ADVANCED CALCULUS

FIRST SEMESTER

FOR THE 2snd CLASS STUDENTS

BY

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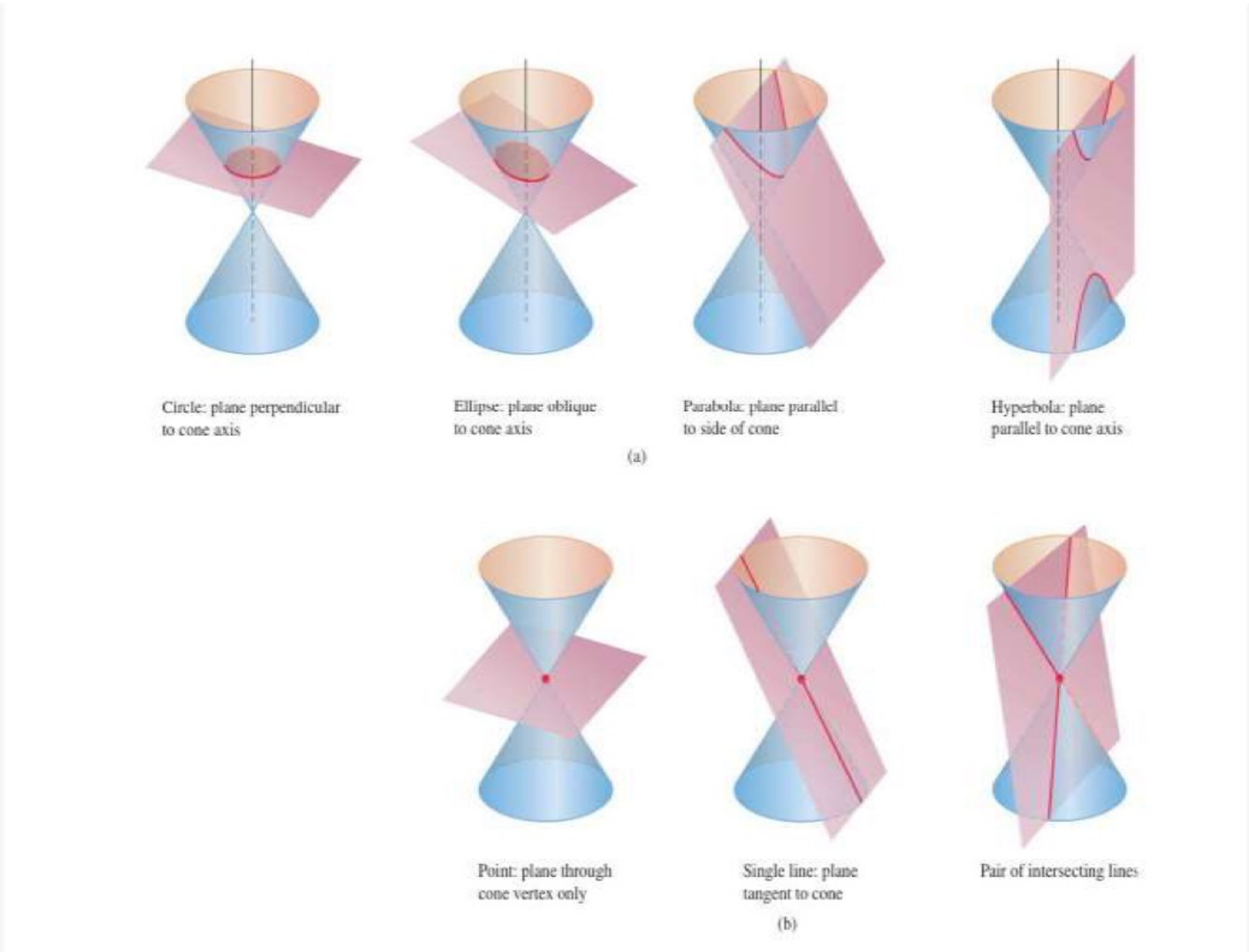
Contents:

Chapter one: Conic Sections and Polar Coordinates

1.1 Conic Sections and Quadratic Equations

Conic Sections

In this section we define and review parabolas, ellipses, and hyperbolas geometrically and derive their standard Cartesian equations. These curves are called conic sections or conics because they are formed by cutting a double cone with a plane (Figure 1). This geometry method was the only way they could be described by Greek mathematicians who did not have our tools of Cartesian or polar coordinates. In the next section we express the conics in polar coordinates.



