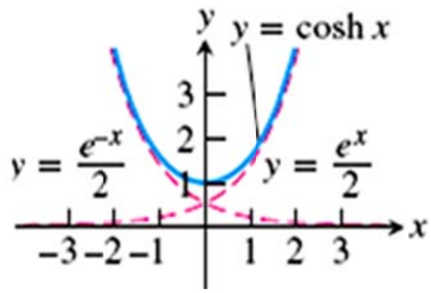
# Hyperbolic Functions

**Definition and Identities:**

The hyperbolic cosine and hyperbolic sine functions are defined by the following equations:

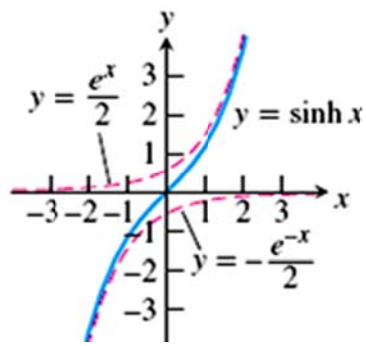



Hyperbolic cosine of $x$: $\cosh x = \dfrac{e^{x}+e^{-x}}{2}$

***Note:*** when $x \to \infty \Rightarrow e^{-x} \to 0$ So $\cosh x \cong \dfrac{e^{x}}{2}$

when $x \to -\infty \Rightarrow e^{x} \to 0$ So $\cosh x \cong \dfrac{e^{-x}}{2}$

So $D_f = (-\infty, \infty)$ and $R_f = [1, \infty)$



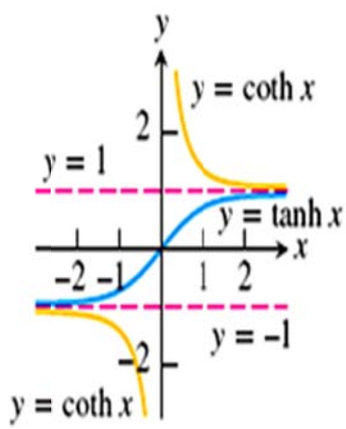

Hyperbolic sine of $x$: $\sinh x = \dfrac{e^{x}-e^{-x}}{2}$

***Note:*** when $x \to \infty \Rightarrow e^{-x} \to 0$ So $\sinh x \cong \dfrac{e^{x}}{2}$

when $x \to -\infty \Rightarrow e^{x} \to 0$ So $\sinh x \cong -\dfrac{e^{-x}}{2}$

So $D_f = (-\infty, \infty)$ and $R_f = (-\infty, \infty)$

The notation $\cosh x$ is often read "kosh $x$" and sinh $x$ is pronounced as if spelled "cinch $x$" or "shine $x$".




Four additional hyperbolic functions are defined in terms of $\cosh x$ and $\sinh x$ as shown below:

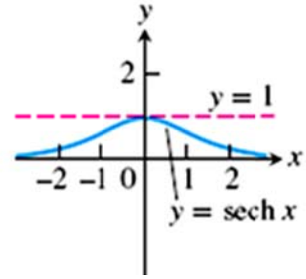
Hyperbolic tangent of $x$: $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$

$D_f = (-\infty, \infty)$ and $R_f = (-1, 1)$

Hyperbolic cotangent of $x$: $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$

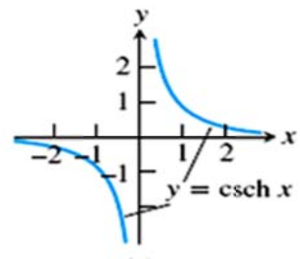
$D_f = (-\infty, \infty) \setminus \{0\}$ and $R_f = (-\infty, -1) \cup (1, \infty)$

Hyperbolic secant of $x$: $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$




$D_f = (-\infty, \infty)$ and $R_f = (0,1]$

Hyperbolic cosecant of $x$: $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$




$D_f = (-\infty, \infty) \setminus \{0\}$ and $R_f = (-\infty, \infty) \setminus \{0\}$

