University of Baghdad
College of Education for Pure Science
(Ibn Al-Haitham)
Mathematics Department

جامعة بغداد
كلية التربية للعلوم الصرفه - ابن الهيثم
قسم الرياضيات - المرحلة الرابعة

محاضرات الاحصاء الرياضي

أساتذة المادة

أ.م.د. بيداء عطية   أ.م.د. علي طالب   أ.م. الاء ماجد

Chapter One

Statistical Distributions

1. Discrete probability Distributions

1.1 Bernoulli distribution

If the random experiment being repeated has only two outcomes such as (success, failure) for example (Male, female), (yes, no), (head, tail) and so on, we have a particularly important case of repeated trials known as Bernoulli trials.

Definition

The discrete r. v. X is said to have a Bernoulli distribution with parameter (p) denoted as $X \sim Ber(1, p)$ if it has probability mass function (p.m.f) and given as follows:

$$f(x) = \begin{cases} p^x(1-p)^{1-x} & x = 0,1 \\ 0 & O.W \end{cases}$$

Properties

1. The mean $\mu_x = E(x) = p$

Proof:

$$E(x) = \sum_{x=0}^{1} x \cdot f(x) = 0 \cdot f(0) + 1 \cdot f(1) = 0 + p^1(1-p)^0 = p$$

2. The variance $var(x) = \sigma_x^2 = p(1-p)$

Proof:

$$var(x) = \sigma_x^2 = E(x^2) - (E(x))^2$$

$$E(x^2) = \sum_{x=0}^{1} x^2 \cdot f(x) = 0^2 \cdot f(0) + 1^2 \cdot f(1) = p^1(1-p)^0 = p$$

$$\sigma_x^2 = E(x^2) - \left(E(x)\right)^2 = p - p^2 = p(1-p)$$

3. The Moment generating function (m.g.f) $M_x(t) = 1 - p + p \cdot e^t$

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Proof:

$$M_x(t) = E(e^{tx}) = \sum_{x=0}^{1} e^{tx} \cdot f(x) = e^0 \cdot f(0) + e^t \cdot f(1) = 1 \cdot (1-p) + e^t p$$
$$= 1 - p + p \cdot e^t$$

1.2 Binomial distribution (هذا التوزيع يستخدم في الحاجات او الحوادث المتناقصة)

The discrete r. v. X is said to have a Bernoulli distribution with parameter (n and p) with ($n\epsilon\mathbb{N}$ and $0 < p < 1$) denoted as $X \sim b(n,p)$ if it has probability mass function (p.m.f) and given as follows:

$$f(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0,1,\cdots,n \\ 0 & O.W \end{cases}$$

Question: Verity that $f(x)$ given above is p.m.f?

Solution: It enough to satisfy two conditions

1. $f(x) > 0$
2. $\sum_{x=0}^{n} f(x) = 1$

It is clear that the first condition is satisfied since ($n\epsilon\mathbb{N}$ and $0 < p < 1$).

For the Second condition we have

$$\sum_{x=0}^{n} f(x) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = (p + (1-p))^n = 1$$

Properties

1. The mean $\mu_x = E(x) = np$

Proof:

$$E(x)=\sum_{x=0}^{n}x\cdot f(x)=\sum_{x=0}^{n}x\cdot\binom{n}{x}p^{x}(1-p)^{n-x}=\sum_{x=0}^{n}x\cdot\frac{n!}{x!\,(n-x)!}\cdot p^{x}(1-p)^{n-x}$$

$$=\sum_{x=0}^{n}\frac{x\cdot n(n-1)!}{x(x-1)!\,(n-x)!}\cdot p\cdot p^{x-1}(1-p)^{n-x}$$

$$=np\cdot\sum_{x=0}^{n}\frac{(n-1)!}{(x-1)!\,(n-x)!}\cdot p^{x-1}(1-p)^{n-x}$$

Now putting $m=n-1$ and $y=x-1$, then $m-y=n-x$ and we have:

$$E(x)=np\cdot\sum_{y=0}^{m}\frac{m!}{y!\,(m-y)!}\cdot p^{y}(1-p)^{m-y}=np\cdot\sum_{y=0}^{m}\binom{m}{y}\cdot p^{y}(1-p)^{m-y}=np$$

2. The variance $var(x)=\sigma_x^2=np(1-p)$

Proof:

$$var(x)=\sigma_x^2=E(x^2)-(E(x))^2$$

$$E(x^2)=E(x^2-x+x)=E(x(x-1)+x)=E\big(x(x-1)\big)+E(x)$$

$$E\big(x(x-1)\big)=\sum_{x=0}^{n}x(x-1)\cdot f(x)=\sum_{x=0}^{n}x(x-1)\cdot\frac{n!}{x!\,(n-x)!}\cdot p^{x}(1-p)^{n-x}$$

$$=\sum_{x=0}^{n}x(x-1)\cdot\frac{n(n-1)(n-2)!}{x(x-1)(x-2)!\,(n-x)!}\cdot p^{2}\cdot p^{x-2}(1-p)^{n-x}$$

$$=n(n-1)\cdot p^{2}\sum_{x=0}^{n}\frac{(n-2)!}{(x-2)!\,(n-x)!}\cdot p^{x-2}(1-p)^{n-x}$$

Now putting $m=n-2$ and $y=x-2$, then $m-y=n-x$ and we have:

$$E\big(x(x-1)\big) = n(n-1)\cdot p^2 \sum_{x=0}^{n} \frac{m!}{y!\,(m-y)!}\cdot p^y(1-p)^{m-y}$$

$$= n(n-1)\cdot p^2 \sum_{x=0}^{n} \binom{m}{y}\cdot p^y(1-p)^{m-y} = n(n-1)\cdot p^2$$

$$E(x^2) = E\big(x(x-1)\big) + E(x) = n(n-1)\cdot p^2 + np$$

$$\sigma_x^2 = E(x^2) - \big(E(x)\big)^2 = n(n-1)\cdot p^2 + np - n^2p^2 = n^2p^2 - np^2 + np - n^2p^2$$

$$= np - np^2 = np(1-p)$$

3. The Moment generating function (m.g.f) $M_x(t) = (1-p+p\cdot e^t)^n$

Proof:

$$M_x(t) = E(e^{tx}) = \sum_{x=0}^{n} e^{tx}\cdot f(x) = \sum_{x=0}^{n} e^{tx}\cdot \binom{n}{x} p^x(1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x}\cdot (pe^t)^x\cdot(1-p)^{n-x}$$

Note: $(a+b)^n = \sum_{x=0}^{n} \binom{n}{x}\cdot a^x\cdot b^{n-x}$, so we can get

$$M_x(t) = E(e^{tx}) = \sum_{x=0}^{n} \binom{n}{x}\cdot(pe^t)^x\cdot(1-p)^{n-x} = (1-p+pe^t)^n$$

Example: let $X \sim b(n,p)$, find $E(x)$ and $var(x)$ using $m.g.f$?