1- The Circle

The set of points in a plane whose distance from some fixed center point is constant radius value. Let $c(\text{h}, \text{k})$ and r are the center and the radius of circle respectively. If p(x,y) be a point on this circle then the standard equation for circle is:

$$cp = r \text{ where } r > 0$$

and

$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

or

$$(x-h)^2 + (y-k)^2 = r^2$$

Note: if the center of circle is (0,0) then $x^2 + y^2 = r^2$

Example : Find the center and radius of the circle has the following equation: $x^2 + y^2 - 4x + 6y - 3 = 0$

Solution: $(x^2 - 4x + 4) + (y^2 + 6y + 9) = 4 + 9 + 3$

$$(x-2)^2 + (y+3)^2 = 16$$

The center is $(2, -3)$ and the radius equal 4.

2. Parabolas

DEFINITIONS: A set that consists of all the points in a plane equidistant from a given fixed point and a given fixed line in the plane is a parabola. The fixed point is the focus of the parabola. The fixed line is the directrix.

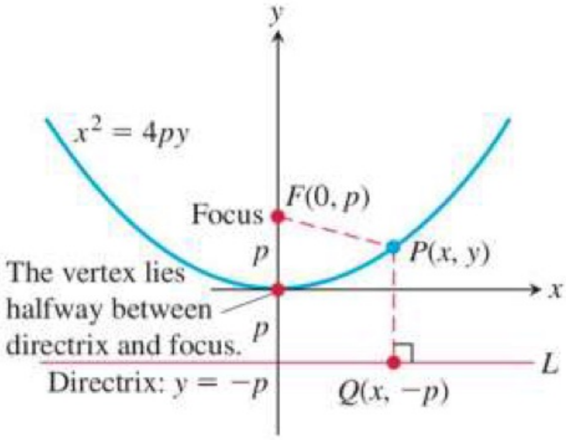
A parabola has its simplest equation when its focus and directrix straddle one of the coordinate axes. For example, suppose that the focus lies at the point $F(0, p)$ on the positive y-axis and that the directrix is the line $y = -p$ (Figure 11.40). In the notation of the figure, a point $P(x, y)$ lies on the parabola if and only if $PF = PQ$. From the distance formula,

$$PF = \sqrt{(x-0)^2 + (y-p)^2} = \sqrt{x^2 + (y-p)^2}$$
$$PQ = \sqrt{(x-x)^2 + (y-(-p))^2} = \sqrt{(y+p)^2}.$$

When we equate these expressions, square, and simplify, we get

$$y = \frac{x^2}{4p} \text{ or } x^2 = 4py. \text{ Standard form} \qquad (1)$$

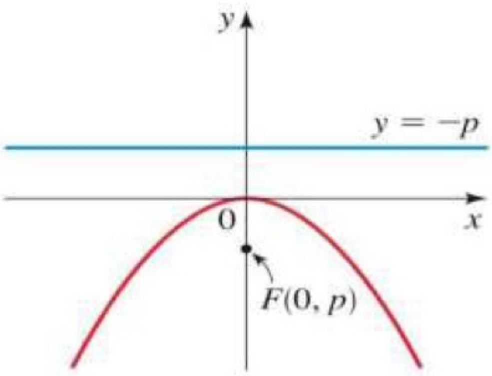
These equations reveal the parabola's symmetry about the y-axis. We call the y-axis the axis of the parabola



The point where a parabola crosses its axis is the vertex. The vertex of the parabola $x^2 = 4py$ lies at the origin . The positive number p is the parabola's focal length.