**Identities:**

| | **Hyperbolic functions** | **Trigonometric Functions** |
|---|---|---|
| 1 | $\cosh^2 x - \sinh^2 x = 1$ | $\cos^2 x + \sin^2 x = 1$ |
| 2 | $\sinh 2x = 2\sinh x \cosh x$ | $\sin 2x = 2\sin x \cos x$ |
| 3 | $\cosh 2x = \cosh^2 x + \sinh^2 x$ | $\cos 2x = \cos^2 x - \sin^2 x$ |
| 4 | $\cosh^2 x = \frac{\cosh 2x + 1}{2}$ | $\cos^2 x = \frac{1 + \cos 2x}{2}$ |
| 5 | $\sinh^2 x = \frac{\cosh 2x - 1}{2}$ | $\sin^2 x = \frac{1 - \cos 2x}{2}$ |
| 6 | $\tanh^2 x = 1 - \operatorname{sech}^2 x$ | $\tan^2 x = \sec^2 x - 1$ |
| **7** | $\coth^2 x = 1 + \operatorname{csch}^2 x$ | $\cot^2 x = \csc^2 x - 1$ |

***Examples:*** Prove that:

1. $\cosh^2 x - \sinh^2 x = 1$

**Sol.:** left side: $\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2$

$$= \frac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4} + \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4}$$

$$= \frac{e^{2x}}{4} + \frac{2e^0}{4} + \frac{e^{-2x}}{4} - \frac{e^{2x}}{4} + \frac{2e^0}{4} - \frac{e^{-2x}}{4} = \frac{4e^0}{4}$$

$= 1 =$ right side o.k.

2. $\operatorname{csch}^2 x = \coth^2 x - 1$

**Sol.:** right side: $\coth^2 x - 1 = \left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)^2 - 1 = \frac{e^{2x} + 2 + e^{-2x}}{e^{2x} - 2 + e^{-2x}} - 1$

$$= \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{e^{2x} - 2 + e^{-2x}}$$

$$\frac{4}{e^{2x} + 2 + e^{-2x}} = \left(\frac{2}{e^x + e^{-x}}\right)^2$$

$= \operatorname{csch}^2 x =$ left side o.k.

3. $\cosh^2 x = \frac{\cosh 2x + 1}{2}$

**Sol.:** right side: $\frac{\cosh 2x + 1}{2} = \frac{\left(e^{2x} + e^{-2x}\right)/2 + 1}{2}$

$$= \frac{e^{2x} + e^{-2x} + 2}{4} = \frac{e^{2x} + 2 + e^{-2x}}{4}$$

$= \left(\frac{e^x + e^{-x}}{2}\right)^2 = \cosh^2 x =$ left side o.k.

4. $\cosh x + \sinh x = e^x$

**Sol.:** left side: $\cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}$

$$= \frac{1}{2}\left(e^x + e^{-x} + e^x - e^{-x}\right) = \frac{2e^x}{2}$$

$= e^x =$ right side o.k.

***Examples:*** Solve the following equations:

1. $5\cosh x - 3\sinh x = 5$

**Sol.:** $5\frac{e^x + e^{-x}}{2} - 3\frac{e^x - e^{-x}}{2} = 5$

$$5e^x + 5e^{-x} - 3e^x + 3e^{-x} = 10 \Rightarrow 2e^x + 8e^{-x} = 10 \Rightarrow e^x + 4e^{-x} = 5$$

$$\Rightarrow e^x + \frac{4}{e^x} = 5 \Rightarrow \frac{e^{2x} + 4}{e^x} = 5 \Rightarrow e^{2x} + 4 = 5e^x \Rightarrow e^{2x} - 5e^x + 4 = 0$$

$\Rightarrow \quad (e^x - 4)(e^x - 1) = 0$

$\therefore$ either $(e^x - 4) = 0 \quad \Rightarrow \quad e^x = 4 \quad \Rightarrow \quad x = \ln 4$

or $(e^x - 1) = 0 \quad \Rightarrow \quad e^x = 1 \quad \Rightarrow \quad x = \ln 1 = 0$

2. $3\cosh x - 2\sinh x = 10$

**Sol.:** $3\dfrac{e^x + e^{-x}}{2} - 2\dfrac{e^x - e^{-x}}{2} = 10$

$$3e^x + 3e^{-x} - 2e^x + 2e^{-x} = 20 \quad \Rightarrow \quad e^x + 5e^{-x} = 20$$

$$\Rightarrow \quad e^{2x} + 5 = 20e^x \quad \Rightarrow \quad e^{2x} - 20e^x + 5 = 0$$

$$e^x = \frac{-(-20) \mp \sqrt{(-20)^2 - 4(1)(5)}}{2(1)} = \frac{20 \mp \sqrt{400 - 20}}{2} = \frac{20 \mp \sqrt{380}}{2}$$