Solution:

$$Note{:}\ E(x^r) = M_x^{(r)}(0) \text{and } r = 1,2,\cdots$$

$$E(x) = M_x'(0)$$

$$M_x(t) = E(e^{tx}) = (1 - p + pe^t)^n$$

$$M_x'(t) = n(1 - p + pe^t)^{n-1} \cdot pe^t$$

$$E(x) = M_x'(0) = np$$

$$\sigma_x^2 = E(x^2) - \left(E(x)\right)^2 = M_x''(0) - \left(M_x'(0)\right)^2$$

$$M_x''(t) = n(n-1)(1 - p + pe^t)^{n-2} \cdot (pe^t)^2 + n(1 - p + pe^t)^{n-1} pe^t$$

$$E(x^2) = M_x''(0) = n(n-1)(1 - p + p)^{n-2} \cdot p^2 + n(1 - p + p)^{n-1} \cdot p$$
$$= n^2p^2 - np^2 + np$$

$$\sigma_x^2 = M_x''(0) - \left(M_x'(0)\right)^2 = n^2p^2 - np^2 + np - n^2p^2 = np(1-p)$$

Example: let $X \sim b(n, p)$, show that

1. $E\left(\frac{x}{n}\right) = p$
2. $E\left(\left(\frac{x}{n} - p\right)^2\right) = \frac{p(1-p)}{n}$

Solution:

1. $E\left(\frac{x}{n}\right) = \frac{1}{n}E(x) = \frac{1}{n}np = p$
2. Let $y = \frac{x}{n} - p$ then $E\left(\left(\frac{x}{n} - p\right)^2\right) = E(y^2)$

since $var(y) = E(y^2) - E(y^2),$ then $E(y^2) = var(y) + E(y^2)$

Note: $var(c) = 0, c$ is constant

$$E\left(\left(\frac{x}{n}-p\right)^2\right)=var\left(\frac{x}{n}-p\right)+\left(E\left(\frac{x}{n}-p\right)\right)^2$$

$$=var\left(\frac{x}{n}\right)-var(p)+\left(E\left(\frac{x}{n}\right)-E(p)\right)^2$$

$$=\frac{1}{n^2}var(x)-0+\left(\frac{1}{n}E(x)-p\right)^2=\frac{1}{n^2}\cdot np(1-p)+\left(\frac{1}{n}np-p\right)^2$$

$$=\frac{p(1-p)}{n}$$

Example: let X_1, X_2, X_3 be an independent r.vs have the same p.d.f

$$f(x)=3x^2, 0<x<1$$

Find the probability that that exactly two of these three variables exceeded $\frac{1}{2}$?

Solution: At the fist we have to find the probability that any one of these three variables exceeded $\frac{1}{2}$ as follows:

$$p=\int_{\frac{1}{2}}^{1} f(x)dx=\int_{\frac{1}{2}}^{1} 3x^2dx=x^3\Big|_{1/2}^{1}=1-\frac{1}{8}=\frac{7}{8}$$

The probability of exactly two of these three variables exceed $\frac{1}{2}$ is:

$$f(2)=Pr(x=2)=\binom{3}{2}\left(\frac{7}{8}\right)^2\left(\frac{1}{8}\right)^{2-1}=\frac{147}{512}$$

Example: let $X_1, X_2, \cdots, X_k$ be an independent r.vs such that $X_i \sim b(n_i, p), i=1,2,\cdots,k$ show that

$$\sum_{i=1}^{k} X_i \sim b\left(\sum_{i=1}^{k} n_i, p\right)$$

Solution: let

$$Y = \sum_{i=1}^{k} X_i$$

By using the m,g.f

$$M_Y(t) = E(e^{tY}) = E\left(e^{t\sum_{i=1}^{k} X_i}\right) = E\left(e^{t(X_1,X_2,\cdots,X_k)}\right) = E(e^{tX_1} \cdot e^{tX_2} \cdots e^{tX_k})$$

Now, since $X_1, X_2, \cdots, X_k$ be an independent r.vs, then

$$\begin{aligned} M_Y(t) = E(e^{tX_1}) \cdot E(e^{tX_2}) \cdots E(e^{tX_k}) &= M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_k}(t) \\ &= (1-p+pe^t)^{n_1} \cdot (1-p+pe^t)^{n_2} \cdots (1-p+pe^t)^{n_k} \\ &= (1-p+pe^t)^{\sum_{i=1}^{k} n_i} \end{aligned}$$

$$\therefore Y = \sum_{i=1}^{k} X_i \sim b\left(\sum_{i=1}^{k} n_i \, , p\right)$$

Example: let $X \sim b(n,p)$ show that:

$$f(x+1) = \left(\frac{p(n-x)}{(x+1)(1-p)}\right) \cdot f(x)$$

Solution:

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$f(x+1) = \binom{n}{x+1} p^{x+1} (1-p)^{n-(x+1)}$$

$$\frac{f(x+1)}{f(x)}=\frac{\frac{n!}{(x+1)!\big(n-(x+1)\big)!}\,p^{x+1}(1-p)^{n-(x+1)}}{\frac{n!}{x!\,(n-x)!}\,p^{x}(1-p)^{n-x}}$$

$$=\frac{\frac{n!}{(x+1)x!\big(n-(x+1)\big)(n-x)!}}{\frac{n!}{x!\,(n-x)!}}\cdot\frac{p^{x}\,p\,(1-p)^{n-x}\,(1-p)^{-1}}{p^{x}\,(1-p)^{n-x}}$$

$$=\frac{n!\,x!\,(n-x)!}{n!\ (x+1)x!\big(n-(x+1)\big)!}\cdot\frac{p}{(1-p)}$$

$$=\frac{n!\,x!\,(n-x)(n-x-1)!}{n!\ (x+1)x!\,(n-x-1)!}\cdot\frac{p}{(1-p)}=\frac{(n-x)}{(x+1)}\cdot\frac{p}{(1-p)}$$

$$\therefore f(x+1)=\left(\frac{p(n-x)}{(x+1)(1-p)}\right)\cdot f(x)$$

1.3 Poisson Distribution

A discrete random variable X is said to have a Poisson distribution, with parameter λ and denoted by $X\sim p(\lambda)$, if it has a probability mass function (p.m.f) given by:

$$f(x)=\begin{cases}\dfrac{e^{-\lambda}\,\lambda^{x}}{x!} & x=0,1,2,\cdots\ \text{ and } \lambda>0\\ 0 & O.W\end{cases}$$