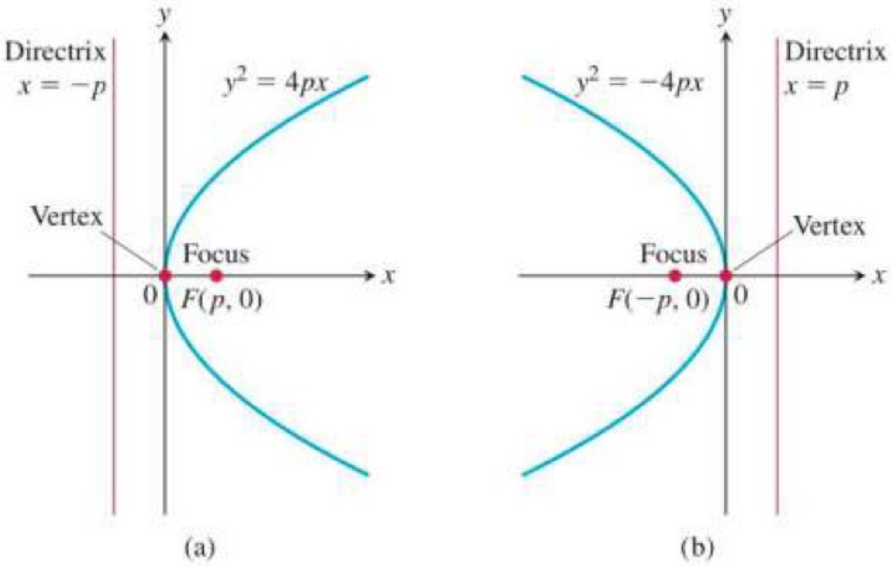
If the parabola opens downward, with its focus at $(0, -p)$ and its directrix the line $y = p$, then Equations (1) become

$$y = -\frac{x^2}{4p} \text{ and } x^2 = -4py.$$



By interchanging the variables x and y, we obtain similar equations for parabolas opening to the right or to the left .

(a) The parabola $y^2 = 4px$. (b) The parabola $y^2 = -4px$.



EXAMPLE: Find the focus and directrix of the parabola $y^2 = 10x$.
Solution : We find the value of p in the standard equation $y^2 = 4px$:

$$4p = 10, \text{ so } p = \frac{10}{4} = \frac{5}{2}.$$

Then we find the focus and directrix for this value of p :

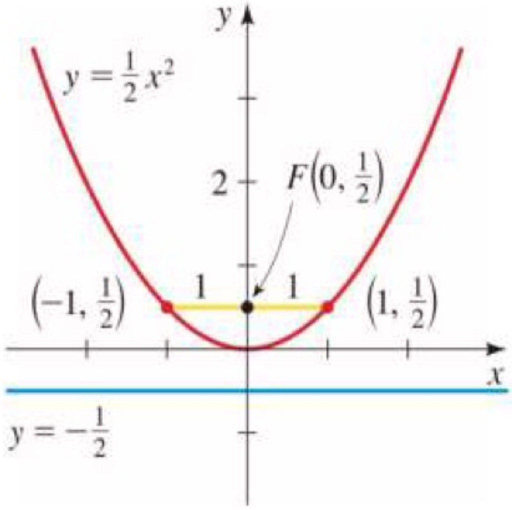
Focus: $(p, 0) = \left(\frac{5}{2}, 0\right)$

Directrix: $x = -p$ or $x = -\frac{5}{2}$.

EXAMPLE: Find the focus, directrix of the parabola $y = \frac{1}{2}x^2$, and sketch its graph.
Solution: We first put the equation in the form $x^2 = 4py$.

$$y = \frac{1}{2}x^2 \quad \text{Multiply by 2}$$
$$x^2 = 2y$$

From this equation we see that $4p = 2$, Solving for p gives $p = \frac{1}{2}$, so the focus is $\left(0, \frac{1}{2}\right)$, and the directrix is $y = -\frac{1}{2}$.