$\therefore$ either $e^x = 19.74 \quad \Rightarrow \quad x = \ln 19.74 = 2.98$

or $e^x = 0.254 \quad \Rightarrow \quad x = \ln 0.254 = -1.373$

**Derivatives of Hyperbolic Function:**

If $u$ is any function of $x$, then:

| | Derivative of hyperbolic functions | Derivative of trigonometric functions |
|---|---|---|
| 1 | $\frac{d}{dx}\sinh u = \cosh u.\frac{du}{dx}$ | $\frac{d}{dx}\sin u = \cos u.\frac{du}{dx}$ |
| 2 | $\frac{d}{dx}\cosh u = \sinh u.\frac{du}{dx}$ | $\frac{d}{dx}\cos u = -\sin u.\frac{du}{dx}$ |
| 3 | $\frac{d}{dx}\tanh u = \sec h^2 u.\frac{du}{dx}$ | $\frac{d}{dx}\tan u = \sec^2 u.\frac{du}{dx}$ |
| 4 | $\frac{d}{dx}\coth u = -\csc h^2 u.\frac{du}{dx}$ | $\frac{d}{dx}\cot u = -\csc^2 u.\frac{du}{dx}$ |
| 5 | $\frac{d}{dx}\sec hu = -\sec hu \tanh u.\frac{du}{dx}$ | $\frac{d}{dx}\sec u = \sec u \tan u.\frac{du}{dx}$ |
| 6 | $\frac{d}{dx}\csc hu = -\csc hu \coth u.\frac{du}{dx}$ | $\frac{d}{dx}\csc u = -\csc u \cot u.\frac{du}{dx}$ |

***Examples:*** Prove that:

1. $\frac{d}{dx}\sinh x = \cosh x$

**Sol.:** $\frac{d}{dx}\sinh x = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x - (-e^{-x})}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$ o.k.

2. $\frac{d}{dx}\csc hx = -\csc hx \coth x$

**Sol.:** $\frac{d}{dx}\csc hu = \frac{d}{dx}\left(\frac{1}{\sinh x}\right) = \frac{\sinh x * (0) - 1 * \cosh x}{\sinh^2 x}$

$$= \frac{-\cosh x}{\sinh^2 x} = \frac{-1}{\sinh x}.\frac{\cosh x}{\sinh x} = -\csc hx \coth x \qquad \text{o.k.}$$

***Examples:*** Find $\frac{dy}{dx}$ of the following:

1. $y = \sinh 3x$

**Sol.:** $\frac{dy}{dx} = \cosh 3x * 3 = 3\cosh 3x$

2. $y = \tanh(1 + x^3)$

**Sol.:** $\frac{dy}{dx} = \sec h^2(1 + x^3) * 3x^2 = 3x^2 \sec h^2(1 + x^3)$

3. $y = \coth \frac{1}{x}$

**Sol.:** $\frac{dy}{dx} = -\csc h^2\left(\frac{1}{x}\right) * \left(\frac{-1}{x^2}\right) = \frac{1}{x^2}\csc h^2\left(\frac{1}{x}\right)$

4. $y = x \sec hx^2$

**Sol.:** $\frac{dy}{dx} = x\left(-\sec hx^2.\tanh x^2.2x\right) + \sec hx^2$

$$= -2x^2 \sec hx^2.\tanh x^2 + \sec hx^2$$

5. $y = \csc h^2(x^2 + 1)$

**Sol.:** $\frac{dy}{dx} = 2\csc h(x^2 + 1)\left[-\csc h(x^2 + 1)\coth(x^2 + 1) * 2x\right] \qquad = -4x \csc h^2(x^2 + 1)\coth(x^2 + 1)$

6. $y = \ln \tanh 2x$

**Sol.:** $\frac{dy}{dx} = \frac{\sec h^2 2x * 2}{\tanh 2x} = \frac{2\frac{1}{\cosh^2 2x}}{\frac{\sinh 2x}{\cosh 2x}} = \frac{2}{\cosh^2 2x} * \frac{\cosh 2x}{\sinh 2x}$

$= \frac{2*2}{2\cosh 2x.\sinh 2x} = \frac{4}{\sinh 4x} = 4\csc h4x$

7. $y = (\sinh x)^x$

**Sol.:** $\ln y = x \ln \sinh x$

$\frac{1}{y}.\frac{dy}{dx} = x.\frac{\cosh x}{\sinh x} + \ln \sinh x$

$\frac{dy}{dx} = y\left(x.\frac{\cosh x}{\sinh x} + \ln \sinh x\right) = (\sinh x)^x (x \coth x + \ln \sinh x)$