Properties

1. The m.g.f of the distribution is $M_X(t)=e^{\lambda(e^{t}-1)}$

Proof:

$$M_X(t)=E(e^{tx})=\sum_{x=0}^{\infty}e^{tx}f(x)=\sum_{x=0}^{\infty}e^{tx}\cdot\frac{e^{-\lambda}\,\lambda^{x}}{x!}=e^{-\lambda}\sum_{x=0}^{\infty}\frac{e^{tx}\,\lambda^{x}}{x!}=e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^{t})^{x}}{x!}$$

Note: $e^x = \sum_{i=0}^{n} \frac{x^i}{i!} = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, then $\sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{\lambda e^t}$, so we get

$$M_X(t) = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda e^t - \lambda} = e^{\lambda(e^t - 1)}$$

2. $\mu_x = \sigma_x^2 = \lambda$

It's possible to using the m.g.f for finding the mean and the variance of Poisson distribution as follows.

$$M_X(t) = e^{\lambda(e^t-1)}$$

$$M'_x(t) = \lambda e^t e^{\lambda(e^t-1)}$$

$$M''_x(t) = \lambda e^t e^{\lambda(e^t-1)} + \lambda^2 e^{2t} e^{\lambda(e^t-1)}$$

$$E(x) = M'_x(0) = \lambda e^0 e^{\lambda(1-1)} = \lambda$$

$$\therefore \mu_x = E(x) = \lambda$$

$$M''_x(0) = \lambda e^0 e^{\lambda(1-1)} + \lambda^2 e^0 e^{\lambda(1-1)} = \lambda + \lambda^2$$

$$\sigma_x^2 = E(x^2) - \big(E(x)\big)^2 = M''_x(0) - \big(M'_x(0)\big)^2 = \lambda + \lambda^2 - \lambda^2 = \lambda$$

$$\therefore \sigma_x^2 = \lambda$$

3. The Poisson distribution is an approximation of binomial distribution as $(\lambda = np)$ and (n) approaches to infinity

Proof: the m.g.f of the binomial distribution is

$$M_x(t) = E(e^{tx}) = (1 - p + pe^t)^n = \big(1 + p(e^t - 1)\big)^n$$

Now putting $\left(p = \frac{\lambda}{n}\right)$, then we have:

$$M_x(t) = \left(1 + \frac{\lambda}{n}(e^t - 1)\right)^n$$

$$\lim_{n\to\infty} M_x(t) = \lim_{n\to\infty} \left(1 + \frac{\lambda(e^t - 1)}{n}\right)^n$$

Using the well-known result from calculus that $\left(\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x\right)$, that we get:

$$\lim_{n\to\infty} \left(1 + \frac{\lambda(e^t - 1)}{n}\right)^n = e^{\lambda(e^t-1)}$$

This is the m.g.f of the Poisson distribution with parameter (λ).

Example: verify that the function $\left(f(x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad x = 0,1,2,\cdots\right)$ is actually the probability function?

Solution: it's enough to satisfy the following two conditions

1. $f(x) > 0$
2. $\sum_{x=0}^{\infty} f(x) = 1$

Firstly, since $(\lambda > 0$ and $x = 0,1,2,\cdots)$, so it's clear that $f(x) > 0$.

Secondly, as we know $\left(e^x = \sum_{i=0}^{n} \frac{x^i}{i!} = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$, so we ge:

$$\sum_{x=0}^{\infty} f(x) = \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda}e^{\lambda} = 1$$

Example: let $X_1, X_2, \cdots, X_n$ be an independent r.vs such that $X_i \sim P(\lambda_i), i = 1,2,\cdots,n$, then $\sum_{i=1}^{n} X_i \sim P(\sum_{i=1}^{n} \lambda_i)$.

Solution: let $(Y = \sum_{i=1}^{n} X_i)$

$$M_Y(t) = E(e^{tY}) = E\left(e^{t\sum_{i=1}^{n} X_i}\right) = E\left(e^{t(X_1,X_2,\cdots,X_n)}\right) = E(e^{tX_1} \cdot e^{tX_2} \cdots e^{tX_n})$$

Now, since $X_1, X_2, \cdots, X_n$ be an independent r.vs, then

$$M_Y(t) = E(e^{tX_1}) \cdot E(e^{tX_2}) \cdots E(e^{tX_n}) = M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_n}(t)$$
$$= e^{\lambda_1(e^t-1)} \cdot e^{\lambda_2(e^t-1)} \cdots e^{\lambda_n(e^t-1)} = e^{\sum_{i=1}^{n}\lambda_i(e^t-1)}$$

$$\therefore Y = \sum_{i=1}^{n} X_i \sim P\left(\sum_{i=1}^{n} \lambda_i\right)$$

1.4 Negative binomial distribution

Consider an experiment of independent Bernoulli trials performed until we get a total of (r) successes and then stops. The probability of each individual trial resulting in a success is (p) where $0 < p < l$. let x denote the number of failures encountered before we get the first r successes, then the p.m.f of x is given by:

$$f(x) = \begin{cases} \binom{x+r-1}{x} P^r (1-P)^x , & x = 0,1,2,\cdots, r = 1,2,\cdots \\ 0 & O.W \end{cases}$$

And we write $X \sim N\, b(r,p)$ where the constants r and p are the parameters of distribution.

Ex. show that $f(x)$ is exactly a p.m.f .

