

3 Partial Derivatives

Functions of 2 or more variables

Functions which have more than one variable arise very commonly. Simple examples are

- formula for the area of a triangle $A = \frac{1}{2}bh$ is a function of the two variables, base b and height h

- formula for electrical resistors in parallel:

$$R = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right)^{-1}$$

is a function of three variables R_1 , R_2 and R_3 , the resistances of the individual resistors.

You should be used to the notation $y = f(x)$ for a function of one variable, and that the graph of $y = f(x)$ is a curve. For the functions of two variables the notation simply becomes

$$z = f(x,y)$$

where the two **independent** variables are x and y , while z is the **dependent** variable. The graph of something like $z = f(x,y)$ is a **surface** in three-dimensional space. Such graphs are usually quite difficult to draw by hand.

3.5 Partial Derivatives of a Function of Two Variables

If y is a function of x then $\frac{dy}{dx}$ is the **derivative** meaning the gradient (slope of the graph) or the rate of change with respect to x .

Since $z = f(x,y)$ is a function of two variables, if we want to differentiate we have to decide whether we are differentiating with respect to x or with respect to y (the answers are different). A special notation is used. We use the symbol ∂ instead of d and introduce the **partial derivatives** of z , which are:

- $\frac{\partial z}{\partial x}$ is read as “partial derivative of z (or f) with respect to x ”, and means differentiate with respect to x holding y constant
- $\frac{\partial z}{\partial y}$ means differentiate with respect to y holding x constant

Another common notation is the subscript notation: z_x means $\frac{\partial z}{\partial x}$; z_y means $\frac{\partial z}{\partial y}$

Note that we cannot use the dash ' symbol for partial differentiation because it would not be clear what we are differentiating with respect to.

Example 1

Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ when $z = x^2 + 3xy + y - 1$.

Solution

To find $\frac{\partial z}{\partial x}$; treat y as a constant and differentiate with respect to x . We have $z = x^2 + 3xy + y - 1$ so

$$\frac{\partial z}{\partial x} = 2x + 3y$$

Similarly

$$\frac{\partial z}{\partial y} = 3x + 1$$

EXAMPLE 2 Find $\partial f/\partial y$ as a function if $f(x, y) = y \sin xy$.

Solution We treat x as a constant and f as a product of y and $\sin xy$:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y \sin xy) = y \frac{\partial}{\partial y} \sin xy + (\sin xy) \frac{\partial}{\partial y}(y) \\ &= (y \cos xy) \frac{\partial}{\partial y}(xy) + \sin xy = xy \cos xy + \sin xy.\end{aligned}$$

EXAMPLE 3 Find f_x and f_y as functions if

$$f(x, y) = \frac{2y}{y + \cos x}.$$

Solution We treat f as a quotient. With y held constant, we get

$$\begin{aligned}f_x &= \frac{\partial}{\partial x} \left(\frac{2y}{y + \cos x} \right) = \frac{(y + \cos x) \frac{\partial}{\partial x}(2y) - 2y \frac{\partial}{\partial x}(y + \cos x)}{(y + \cos x)^2} \\ &= \frac{(y + \cos x)(0) - 2y(-\sin x)}{(y + \cos x)^2} = \frac{2y \sin x}{(y + \cos x)^2}.\end{aligned}$$

With x held constant, we get

$$\begin{aligned}f_y &= \frac{\partial}{\partial y} \left(\frac{2y}{y + \cos x} \right) = \frac{(y + \cos x) \frac{\partial}{\partial y}(2y) - 2y \frac{\partial}{\partial y}(y + \cos x)}{(y + \cos x)^2} \\ &= \frac{(y + \cos x)(2) - 2y(1)}{(y + \cos x)^2} = \frac{2 \cos x}{(y + \cos x)^2}.\end{aligned}$$

3.6 Implicit differentiation

It works for partial derivatives the way it works for ordinary derivatives, as the next example illustrates.

EXAMPLE 4 Find $\partial z/\partial x$ if the equation

$$yz - \ln z = x + y$$

defines z as a function of the two independent variables x and y and the partial derivative exists.