## Integrals of Hyperbolic Function:

If $u$ is any function of $x$, then:

1. $\int \sinh u.du = \cosh u + C$
2. $\int \cosh u.du = \sinh u + C$
3. $\int \sec h^2 u.du = \tanh u + C$
4. $\int \csc h^2 u.du = -\coth u + C$
5. $\int \sec hu.\tanh u.du = -\sec hu + C$
6. $\int \csc hu.\coth u.du = -\csc hu + C$

***Examples:*** Evaluate the following integrals:

1. $\int \coth 5x.dx = \int \frac{\cosh 5x}{\sinh 5x}.dx$

Let $u = \sinh 5x \Rightarrow du = \cosh 5x.5dx$ ➔ $\therefore \cosh 5x.dx = \frac{du}{5}$

$\int \frac{1}{u}.\frac{du}{5} = \frac{1}{5}\ln|u| + C = \frac{1}{5}\ln|\sinh 5x| + C$

2. $\int_0^{\ln 2} 4e^x \sinh x.dx = \int_0^{\ln 2} 4e^x \frac{e^x - e^{-x}}{2}.dx = \int_0^{\ln 2} \left(2e^{2x} - 2\right).dx$

$$= \left[e^{2x} - 2x\right]_0^{\ln 2} = \left[\left(e^{2\ln 2} - 2\ln 2\right) - \left(e^0 - 2*0\right)\right]$$

$$= e^{\ln 2^2} - 2\ln 2 - 1 = 4 - 2\ln 2 - 1 = 3 - 2\ln 2$$

3. $\int_0^1 \sinh^2 x.dx = \int_0^1 \frac{\cosh 2x - 1}{2}.dx = \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1$

$$= \frac{1}{2}\left[\left(\frac{\sinh 2}{2} - 1\right) - \left(\frac{\sinh 0}{2} - 0\right)\right] = \frac{\sinh 2}{4} - \frac{1}{2} = 0.40672$$

4. $\int \tanh 3x.\sec h^2 3x.dx$

**Sol.:** let $u = \tanh 3x \quad \Rightarrow \quad du = 3\sec h^2 3x.dx$ ➔ $\therefore \sec h^2 3x.dx = \frac{du}{3}$

$$\int u.\frac{du}{3} = \frac{u^2}{2*3} + C = \frac{\tanh^2 3x}{6} + C$$

5. $\int e^{\coth x}.\csc h^2 x.dx$

**Sol.:** let $u = \coth x \quad \Rightarrow \quad du = -\csc h^2 x.dx$ ➔ $\therefore \csc h^2 x.dx = -du$

$$\int e^u(-du) = -e^u + C = -e^{\coth x} + C$$

6. $\int \frac{\csc h^2 \sqrt{x}}{\sqrt{x}}.dx$

**Sol.:** let $u^2 = x \quad \Rightarrow \quad 2u.du = dx$

$$\int \frac{\csc h^2 u}{u}.2u.du = 2\int \csc h^2 u.du = -2\coth u + C = -2\coth\sqrt{x} + C$$

***Homework:***

I. Verify the following identities:

1. $\cosh x - \sinh x = e^{-x}$
2. $\sinh(-x) = -\sinh x$
3. $\cosh(-x) = \cosh x$
4. $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
5. $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$
6. $\tanh(x+y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

II. Find *dy*/*dx* of the following functions:

1. $y = \cosh\sqrt{4x^2 + 3}$
2. $y = \dfrac{\sec hx^2}{x^2 + 1}$
3. $y = x \csc he^{4x}$
4. $y = \tan^{-1}(\tanh x)$
5. $y = e^{3x} \sec hx$
6. $y = \sqrt{\sec h5x}$

III. Evaluate the following integrals:

1. $\int \dfrac{\sinh\sqrt{x}}{\sqrt{x}}.dx$
2. $\int \dfrac{\cosh \ln x}{x}.dx$
3. $\int \dfrac{1}{\coth^2 3x}.dx$
4. $\int \sinh x \cosh x.dx$
5. $\int \tanh 3x \sec h3x.dx$
6. $\int \sinh x\sqrt{\cosh x}.dx$
7. $\int \dfrac{\sec h^2 x}{1 - 2\tanh x}.dx$
8. $\int \dfrac{e^{\sinh x}}{\sec hx}.dx$
9. $\int \cosh^2 3x.dx$