Solution:

1. It is clear that f(x) > 0 since each x, r are positive and 0 < p < 1.

2. To show that $\sum f(x) = 1$

By applying the following rule:

$$\sum_{j=0}^{\infty} \binom{n+j-1}{j} Z^j = (1-Z)^{-n}$$

$$\sum_{x=0}^{\infty} xf(x) = \sum_{x=0}^{\infty} \binom{x+r-1}{x} P^r\,(1-P)^x = P^r \sum_{x=0}^{\infty} \binom{x+r-1}{x}\,(1-P)^x$$

$$= P^r\big(1-(1-P)\big)^{-r} = P^r\,P^{-r} = 1$$

Properties

1. The Moment Generating Function (m.g.f)

$$M_x(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r$$

Proof:

$$M_x(t) = E(e^{tx})$$

$$= \sum_{x=0}^{\infty} e^{tx} f(x) = \sum_{x=0}^{\infty} e^{tx} \binom{x+r-1}{x} P^r\,(1-P)^x$$

$$= P^r \sum_{x=0}^{\infty} \binom{x+r-1}{x}\,\big((1-P)e^t\big)^x = P^r(1-(1-p)e^t)^{-r}$$

$$= \left(\frac{p}{1-(1-p)e^t}\right)^r$$

2. The mean of the distribution is given by:

$$\mu_x = \frac{r(1-p)}{p}$$

Proof:

$$\mu_x = E(x) = M_x'(0)$$

$$M_x(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r$$

$$M_x'(t) = r\left(\frac{p}{1-(1-p)e^t}\right)^{r-1} \frac{p(1-p)e^t}{(1-(1-p)e^t)^2}$$

$$\mu_x = E(x) = M_x'(0) = r\left(\frac{p}{1-(1-p)}\right)^{r-1} \frac{p(1-p)}{\left(1-(1-p)\right)^2} = r\left(\frac{p}{p}\right)^{r-1} \frac{p(1-p)}{p^2}$$

$$\mu_x = \frac{r(1-p)}{p}$$

3. the variance of the distribution is $\sigma_x^2 = \frac{r(1-p)}{p^2}$

Proof:

$$\sigma_x^2 = M_x''(0) - (M_x'(0))^2$$

$$M_x'(t) = r\left(\frac{p}{1-(1-p)e^t}\right)^{r-1} \frac{p(1-p)e^t}{(1-(1-p)e^t)^2}$$

$$= r\left(\frac{p}{1-(1-p)e^t}\right)^{r} \left(\frac{p}{1-(1-p)e^t}\right)^{-1} \frac{p(1-p)e^t}{(1-(1-p)e^t)^2}$$

but we have $M_x(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r$ Then

$$M_x'(t) = r\,M_x(t)\left(\frac{p}{1-(1-p)e^t}\right)^{-1} \frac{p(1-p)e^t}{(1-(1-p)e^t)^2}$$

$$= r\,M_x(t)\;\frac{1-(1-p)e^t}{p}\;\frac{p(1-p)e^t}{(1-(1-p)e^t)^2} = r\,M_x(t)\,\frac{(1-p)e^t}{1-(1-p)e^t}$$

putting $(u = (1-p)e^t)$ with $\left(\frac{du}{dt} = (1-p)e^t = u\right)$ and Rewrite $\left(M_x'(t)\right)$

$$M_x'(t) = r\,M_x(t)\left(\frac{u}{1-u}\right)$$

$$M_x''(t) = r\,M_x(t)\,\left(\frac{1-u+u}{(1-u)^2}\right)\left(\frac{du}{dt}\right) + \left(\frac{u}{1-u}\right) rM_x'(t)$$

$$= M_x(t)\,\left(\frac{ru}{(1-u)^2}\right) + \left(\frac{ru}{1-u}\right) M_x'(t)$$

$$= \frac{ru}{1-u}\left(M_x(t)\,\left(\frac{1}{1-u}\right) + M_x'(t)\right)$$

$$= \frac{r(1-p)e^t}{1-(1-p)e^t}\left(M_x(t)\,\left(\frac{1}{1-(1-p)e^t}\right) + M_x'(t)\right)$$

$$M_x''(0) = \frac{r(1-p)}{p}\left(M_x(0)\frac{1}{p} + M_x'(0)\right)$$

Since $M_x(0) = 1\ and\ M_x'(0) = \frac{r(1-p)}{p}$, so we can get

$$M_x''(0) = \frac{r(1-p)}{p}\left(\frac{1}{p} + \frac{r(1-p)}{p}\right) = \frac{r(1-p)}{p^2} + \frac{r^2(1-p)^2}{p^2}$$

$$\sigma_x^2 = M_x''(0) - \left(M_x'(0)\right)^2 = \frac{r(1-p)}{p^2} + \frac{r^2(1-p)^2}{p^2} - \frac{r^2(1-p)^2}{p^2} = \frac{r(1-p)}{p^2}$$

1.5 <u>Geometric distribution</u>

The geometric distribution is special case of negative binomial distribution when $(r = 1)$ hence:

$$f(x) = \begin{cases} p\,(1-p)^x\,, & x = 0,1,2,\cdots \\ 0 & O.W \end{cases}$$

The properties of geometric distribution can be obtained from the corresponding properties of negative binomial distribution by putting $(r = 1)$ it follows that:

$$M_x(t) = \frac{p}{1-(1-p)e^t}$$

$$\mu_x = \frac{1-p}{p}$$

$$\sigma_x^2 = \frac{1-p}{p^2}$$

Example: A fair die is thrown is successive independent trials until the second three are observed. Let x be a r.v that denotes the number of failures before the second three observed.

i. Find the distribution of x.
ii. Find the probability of observing 10 no three is before the second three is observed.
iii. Find the mean, variance, and m.g.f of the distribution.

Solution:

i. $x \sim Nb\left(2, \frac{1}{6}\right)$, that is $r = 2$ and $p = \frac{1}{6}$

$$f(x) = \binom{x+1}{x} P^2 (1-P)^x, \qquad x = 0,1,2,\cdots, r = 2$$

ii. $P_r(x = 10) = f(10) = \binom{11}{10} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{10}$

iii.

$$\mu_x = E(x) = \frac{r(1-p)}{p} = \frac{2\,\frac{5}{6}}{\frac{1}{6}} = 10$$

$$\sigma_x^2 = \frac{r(1-p)}{p^2} = \frac{2\,\frac{5}{6}}{\left(\frac{1}{6}\right)^2} = 60$$

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$$M_x(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r = \left(\frac{\frac{1}{6}}{1-\frac{5}{6}e^t}\right)^2$$

Example: suppose we flip affair coin until we get ahead. Let (x) be the number of tails before we get ahead.

i. Find the p.m.f of (x).
ii. Find the mean, variance, and m.g.f of (x).

Solution:

i. Sine $(r = 1\ )$(first head), then we have a geometric distribution with $\left(p = \frac{1}{2}\right)$ and hence

$$f(x) = p(1-p)^x = \frac{1}{2}\left(\frac{1}{2}\right)^x$$

ii.

$$\mu_x = \frac{1-p}{p} = \frac{1-\frac{1}{2}}{\frac{1}{2}} = 1$$

$$\sigma_x^2 = \frac{1-p}{p^2} = \frac{1/2}{1/4} = 2$$

$$M_x(t) = \frac{p}{1-(1-p)e^t} = \frac{1/2}{1-(1/2)e^t}$$

Exercises (1)

1. Given $X \sim Ber\left(1, \frac{1}{3}\right)$, find the following:

i. The p.m.f of x?

ii. $M_x(t), \sigma_x^2$, and μ_x?

