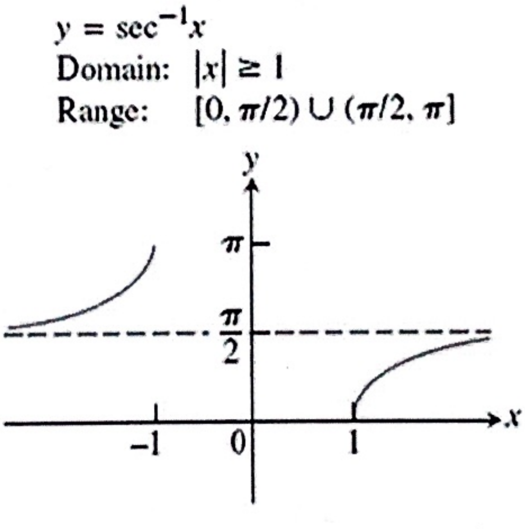
5. The function y=sec x is one-to-one, if we restrict its domain to the interval $\{x: 0 \le x \le \pi\} \setminus \{\frac{\pi}{2}\}$. It has an inverse which is denoted by:

$$\boxed{y = \sec^{-1} x}$$

and is sometimes written as $y =$ **arc**sec x and for the function y=sec^{-1}x

$$D_f = R\setminus(-1,1) \text{ and } \quad R_f = [0, \pi] \setminus \{\frac{\pi}{2}\}.$$



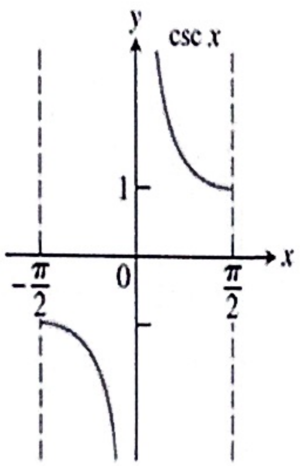


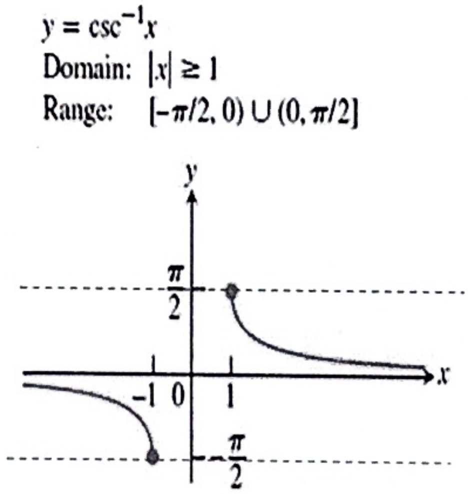
6. The function $y = \csc x$ is one-to-one, if we restrict its domain to the interval $\{x: -\frac{\pi}{2} \le x \le \frac{\pi}{2}\} \setminus \{0\}$. It has an inverse which is denoted by:

$$\boxed{y = \csc^{-1} x}$$

and is sometimes written as $y=$**arc**$\csc x$ and for the function $y = \csc^{-1} x$

$D_f = R \setminus (-1,1)$ and $\qquad R_f = [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$.





Note: To find $\sec^{-1}x$, $\csc^{-1}x$ and $\cot^{-1}x$, use the following identities:

1. $\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$

2. $\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$

3. $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$

The Derivative of Inverse Trigonometric Functions:

In general: If u is a function of x:

1. $\frac{d}{dx}\sin^{-1}u = \frac{du/dx}{\sqrt{1-u^2}} \quad |u|<1$

2. $\frac{d}{dx}\cos^{-1}u = \frac{-du/dx}{\sqrt{1-u^2}} \quad |u|<1$

3. $\frac{d}{dx}\tan^{-1}u = \frac{du/dx}{1+u^2}$

4. $\frac{d}{dx}\cot^{-1}u = \frac{-du/dx}{1+u^2}$

5. $\frac{d}{dx}\sec^{-1}u = \frac{du/dx}{|u|\sqrt{u^2-1}} \quad |u|>1$

6. $\frac{d}{dx}\csc^{-1}u = \frac{-du/dx}{|u|\sqrt{u^2-1}} \quad |u|>1$

Examples: Find dy/dx of the following functions:

1. $y=\sin^{-1}x^2$

Sol.: $\frac{dy}{dx}=\frac{2x}{\sqrt{1-(x^2)^2}}=\frac{2x}{\sqrt{1-x^4}}$

2. $y=\tan^{-1}\sqrt{x+1}$

Sol.: $\frac{dy}{dx}=\frac{\frac{1}{2\sqrt{x+1}}}{1+\left(\sqrt{x+1}\right)^2}=\frac{1}{2\sqrt{x+1}}*\frac{1}{1+x+1}=\frac{1}{2\sqrt{x+1}}*\frac{1}{2+x}$

3. $y=\sec^{-1}3x$

Sol.: $\frac{dy}{dx}=\frac{3}{|3x|\sqrt{(3x)^2-1}}=\frac{1}{|x|\sqrt{9x^2-1}}$

4. $y=x\sin^{-1}3x$

Sol.: $\frac{dy}{dx}=x*\frac{3}{\sqrt{1-(3x)^2}}+\sin^{-1}3x*1=\frac{3x}{\sqrt{1-9x^2}}+\sin^{-1}3x$

5. $y^2\sin x+y=\arctan y$

Sol.: $2y.y`\sin x+y^2\cos x+y`=\frac{y`}{1+y^2}$

$y`(2y\sin x+1-\frac{1}{1+y^2})=-y^2\cos x$

$y`(\frac{2y\sin x(1+y^2)+(1+y^2)-1}{1+y^2})=-y^2\cos x$

$$y`(\frac{2y\sin x+2y^3\sin x+1+y^2-1}{1+y^2})=-y^2\cos x$$

$$y`=\frac{-y^2\cos x(1+y^2)}{2y\sin x+2y^3\sin x+y^2}=\frac{-y\cos x(1+y^2)}{2\sin x+2y^2\sin x+y}$$

Example: If $y=\sin^{-1}(\cos x)$, show that $y``+y`+1=0$.

Sol.: $y`=\frac{-\sin x}{\sqrt{1-\cos^2 x}}=\frac{-\sin x}{\sqrt{\sin^2 x}}=\frac{-\sin x}{\sin x}=-1$

$\therefore y``=0$

So $y``+y`+1=0+(-1)+1=0$ o.k.

Example: If $y=\tan^{-1}x+\tan^{-1}(\frac{1}{x})$, show that $y``+y`=0$.

Sol.: $y`=\frac{1}{1+x^2}+\frac{-1/x^2}{1+(1/x)^2}=\frac{1}{1+x^2}-\frac{1/x^2}{1+1/x^2}=\frac{1}{1+x^2}-\frac{1/\cancel{x^2}}{(x^2+1)/\cancel{x^2}}=\frac{1}{1+x^2}-\frac{1}{x^2+1}=0$

$\therefore y``=0$

So $y``+y`=0+0=0$ o.k.

Homework

I. Verify the following identities.

1. $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$

2. $\sin^{-1} x = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$

3. $2\cos^{-1}\sqrt{x} = \cos^{-1}(2x-1) \quad 0 \le x \le 1$

II. Find *dy/dx* of the following.

1. $y = \tan^{-1}(3x-1)$
2. $y = e^x \sec^{-1}(e^x)$
3. $y = \dfrac{\tan^2 x}{x^2+1}$
4. $y = \sqrt{x}\sec^{-1}(\sqrt{x})$
5. $y = x\cos^{-1}\sqrt{4x+1}$
6. $y = (\tan x)^{\tan^{-1} x}$
7. $y = \sin^{-1}(\ln x)$
8. $y = \ln(\tan^{-1} x^2)$
9. $y = (1+\cos^{-1} 3x)^3$
10. $y = \left(\dfrac{1}{x} - \sin^{-1}\dfrac{1}{x}\right)^4$
11. $x^3 + x\sin^{-1} y = ye^x$
12. $\ln(x+y) = \tan^{-1}(xy)$