## 5. Inverse of Hyperbolic Functions:

All hyperbolic functions have inverses, they are:

| | Inverse of hyperbolic functions | Their domains |
|---|---|---|
| 1 | $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$ | $(-\infty, \infty)$ |
| 2 | $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$ | $[1, \infty)$ |
| 3 | $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ | $(-1,1)$ |
| 4 | $\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$ | $(-\infty, \infty) \setminus [-1,1]$ |
| 5 | $\sec h^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$ | $(0,1]$ |
| 6 | $\csc h^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{\mid x \mid}\right)$ | $(-\infty, \infty) \setminus \{0\}$ |

***Example:*** Prove that: $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$.

**Sol.:** let $y = \sinh^{-1} x \quad \Rightarrow \quad x = \sinh y \quad \Rightarrow \quad x = \frac{e^y - e^{-y}}{2}$

$\Rightarrow \quad e^y - 2x - e^{-y} = 0$ multiply both sides by $e^y$

$\Rightarrow \quad e^{2y} - 2xe^y - 1 = 0 \Rightarrow \quad e^y = \frac{2x \mp \sqrt{4x^2 + 4}}{2} = x \mp \sqrt{x^2 + 1}$

Since $e^y$ is never negative, we must discard the minus sign.

$\therefore y = \ln(x + \sqrt{x^2 + 1})$

That is $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$

## Derivatives and Integrals:

### a. Derivatives

If $u$ is any function of $x$, then

1. $\frac{d}{dx}\sinh^{-1}u = \frac{du/dx}{\sqrt{1+u^2}}$
2. $\frac{d}{dx}\cosh^{-1}u = \frac{du/dx}{\sqrt{u^2-1}}$ $\quad u>1$
3. $\frac{d}{dx}\tanh^{-1}u = \frac{du/dx}{1-u^2}$ $\quad |u|<1$
4. $\frac{d}{dx}\coth^{-1}u = \frac{du/dx}{1-u^2}$ $\quad |u|>1$
5. $\frac{d}{dx}\operatorname{sech}^{-1}u = \frac{-du/dx}{u\sqrt{1-u^2}}$ $\quad 0<u<1$
6. $\frac{d}{dx}\operatorname{csch}^{-1}u = \frac{-du/dx}{u\sqrt{1+u^2}}$ $\quad u\neq 0$

### b. Integrals:

If $u$ is any function of $x$, then

1. $\int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1}u + C$
2. $\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1}u + C$
3. $\int \frac{du}{1-u^2} = \begin{cases} \tanh^{-1}x + C & if \quad |u|<1 \\ \coth^{-1}x + C & if \quad |u|>1 \end{cases}$
4. $\int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sech}^{-1}|u| + C$
5. $\int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1}|u| + C$

## Useful identities

**1.** $\operatorname{sech}^{-1}x = \cosh^{-1}\left(\frac{1}{x}\right)$ **2.** $\operatorname{csch}^{-1}x = \sinh^{-1}\left(\frac{1}{x}\right)$ **3.** $\coth^{-1}x = \tanh^{-1}\left(\frac{1}{x}\right)$

***Example:*** Show that $\frac{d}{dx}\cosh^{-1}x = \frac{1}{\sqrt{x^2-1}}$.

**Sol.:** Let $y=\cosh^{-1}x \;\Rightarrow\; \therefore x=\cosh y$

And by implicit differentiation:

$$1=\sinh y.\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}=\frac{1}{\sinh y}=\frac{1}{\sqrt{\cosh^2 y-1}}=\frac{1}{\sqrt{x^2-1}} \text{ o.k.}$$

***Examples:*** Find $dy/dx$ of the following functions:

1. $y=\sinh^{-1}3x$

**Sol.:** $\frac{dy}{dx}=\frac{3}{\sqrt{1+(3x)^2}}=\frac{3}{\sqrt{1+9x^2}}$

2. $y=\cosh^{-1}e^x$

**Sol.:** $\frac{dy}{dx}=\frac{e^x}{\sqrt{e^{2x}-1}}$

3. $y=2\tanh^{-1}\left(\tan\frac{x}{2}\right)$

**Sol.:** $\frac{dy}{dx}=\frac{2\sec^2\left(\frac{x}{2}\right)*\frac{1}{2}}{1-\tan^2\left(\frac{x}{2}\right)}=\frac{\frac{1}{\cos^2\left(\frac{x}{2}\right)}}{1-\frac{\sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)}}=\frac{\frac{1}{\cos^2\left(\frac{x}{2}\right)}}{\frac{\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)}}=\frac{1}{\cos\left(2.\frac{x}{2}\right)}=\frac{1}{\cos x}=\sec x$