2. The m. g. f of ar.v x is $M_x(t) = \left(\frac{2}{3} + \frac{1}{3}e^t\right)^9$, find the following:

i. The p.mf of x?

ii. σ_x^2, and μ_x?

iii. Show that $Pr(\mu_x - 2\sigma_x < x < \mu_x + 2\sigma_x) = \sum_{i=1}^{n} x^9 \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{9-x}$?

3. Let $X \sim b(2, p)$ and $Y \sim b(4, p)$, if $Pr(X \geq 1) = \frac{5}{9}$, find $Pr(Y \geq 1)$?

4. Let X_1 and X_2 are independent r.vs such that $X_1 \sim P(4)$ *and* $X_2 \sim P(6)$, if $Y = X_1 + X_2$ find the following:

i. The p.m.f of Y?

ii. σ_y^2, and μ_y?

iii. $Pr(Y \leq 1)$?

5. Given $X \sim P(\lambda)$ find the value of (λ), if you know that $f(x) = \frac{4}{x} \cdot f(x-1), x \in \mathbb{N}$?

6. Let $X_i \sim Nb(r_i, p), i = 1,2, \cdots, n$, show that $\sum_{i=1}^{n} X_i \sim Nb(\sum_{i=1}^{n} r_i, p)$?

7. Let $X \sim Nb(4,0.3)$, find the following:

i. P.m.f of x?

ii. $M_x(t), \sigma_x^2$, and μ_x?

iii. Let $y = 4 + 5x$, find σ_y^2, and μ_y?

2. Continuous probability Distributions

2.1 The uniform distribution

A continuous r. v X is said to follow a uniform distribution denoted as $X \sim u(a, b)$ if the p.d.f of x is

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \leq x \leq b \\ 0 & O.w \end{cases}$$

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The real numbers a, b & $b > a$ are the parameters of the distribution It can be shown that $f(x)$ is actually a p.d.f since $f(x) = \frac{1}{b-a} > 0$ (because $a < b$) and

$$\int_a^b f(x)dx = \frac{1}{b-a}\int_a^b dx = 1$$

Properties

1. The mean $\mu_x = \frac{b+a}{2}$

Proof:

$$\mu_x = E(x) = \int_a^b x\, f(x)dx = \frac{1}{b-a}\int_a^b x\, dx = \frac{1}{b-a}\, \frac{x^2}{2}\Big]_a^b = \frac{1}{b-a}\, \frac{b^2-a^2}{2} = \frac{b+a}{2}$$

2. The variance $\sigma_x^2 = \frac{(b-a)^2}{12}$

Proof:

$$\sigma_x^2 = E(x^2) - (E(x))^2$$

$$E(x^2) = \int_a^b x^2\, f(x)dx = \frac{1}{b-a}\int_a^b x^2\, dx = \frac{1}{b-a}\, \frac{x^3}{3}\Big]_a^b = \frac{1}{b-a}\, \frac{b^3-a^3}{3}$$

$$= \frac{1}{b-a}\, \frac{(b-a)(b^2+ab+a^2)}{3} = \frac{(b^2+ab+a^2)}{3}$$

$$\sigma_x^2 = \frac{(b^2+ab+a^2)}{3} - \left(\frac{b+a}{2}\right)^2 = \frac{(b^2+ab+a^2)}{3} - \frac{b^2+2ab+a^2}{4}$$

$$= \frac{4b^2+4ab+4a^2-3b^2-6ab-3a^2}{12} = \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}$$

3. The m.g.f of the distribution is

$$M_x(t) = \frac{e^{bt} - e^{at}}{b(b-a)}, \qquad t > 0.$$

4. The k^{th} moment about origin is

$$E(x^k) = \frac{b^{k+1} - a^{k+1}}{(k+1)(b-a)}$$

Proof:

$$E(x^k) = \int_a^b x^k\, f(x)dx = \frac{1}{b-a}\int_a^b x^k\, dx = \frac{1}{b-a}\ \frac{x^{k+1}}{k+1}\Big]_a^b = \frac{b^{k+1} - a^{k+1}}{(k+1)(b-a)}$$

Example: let $x \sim u\,(-a\,, a)\,, a\ > 0$ find the value of (a) if it is known that

$$p_r\,(x > 1) = \frac{1}{3}$$

Solution:

$$f(x) = \frac{1}{a - (-a)} = \frac{1}{2a}$$

$$p_r\,(x > 1) = \int_1^a \frac{1}{2a}\, dx = \frac{1}{2a}\ x]_1^a = \frac{a-1}{2a} = \frac{1}{3}$$

$$2a = 3a - 3\ \Rightarrow a = 3$$

2.2 Gamma distribution

Definition: If $\propto > 0$, we define the gamma function

$$\Gamma_\propto = \int_0^\infty y^{\propto - 1}\, e^{-y}\, dy$$

Properties of gamma function

1. $\Gamma_{\propto+1} = \propto \Gamma_{\propto}$, If $\propto \neq 0$ is positive real number
2. If $\propto$ is positive number then $\Gamma_{\propto+1} = \propto !$
3. $\Gamma_{\propto} = \int_0^\infty x^{\propto-1}\, e^{-x}\, dx = 2\int_0^\infty x^{2\propto-1}\, e^{-x^2}\, dx$
4. $\Gamma_{\frac{1}{2}} = \sqrt{\pi}$

Note that:

- If $\propto = 1 \Rightarrow \Gamma_1 = \int_0^\infty x^{1-1}\, e^{-x}\, dx = \int_0^\infty e^{-x}\, dx = -e^{-x}]_0^\infty = 1$
- If $\propto > 1 \Rightarrow \Gamma_{\propto} = (\propto -1)\int_0^\infty x^{\propto-2}\, e^{-x}\, dx = (\propto -1)\Gamma_{\propto-1} = (\propto -1)!$

Definition: the continuous $r.v\ X$ is said to have a gamma distribution with parameters $\propto, \beta > 0$ denoted as $X \sim G\ (\propto, \beta)$ if the p.d.f of x is:

$$f(x) = \begin{cases} \dfrac{1}{\Gamma_{\propto}\, \beta^{\propto}}\, x^{\propto-1}\, e^{-x/\beta} & x > 0 \\ 0 & O.W \end{cases}$$