Remark

1 If vertix is (0,0), then:

$$y^2 = 4p(x) \quad , \quad x^2 = 4py$$
$$y^2 = -4px \quad , \quad x^2 = -4py$$

2 If the vertex is (h, k) then:

$$(y-k)^2 = 4p(x-h) \quad , \quad (x-h)^2 = 4p(y-k)$$
$$(y-k)^2 = -4p(x-h) \quad , \quad (x-h)^2 = -4p(y-k)$$

Ex: Find the equation of Parabola if the focus is (0,2).
Sol: The focus is (0,2), then focus is y-axis and the directrix is $y = -2$

$$x^2 = 4py$$
$$x^2 = 4\,(2)y \Rightarrow \quad x^2 = 8y \quad \text{the equation of Parabola}$$

Ex: Find the equation of Parabola if the vertix is $(-1,4)$ and Pass through $(4,-2)$,

directrix this Parabola is parallel the y-axis

Sol: The vertix is $(-1,4)$, from remark

$$(y-k)^2 = 4p(x-h)$$
$$\therefore (-2-4)^2 = 4p(4+1)$$
$$(-6)^2 = 4p(5)$$
$$36 = 20p \Rightarrow p = \frac{36}{20}$$
$$\therefore (y-k)^2 = 4p(x-h)$$
$$(y-4)^2 = 4 * \frac{36}{20}(x+1)$$
$$(y-4)^2 = \frac{36}{5}(x+1)$$ the Parabola eq. where vertix is $(-1,4)$.

Ex: The Point $(1,-2)$ is belong to the Parabola $x^2 = 4$ Py find the focus and directrix.

Sol:

$x^2 = 4$ Py , the parabola lie in y-axis (Positive)

$\because (x,y) = (1,-2)$, $\therefore 1^2 = 4P(-2)$

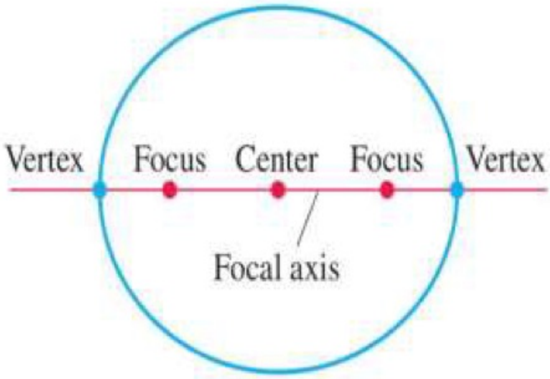
$1 = -8P \Rightarrow P = -1/8$

The focus $F(0,-1/8)$, directrix is $y = 1/8$

3. Ellipses

DEFINITIONS: An ellipse is the set of points in a plane whose distances from two fixed points in the plane have a constant sum. The two fixed points are the foci of the ellipse.

The line through the foci of an ellipse is the ellipse's focal axis. The point on the axis halfway be-tween the foci is the center. The points where the focal axis and ellipse cross are the ellipse's vertices .



If the foci are $F_1(-c, 0)$ and $F_2(c, 0)$ (Figure 11.43), and $PF_1 + PF_2$ is denoted by $2a$, then the coordinates of a point P on the ellipse satisfy the equation

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a.$$

To simplify this equation, we move the second radical to the right-hand side, square, isolate the remaining radical, and square again, obtaining

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1 \qquad \dots . *$$

Since $PF_1 + PF_2$ is greater than the length F_1F_2 (by the triangle inequality for triangle F_1F_2), the number $2a$ is greater than $2c$. Accordingly, $a > c$ and the number $a^2 - c^2$ in Equation (*) is positive.