Solution We differentiate both sides of the equation with respect to x , holding y constant and treating z as a differentiable function of x :

$$\begin{aligned}\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x} \ln z &= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x} \\ y \frac{\partial z}{\partial x} - \frac{1}{z} \frac{\partial z}{\partial x} &= 1 + 0 && \text{With } y \text{ constant,} \\ \left(y - \frac{1}{z}\right) \frac{\partial z}{\partial x} &= 1 && \frac{\partial}{\partial x}(yz) = y \frac{\partial z}{\partial x}. \\ \frac{\partial z}{\partial x} &= \frac{z}{yz - 1}. && \blacksquare\end{aligned}$$

3.7 Functions of More Than Two Variables

EXAMPLE 6 If x , y , and z are independent variables and

$$f(x, y, z) = x \sin(y + 3z),$$

then

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{\partial}{\partial z} [x \sin(y + 3z)] = x \frac{\partial}{\partial z} \sin(y + 3z) \\ &= x \cos(y + 3z) \frac{\partial}{\partial z} (y + 3z) = 3x \cos(y + 3z).\end{aligned}$$

Other examples of evaluating partial derivatives

- (i) $z = \ln(x^2 - y)$. Then $\frac{\partial z}{\partial x} = \frac{2x}{x^2 - y}$ and $\frac{\partial z}{\partial y} = \frac{-1}{x^2 - y}$.
- (ii) $z = x \cos y + ye^x$. Then $\frac{\partial z}{\partial x} = \cos y + ye^x$ and $\frac{\partial z}{\partial y} = -x \sin y + e^x$.
- (iii) $z = y \sin xy$. Then $\frac{\partial z}{\partial x} = y(y \cos xy) = y^2 \cos xy$ and $\frac{\partial z}{\partial y} = yx \cos xy + \sin xy$.

(iv) If $x^2 + y^2 + z^2 = 1$ find the rate at which z is changing with respect to y at the point $(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$.

Solution. We have $z = (1 - x^2 - y^2)^{1/2}$. We want $\frac{\partial z}{\partial y}$ when

$$(x, y) = (\frac{2}{3}, \frac{1}{3}).$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}(1 - x^2 - y^2)^{-1/2}(-2y) = -\frac{y}{(1 - x^2 - y^2)^{1/2}}$$

Putting in $(x, y) = (\frac{2}{3}, \frac{1}{3})$ gives

$$\frac{\partial z}{\partial y} = -\frac{1/3}{(1 - (2/3)^2 - (1/3)^2)^{1/2}} = -\frac{1}{2}.$$

3.8 Second-Order Partial Derivatives

When we differentiate a function $f(x, y)$ twice, we produce its second-order derivatives. These derivatives are usually denoted by

$$\frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx}, \quad \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy},$$

$$\frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx}, \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy}.$$

The defining equations are

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right),$$

and so on. Notice the order in which the mixed partial derivatives are taken:

$$\frac{\partial^2 f}{\partial x \partial y} \quad \text{Differentiate first with respect to } y, \text{ then with respect to } x.$$

$$f_{yx} = (f_y)_x \quad \text{Means the same thing.}$$

Example

If $f(x, y) = x \cos y + ye^x$, find the second-order derivatives

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}.$$

Solution The first step is to calculate both first partial derivatives.

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x} (x \cos y + ye^x) & \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y} (x \cos y + ye^x) \\ &= \cos y + ye^x & &= -x \sin y + e^x \end{aligned}$$

Now we find both partial derivatives of each first partial:

$$\begin{aligned} \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = -\sin y + e^x & \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = -\sin y + e^x \\ \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = ye^x. & \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = -x \cos y. \end{aligned}$$

Example

Let $z = 4x^2 - 8xy^4 + 7y^5 - 3$. Find all the first and second order partial derivatives of z . the first and second order partial derivatives of the function $z = \sin xy$.

$$\begin{aligned} \frac{\partial z}{\partial x} &= y \cos xy \\ \frac{\partial z}{\partial y} &= x \cos xy \\ \frac{\partial^2 z}{\partial x^2} &= -y^2 \sin xy \\ \frac{\partial^2 z}{\partial y^2} &= -x^2 \sin xy \\ \frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} (x \cos xy) = x(-y \sin xy) + \cos xy = -xy \sin xy + \cos xy \\ \frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} (y \cos xy) = y(-x \sin xy) + \cos xy = -xy \sin xy + \cos xy \end{aligned}$$

THEOREM

The Mixed Derivative Theorem If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy} , and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b) , then $f_{xy}(a, b) = f_{yx}(a, b)$.