Properties of Gamma distribution

1. The m. g. f of the distribution is $M_x(t) = (1-\beta t)^{-\propto},\ t > \frac{1}{\beta}$

Proof:

$$M_x(t) = E(e^{tx})$$

$$= \int_0^\infty e^{tx} f(x)dx = \int_0^\infty e^{tx}\, \frac{1}{\Gamma_{\propto}\, \beta^{\propto}}\, x^{\propto-1}\, e^{-x/\beta}\, dx$$

$$= \frac{1}{\Gamma_{\propto}\, \beta^{\propto}} \int_0^\infty x^{\propto-1}\, e^{tx-\frac{x}{\beta}}\, dx = \frac{1}{\Gamma_{\propto}\, \beta^{\propto}} \int_0^\infty x^{\propto-1}\, e^{\frac{\beta tx - x}{\beta}}\, dx$$

$$= \frac{1}{\Gamma_{\propto}\, \beta^{\propto}} \int_0^\infty x^{\propto-1}\, e^{\frac{-x(1-\beta t)}{\beta}}\, dx$$

Putting $\left(y = \frac{x(1-\beta t)}{\beta} \Rightarrow x = \frac{\beta}{1-\beta t}\, y \Rightarrow \ dx = \frac{\beta}{1-\beta t}\, dy\right)$, and rewrite $M_x(t)$:

$$M_x(t) = \frac{1}{\Gamma_\propto\, \beta^\propto} \int_0^\infty \left(\frac{\beta}{1-\beta t}\, y\right)^{\propto-1} e^{-y}\, \frac{\beta}{1-\beta t}\, dy$$

$$= \frac{1}{\Gamma_\propto\, \beta^\propto} \int_0^\infty \left(\frac{\beta}{1-\beta t}\right)^{\propto-1} (\,y)^{\propto-1}\, e^{-y}\, \frac{\beta}{1-\beta t}\, dy$$

$$= \frac{1}{\Gamma_\propto\, \beta^\propto} \left(\frac{\beta}{1-\beta t}\right)^{\propto} \int_0^\infty (\,y)^{\propto-1}\, e^{-y}\ dy = \frac{1}{\Gamma_\propto\, \beta^\propto} \left(\frac{\beta}{1-\beta t}\right)^{\propto} \Gamma_\propto$$

$$= \frac{1}{\beta^\propto}\left(\frac{\beta}{1-\beta t}\right)^{\propto} = \left(\frac{1}{\beta}\frac{\beta}{(1-\beta t)}\right)^{\propto} = \left(\frac{1}{(1-\beta t)}\right)^{\propto} = (1-\beta t)^{-\propto}$$

2. $\mu_x = E(x) = \propto \beta$
3. $\sigma_x^2 = ver(x) = \propto \beta^2$
4. The k^{th} moment about origin is

$$E(x^k) = \frac{\beta^k \Gamma_{\propto+k}}{\Gamma_\propto}, k = 1,2,3,\cdots$$

Proof:

$$E(x^k) = \int_0^\infty x^k\, f(x)dx = \int_0^\infty x^k\, \frac{1}{\Gamma_\propto\, \beta^\propto}\, x^{\propto-1}\, e^{-x/\beta}\, dx = \frac{1}{\Gamma_\propto\, \beta^\propto}\int_0^\infty\ x^{k+\propto-1}\, e^{-x/\beta}\, dx$$

Let $y = \frac{x}{\beta} \Rightarrow x = \beta y \Rightarrow dx = \beta\, dy$, and rewrite $E(x^k)$

$$E(x^k) = \frac{1}{\Gamma_\propto \beta^\propto} \int_0^\infty (\beta y)^{k+\propto-1} e^{-y} \beta \, dy = \frac{1}{\Gamma_\propto \beta^\propto} \int_0^\infty \beta^{k+\propto-1} y^{k+\propto-1} e^{-y} \beta \, dy$$

$$= \frac{\beta^{k+\propto}}{\Gamma_\propto \beta^\propto} \int_0^\infty y^{k+\propto-1} e^{-y} \, dy = \frac{\beta^\propto \beta^k}{\Gamma_\propto \beta^\propto} \Gamma_{\propto+k} = \frac{\beta^k \Gamma_{\propto+k}}{\Gamma_\propto}$$

Example: use the formula of $E(x^k)$, to find μ_x , σ_x^2?

Solution: putting k=1 $then$

$$E(x) = \frac{\beta \, \Gamma_{\propto+1}}{\Gamma_\propto} = \frac{\beta \propto \Gamma_\propto}{\Gamma_\propto} = \propto \beta$$

Putting k=2, $we\ get$

$$E(x^2) = \frac{\beta^2 \, \Gamma_{\propto+2}}{\Gamma_\propto} = \frac{\beta^2 \, (\propto+1)\Gamma_{\propto+1}}{\Gamma_\propto} = \frac{\beta^2 \, (\propto+1) \propto \Gamma_\propto}{\Gamma_\propto} = \beta^2 \propto (\propto+1)$$

$$\sigma_x^2 = E(x^2) - \big(E(x)\big)^2 = \beta^2 \propto (\propto+1) - (\propto \beta)^2 = \beta^2 \propto^2 + \beta^2 \propto - \beta^2 \propto^2 = \beta^2 \propto$$

2.3 <u>Chi-Square Distribution</u>

Chi Square Distribution define as a special case of Gamma distribution when $\propto = \frac{r}{2}$, and

β=2 where r is positive integer. Hence the p.d.f of the r. v. x is

$$f(x) = \begin{cases} \frac{1}{\Gamma_{\frac{r}{2}} \, 2^{\frac{r}{2}}} \, x^{\frac{r}{2}-1} \, e^{-\frac{x}{2}} & x > 0 \\ 0 & O.W \end{cases}$$

We write $X \sim X^2(r)$ where r is the number of degrees freedom representing the parameter of the distribution.