The algebraic steps leading to Equation (*) can be reversed to show that every point P whose coordinates satisfy an equation of this form with $0 < c < a$ also satisfies the equation $PF_1 + PF_2 = 2a$. A point therefore lies on the ellipse if and only if its coordinates satisfy Equation (*). If we let b denote the positive square root of $a^2 - c^2$,

$$b = \sqrt{a^2 - c^2} \qquad \dots \dots **$$

then $a^2 - c^2 = b^2$ and Equation (*) takes the form

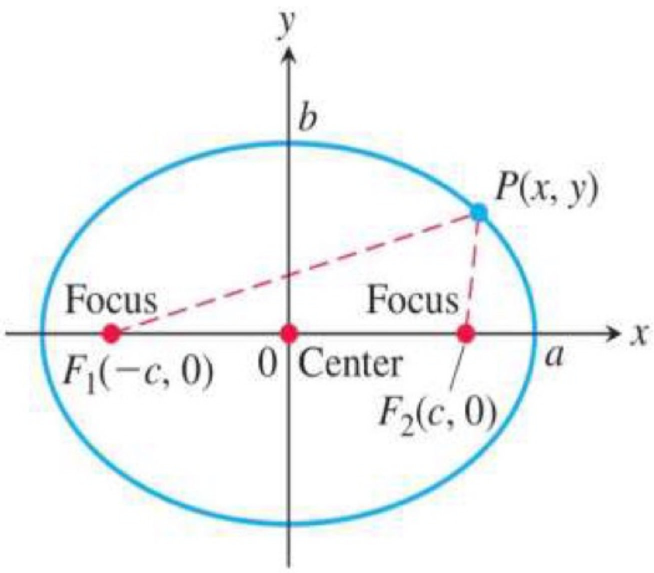
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \dots\dots ***$$

Equation (***) reveals that this ellipse is symmetric with respect to the origin and both coordinate axes. It lies inside the rectangle bounded by the lines $x = \pm a$ and $y = \pm b$. It crosses the axes at the points $(\pm a, 0)$ and $(0, \pm b)$.

The major axis of the ellipse in Equation (***) is the line segment of length $2a$ joining the points $(\pm a, 0)$. **The minor axis** is the line segment of length $2b$ joining the points $(0, \pm b)$. The number a itself is **the semimajor axis**, the number b **the semiminor axis**. The number c, found from Equation (**) as

$$c = \sqrt{a^2 - b^2},$$

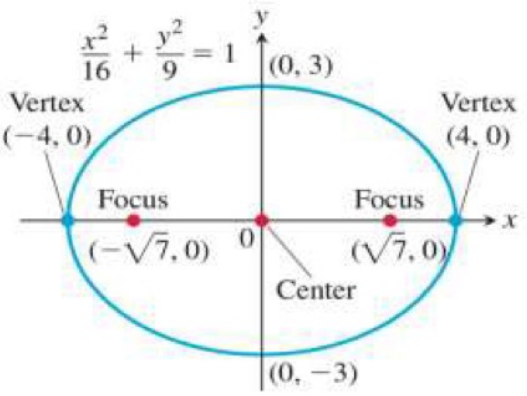
is **the center-to-focus distance** of the ellipse. If $a = b$ then the ellipse is a circle.



EXAMPLE : The ellipse

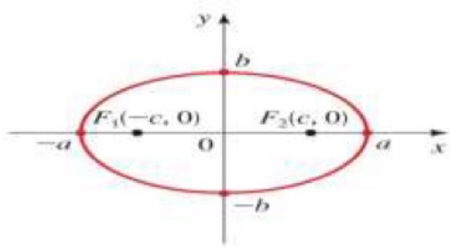
$$\frac{x^2}{16}+\frac{y^2}{9}=1$$

Semimajor axis: $a=\sqrt{16}=4$, Semiminor axis: $b=\sqrt{9}=3$,
Center-to-focus distance: $c=\sqrt{16-9}=\sqrt{7}$,
Foci: $(\pm c,0)=(\pm\sqrt{7},0)$,
Vertices: $(\pm a,0)=(\pm 4,0)$,
Center: $(0,0)$.