4. $y=\coth^{-1}\left(\frac{1}{x}\right)$

**Sol.:** $\frac{dy}{dx}=\frac{-1/x^2}{1-1/x^2}=\frac{-1/x^2}{\frac{x^2-1}{x^2}}=\frac{-1}{x^2-1}=\frac{1}{1-x^2}$

5. $y = \sec h^{-1}(\cos x)$

**Sol.:** $\frac{dy}{dx} = \frac{-(-\sin x)}{\cos x\sqrt{1-\cos^2 x}} = \frac{\sin x}{\cos x \sin x} = \frac{1}{\cos x} = \sec x$

***Examples:*** Evaluate the following integrals:

1. $\int_0^1 \frac{2.dx}{\sqrt{1+4x^2}} = \int_0^1 \frac{2.dx}{\sqrt{1+(2x)^2}}$

Let $u = 2x \Rightarrow du = 2.dx$ , at $x = 0 \Rightarrow u = 0$

at $x = 1 \Rightarrow u = 2$

$$\int_0^2 \frac{du}{\sqrt{1+u^2}} = \left[\sinh^{-1} u\right]_0^2 = \left[\sinh^{-1} 2 - \sinh^{-1} 0\right] = 1.4436$$

2. $\int \frac{dx}{9x^2-25} = \int \frac{dx}{25\left(\frac{9}{25}x^2-1\right)} = \frac{1}{25}\int \frac{dx}{\left(\frac{3x}{5}\right)^2-1}$

Let $u = \frac{3x}{5} \Rightarrow du = \frac{3.dx}{5} \Rightarrow dx = \frac{5.du}{3}$

$$\frac{1}{25}\int \frac{5.du/3}{u^2-1} = \frac{1}{15}\int \frac{du}{u^2-1} = \frac{-1}{15}\int \frac{du}{1-u^2} = \begin{cases} \frac{-1}{15}\tanh^{-1} u + C = \frac{-1}{15}\tanh^{-1}\left(\frac{3x}{5}\right) + C \\ \frac{-1}{15}\coth^{-1} u + C = \frac{-1}{15}\coth^{-1}\left(\frac{3x}{5}\right) + C \end{cases}$$

3. $\int \frac{dx}{\sqrt{4x^2-9}} = \int \frac{dx}{\sqrt{9\left(\frac{4}{9}x^2-1\right)}} = \int \frac{dx}{\sqrt{9\left[\left(\frac{2x}{3}\right)^2-1\right]}}$

$$= \frac{1}{3}\int \frac{\frac{2}{3}dx}{\sqrt{\left(\frac{2x}{3}\right)^2-1}} * \frac{3}{2} = \frac{1}{2}\cosh^{-1}\left(\frac{2x}{3}\right) + C$$

4. $\int \frac{\sin x.dx}{\sqrt{1+\cos^2 x}}$

Let $u = \cos x \quad \Rightarrow \quad du = -\sin x.dx$

$\Rightarrow \quad \sin x.dx = -du$

$$\int \frac{-du}{\sqrt{1+u^2}} = -\sinh^{-1} u + C = -\sinh^{-1}(\cos x) + C$$

5. $\int \frac{4\tanh^{-1} x}{1-x^2}.dx$

Let $u = \tanh^{-1} x \quad \Rightarrow \quad du = \frac{dx}{1-x^2}$

$$\int 4u.du = \frac{4u^2}{2} + C = 2u^2 + C = 2\left(\tanh^{-1} x\right)^2 + C$$

## ***Homework:***

1. Find $dy/dx$ of the following:

   a. $y = \sinh^{-1} 5x$
   b. $y = \sinh^{-1} e^x$
   c. $y = \cosh^{-1} \sqrt{x}$
   d. $y = \tanh^{-1}(x^2 - 1)$
   e. $y = \tanh^{-1} \sin 3x$
   f. $y = x \sinh^{-1}\left(\frac{1}{x}\right)$

2. Evaluate the following integrals:

   a. $\int \frac{dx}{\sqrt{81+16x^2}}$
   b. $\int \frac{dx}{\sqrt{16x^2-9}}$
   c. $\int \frac{dx}{49-4x^2}$
   d. $\int \frac{e^x.dx}{\sqrt{e^{2x}-16}}$
   e. $\int \frac{2dx}{5-3x^2}$
   f. $\int \frac{dx}{x\sqrt{9-x^4}}$