Properties

The properties of chi square dist. are the same of properties of gamma dist. when $\propto = \frac{r}{2}$, and β=2 that is:

1. $M_x(t) = (1-2t)^{\frac{-r}{2}}$
2. $\mu_x = E(x) = \propto \beta = \frac{r}{2}\, 2 = r$
3. $\sigma_x^2 = ver(x) = \propto \beta^2 = \frac{r}{2}\, 4 = 2r$
4. $E(x^k) = \frac{2^k\, \Gamma_{\frac{r}{2}+k}}{\Gamma_{\frac{r}{2}}}, k = 1,2, \ldots$

2.4 Beta distribution

Definition: If $\propto > 0\,, \beta > 0$, Beta function could be define as follows:

$$\beta(\propto, \beta) = \int_0^1 x^{\propto - 1}\,(1-x)^{\beta - 1}\,dx$$

It can be shown that

$$\beta(\propto, \beta) = \frac{\Gamma_\propto\, \Gamma_\beta}{\Gamma_{\propto+\beta}}, \propto > 0\,, \beta > 0$$

Definition: the continuous r. v. x is said to have a Beta distribution denoted as $x \sim \beta\,(\propto, \beta)$, if the p. d. f of x is

$$f(x) = \begin{cases} \frac{\Gamma_{\propto+\beta}}{\Gamma_\propto\, \Gamma_\beta}\, x^{\propto-1}\,(1-x)^{\beta-1} & 0 < x < 1 \\ 0 & O.W \end{cases}$$

Properties

1. The k^{th} moment about origin is $E(x^k) = \frac{\Gamma_{(k+\propto)}\, \Gamma_{(\propto+\beta)}}{\Gamma_\propto\, \Gamma_{(k+\propto+\beta)}}, k = 1,2, \ldots$

Proof:

$$E(x^k) = \int_0^1 x^k\, f(x)dx = \int_0^1 x^k\, \frac{\Gamma_{\propto+\beta}}{\Gamma_\propto\, \Gamma_\beta}\, x^{\propto-1}\,(1-x)^{\beta-1}dx$$

$$= \frac{\Gamma_{\propto+\beta}}{\Gamma_\propto\, \Gamma_\beta}\int_0^1 x^{k+\propto-1}\,(1-x)^{\beta-1}dx = \frac{\Gamma_{\propto+\beta}}{\Gamma_\propto\, \Gamma_\beta}\;\beta\,(\propto+k\,,\beta) = \frac{\Gamma_{\propto+\beta}}{\Gamma_\propto\, \Gamma_\beta}\,\frac{\Gamma_{\propto+k}\,\Gamma_\beta}{\Gamma_{k+\propto+\beta}}$$

$$= \frac{\Gamma_{(k+\propto)}\,\Gamma_{(\propto+\beta)}}{\Gamma_\propto\,\Gamma_{(k+\propto+\beta)}}$$

2. From the above formula, the mean and variance of the distribution can be deriving as follows.

Putting k=1 we obtain:

$$\mu_x = E(x) = \frac{\Gamma_{(\propto+1)}\,\Gamma_{(\propto+\beta)}}{\Gamma_\propto\,\Gamma_{(\propto+\beta+1)}} = \frac{\propto\;\Gamma_\propto\,\Gamma_{(\propto+\beta)}}{\Gamma_\propto\,(\propto+\beta)\Gamma_{(\propto+\beta)}} = \frac{\propto}{\propto+\beta}$$

Putting k=2 we obtain:

$$E(x^2) = \frac{\Gamma_{(\propto+2)}\,\Gamma_{(\propto+\beta)}}{\Gamma_\propto\,\Gamma_{(\propto+\beta+2)}} = \frac{\propto(\propto+1)\Gamma_\propto\Gamma_{(\propto+\beta)}}{\Gamma_\propto(\propto+\beta+1)(\propto+\beta)\Gamma_{(\propto+\beta)}} = \frac{\propto(\propto+1)}{(\propto+\beta+1)(\propto+\beta)}$$

$$var(x) = \sigma_x^2 = E(x^2) - \left(E(x)\right)^2 = \frac{\propto(\propto+1)}{(\propto+\beta+1)(\propto+\beta)} - \left(\frac{\propto}{\propto+\beta}\right)^2$$

$$= \frac{\propto(\propto+1)}{(\propto+\beta+1)(\propto+\beta)} - \frac{\propto^2}{(\propto+\beta)^2} = \frac{\propto(\propto+1)(\propto+\beta) - \propto^2(\propto+\beta+1)}{(\propto+\beta+1)(\propto+\beta)^2}$$

$$= \frac{\propto^3 + \propto^2\beta + \propto^2 + \propto\beta - \propto^3 - \propto^2\beta - \propto^2}{(\propto+\beta+1)(\propto+\beta)^2} = \frac{\propto\beta}{(\propto+\beta+1)(\propto+\beta)^2}$$

2.5 Normal distribution

A continuous r.v x is said to have normal distribution with parameters μ , σ^2 denoted as $X \sim N(\mu, \sigma^2)$ if the p. d. f of x is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \, e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty$$

Properties

1. The M. g. f of normal dist. is $M_x(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} \, f(x) dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{tx} \, e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let $y = \frac{x-\mu}{\sigma} \Rightarrow x = \sigma y + \mu \Rightarrow dx = \sigma \, dy$

$$M_y(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{t(\sigma y+\mu)} \, e^{\frac{-y^2}{2}} \sigma \, dy = \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\sigma y} \, e^{\frac{-y^2}{2}} \, dy$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(y^2-2t\sigma y)}{2}} \, dy$$

Using the complete square method by adding and subtracting $\left(\frac{2t\sigma}{2}\right)^2 = \sigma^2 t^2$

$$M_y(t) = \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(y^2-2t\sigma y+\sigma^2 t^2-\sigma^2 t^2)}{2}} \, dy = \frac{e^{t\mu}}{\sqrt{2\pi}} e^{\frac{\sigma^2 t^2}{2}} \int_{-\infty}^{\infty} e^{\frac{-(y^2-2t\sigma y+\sigma^2 t^2)}{2}} \, dy$$

$$= \frac{e^{t\mu + \frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(y-\sigma t)^2}{2}} \, dy$$

Let $z = y - \sigma t \Rightarrow y = z + \sigma t \Rightarrow dy = dz,$ and rewrite m.g.f as follows

$$M_z(t) = e^{t\mu + \frac{\sigma^2 t^2}{2}} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-z^2}{2}} dz \right) = e^{t\mu + \frac{\sigma^2 t^2}{2}}$$

p.d.f of N(0,1)

$$M_x(t) = e^{t\mu + \frac{\sigma^2 t^2}{2}}$$

2. The mean of the normal dist. is $M'_x(0) = E(x) = \mu$

$$M'_x(t) = (\mu + \sigma^2 t)\ e^{t\mu + \frac{\sigma^2 t^2}{2}}$$

$$M'_x(0) = (\mu + 0)e^0 = \mu$$

3. The variance of the distribution is $Var(x) = \sigma^2$

$$Var(x) = M''_x(0) - \left(M'_x(0)\right)^2$$

$$M''_x(t) = (\mu + \sigma^2 t)(\mu + \sigma^2 t)\ e^{t\mu + \frac{\sigma^2 t^2}{2}} + e^{t\mu + \frac{\sigma^2 t^2}{2}}\ \sigma^2$$

$$M''_x(0) = (\mu + 0)(\mu + 0)e^0 + e^0\ \sigma^2 = \mu^2 + \sigma^2$$

$$Var(x) = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$$

Definition: If the r.v $Z \sim N(0,1)$, then we say that Z distributed as standard normal distribution with p. d. f.