Example

Find $\partial^2 w / \partial x \partial y$ if

$$w = xy + \frac{e^y}{y^2 + 1}.$$

Sol

$$\frac{\partial w}{\partial x} = y \quad \text{and} \quad \frac{\partial^2 w}{\partial y \partial x} = 1.$$

If we differentiate first with respect to y , we obtain $\partial^2 w / \partial x \partial y = 1$ as well.

3.9 Chain rule for partial derivatives

Recall the chain rule for ordinary derivatives:

$$\text{if } y = f(u) \text{ and } u = g(x) \text{ then } \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

In the above we call u the **intermediate variable** and x the **independent variable**. For partial derivatives the chain rule is more complicated. It depends on how many intermediate variables and how many independent variables are present.

3.9.1 Chain Rule For Functions of One Independent Variable and Two Intermediate Variables

If $z = f(x, y)$ and x and y are functions of t ($x = x(t)$ and $y = y(t)$) then z is ultimately a function of t only and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

Example

Let $z = x^2y$, $x = t^2$ and $y = t^3$. Calculate dz/dt by (a) the chain rule, (b) expressing z as a function of t and finding dz/dt directly.

Solution

(a) by the chain rule

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\ &= (2xy)(2t) + (x^2)(3t^2) \\ &= 4xyt + 3x^2t^2 \\ &= 4t^2t^3t + 3t^4t^2 \\ &= 7t^6 \end{aligned}$$

Example:

Use the Chain Rule to find the derivative of $w = xy$

with respect to t along the path $x = \cos t, y = \sin t$. What is the derivative's value at $t = \pi/2$?

Solution We apply the Chain Rule to find dw/dt as follows:

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} \\ &= \frac{\partial(xy)}{\partial x} \cdot \frac{d}{dt}(\cos t) + \frac{\partial(xy)}{\partial y} \cdot \frac{d}{dt}(\sin t) \\ &= (y)(-\sin t) + (x)(\cos t) \\ &= (\sin t)(-\sin t) + (\cos t)(\cos t) \\ &= -\sin^2 t + \cos^2 t \\ &= \cos 2t.\end{aligned}$$

In this example, we can check the result with a more direct calculation. As a function of t ,

$$w = xy = \cos t \sin t = \frac{1}{2} \sin 2t,$$

so

$$\frac{dw}{dt} = \frac{d}{dt} \left(\frac{1}{2} \sin 2t \right) = \frac{1}{2} \cdot 2 \cos 2t = \cos 2t.$$

In either case, at the given value of t ,

$$\left(\frac{dw}{dt} \right)_{t=\pi/2} = \cos \left(2 \cdot \frac{\pi}{2} \right) = \cos \pi = -1. \quad \blacksquare$$

3.9.2 Chain Rule for Functions of One Independent Variable and Three Intermediate Variables

If $w = f(x, y, z)$ and $x = x(t), y = y(t), z = z(t)$ then w is ultimately a function of t only and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$$

Example

Let $w = xy + z$ with $x = \cos t, y = \sin t$ and $z = t$. Calculate dw/dt .

What is the derivative's value at $t = 0$?

Solution.

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} \\ &= y(-\sin t) + x(\cos t) + (1)(1) \\ &= -\sin^2 t + \cos^2 t + 1\end{aligned}$$

$$= 1 + \cos 2t$$

$$= 1 + \cos (0) = 2.$$

At $t=0$

3.9.3 Chain Rule for Functions of two Independent Variable and two Intermediate Variables

If $z = f(x,y)$ and $x = x(u,v)$, $y = y(u,v)$ then z is a function of u and v and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}$$

Example

Let $u = x^2 - 2xy + 2y^3$ with $x = s^2 \ln t$ and $y = 2st^3$. Find $\partial u/\partial s$ and $\partial u/\partial t$.

Solution

This time u is a function of 2 variables x and y , each of which is itself a function of 2 variables s and t .

$$\begin{aligned} \frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} \\ &= (2x - 2y)(2s \ln t) + (-2x + 6y^2)(2t^3) \\ &= (2s^2 \ln t - 4st^3)(2s \ln t) + (-2s^2 \ln t + 24s^2t^6)(2t^3) \\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} \\ &= (2x - 2y) \left(\frac{s^2}{t} \right) + (-2x + 6y^2)(6st^2) \\ &= (2s^2 \ln t - 4st^3) \left(\frac{s^2}{t} \right) + (-2s^2 \ln t + 24s^2t^6)(6st^2) \end{aligned}$$