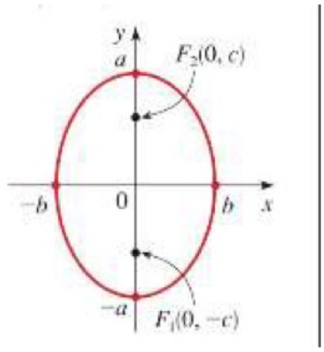
1.Standard-Form Equations for Ellipses Centered at the Origin

Foci on the x-axis: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ (a>b)$



Center-to-focus distance: $c=\sqrt{a^2-b^2}$
Foci: $(\pm c,0)$
Vertices: $(\pm a,0)$

Foci on the y-axis: $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1\ (a>b)$
Center-to-focus distance: $c=\sqrt{a^2-b^2}$
Foci: $(0,\pm c)$
Vertices: $(0,\pm a)$
In each case, a is the semimajor axis and b is the semiminor axis.



2. The standard form equation for ellipses centered at the Point (h,k)

A. foci on the x-axis

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1,\ a>b$$

the foci is: $(h+c,k),(h-c,k)$

B. foci on the y-axis

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1,\ a<b$$

the loci is: $(h,k+c),(h,k-c)$

EXAMPLE :

Find the foci of the ellipse $16x^2+9y^2=144$, and sketch its graph.

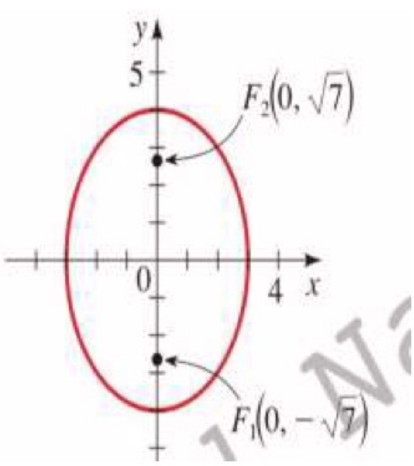
SOLUTION: First we put the equation in standard form. Dividing by 144, we get = Conic

$$\frac{x^2}{9}+\frac{y^2}{16}=1$$

Since $16>9$, this is an ellipse with its foci on the y-axis and with $a=4$ and $b=3$. We have

$$\begin{aligned} c^2 &= a^2-b^2=16-9=7 \\ c &= \sqrt{7} \end{aligned}$$

Thus the foci are $(0,\pm\sqrt{7})$. The graph is shown in figure



Ex: Find the equation of ellipse

$$4x^2 + 9y^2 - 48x + 72y + 144 = 0$$

Sol :

$$(4x^2 - 48x) + (9y^2 + 72y) + 144 = 0$$
$$4(x^2 - 12x) + 9(y^2 + 8y) + 144 = 0$$
$$4(x^2 - 12x + 36 - 36) + 9(y^2 + 8y + 16 - 16) + 144 = 0$$
$$4(x^2 - 12x + 36) - 144 + 9(y^2 + 8y + 16) - 144 + 144 = 0$$
$$4(x-6)^2 + 9(y+4)^2 = 144] \div 144$$
$$\frac{(x-6)^2}{36} + \frac{(y+4)^2}{16} = 1$$

∴ the center is $(h, k) = (6, -4)$

$a^2 = 36 \Rightarrow a = 6\,,\qquad (6,0), (-6,0),$
$b^2 = 16 \Rightarrow b = 4,\qquad (0,4), (0) - 4)$
$c = \sqrt{a^2 - b^2} = 2\sqrt{5},$ foci is $(6 + 2\sqrt{5}, -4), (6 - 2\sqrt{5}, -4)$

the length of simemajor is $2a = 12$
simeminer is $2b = 8$

Ex: Find the vertices, foci, semimajor, semiminor, center, length of major, length of minor

1. $\frac{x^2}{64} + \frac{y^2}{100} = 1$
2. $\frac{x^2}{81} + \frac{y^2}{36} = 1$
3. $4x^2 + 25y^2 = 100$