$$f(z) = \frac{1}{\sqrt{2\pi}}\ e^{\frac{-z^2}{2}}, -\infty < z < \infty$$

The mean, variance and moment generating function of the r.v Z are

$$M_Z(t) = e^{\frac{t^2}{2}}, \qquad \mu_z = 0, \qquad \sigma_z^2 = 1$$

<u>**Theorem (1):**</u> if the r.v $X \sim N(\mu, \sigma_x^2)$ then $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$

Proof: By using the transformation method we have

$$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\ e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty<x<\infty$$

Assuming that the space of X denoted by A and the space of Z denoted by B and are defined as follows

$$A=\{X;-\infty<X<\infty\}\ and\ B=\{Z;-\infty<Z<\infty\}$$

$$Z=u(X)=\frac{X-\mu}{\sigma}\ is\ (1-1)\ \text{transformation maps A onto B}$$

$$X=u^{-1}(Z)=\mu+\sigma Z\ \ is\ (1-1)\ \text{transformation maps B onto A}$$

$$|J|=\left|\frac{dx}{dz}\right|=\sigma\ \Rightarrow g(z)=f(u^{-1}(z))\cdot|J|$$

$$g(z)=\frac{1}{\sqrt{2\pi\sigma^2}}\ e^{\frac{-z^2}{2}}\cdot\ \sigma=\frac{1}{\sqrt{2\pi}}\ e^{\frac{-z^2}{2}}$$

$$\therefore Z=\frac{X-\mu}{\sigma}\sim N(0,1)$$

2.5.1 Calculating the probabilities

The probabilities concerning the r.v X which distributed as $N\,(\mu,\sigma^2)$ can be expressed in terms of probabilities concerning $(Z=\ X-\mu\sigma)$ which distributed as$N(0,1)$, however an integral like $\left(\int_{-\infty}^{k}\frac{1}{\sqrt{2\pi}}e^{\frac{-z^2}{2}}\,dz\right)$ cannot be evaluated. Instead we use tables which approximate the value of this integral for different values of k. In general, the following rules are important.

1. $Pr(Z<0)=Pr(Z>0)=0.5$
2. $Pr(Z<-Z_1)=1-Pr(Z<Z_1),\ Z_1>0$
3. $Pr(Z_1<Z<Z_2)=Pr(Z<Z_2)-P_r(Z<Z_1)=N(\,Z_2)-N(Z_1)$

Example: given that $X\sim N(2,25)$, find $P_r(0<X<10)$

Solution:

$$Pr(0 < X < 10) = Pr\left(\frac{0-2}{5} < Z < \frac{10-2}{5}\right) = Pr(-0.4 < Z < 1.6)$$
$$= Pr(Z < 1.6) - P_r(Z < -0.4) = Pr(Z < 1.6) - \big(1 - Pr(Z < 0.4)\big)$$
$$= N(1.6) - \big(1 - N(0.4)\big) = 0.945 - (1 - 0.655) = 0.6 \ \ (from\ table)$$

Theorem (2): if the r.v $X \sim N(\mu, \sigma_z^2)$ then $Y = \left(\frac{X-\mu}{\sigma}\right)^2 \sim X^2(1)$

Proof: By using the m. g. f. method

$$M_y(t) = E(e^{ty}) = E\left(e^{t\left(\frac{x-\mu}{\sigma}\right)^2}\right) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{t\left(\frac{x-\mu}{\sigma}\right)^2}\, e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Putting $z = \frac{x-\mu}{\sigma} \Rightarrow x = \sigma z + \mu \Rightarrow dx = \sigma\, dy$, then

$$M_y(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz^2}\, e^{\frac{-1}{2}z^2}\, \sigma\, dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz^2 - \frac{1}{2}z^2}\, dy$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{z^2\left(t-\frac{1}{2}\right)}\, dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-z^2}{2}(1-2t)}\, dy$$

Let $w = z\sqrt{1-2t} \Rightarrow z = \frac{1}{\sqrt{1-2t}} w \Rightarrow dz = \frac{1}{\sqrt{1-2t}} dw$

$$M_y(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-w^2}{2}}\, \frac{1}{\sqrt{1-2t}} dw = \frac{1}{\sqrt{1-2t}} \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{\frac{-w^2}{2}}\, dw\right) = (1-2t)^{-\frac{1}{2}}$$

This is the m. g. f. of chi square dist. with 1 degree of freedom

$$Y = \left(\frac{X-\mu}{\sigma}\right)^2 \sim X^2(1)$$

Theorem (3): If $X_i, i = 1,2, \dots, n$ are independent r.vs distributed as $N(0,1)$, then

$$\sum_{i=1}^{n} X_i^2 \sim X^2(n)$$

Proof: let $Y = \sum_{i=1}^{n} X_i^2$

$$M_y(t) = E(e^{ty}) = E\left(e^{t\sum_{i=1}^{n} X_i^2}\right) = E\left(e^{t\left(x_1^2+x_2^2+\cdots+x_n^2\right)}\right) = E\left(e^{tx_1^2+tx_2^2+\cdots+tx_n^2}\right)$$

$$= E\left(e^{tx_1^2} e^{tx_2^2} \cdots e^{tx_n^2}\right) = E\left(e^{tx_1^2}\right)E\left(e^{tx_2^2}\right)\cdots E\left(e^{tx_n^2}\right)$$

Since each of $X_1, X_2, \cdots, X_n \sim N(0,1)$ then each of $X_1^2, X_2^2, \cdots X_n^2 \sim X^2(1)$ (Theorem (2))

$$M_Y(t) = E\left(e^{tX_1^2}\right)E\left(e^{tX_2^2}\right)\cdots E\left(e^{tX_n^2}\right) = (1-2t)^{\frac{-1}{2}}(1-2t)^{\frac{-1}{2}}\cdots(1-2t)^{\frac{-1}{2}}$$

$$= \left((1-2t)^{-\frac{1}{2}}\right)^n = (1-2t)^{\frac{-n}{2}}$$

$$\therefore Y = \sum_{i=1}^{n} X_i^2 \sim X^2(n)$$

2.2 The student t distribution

Let W and V are stochastically independent random variables such that $W \sim N(0,1)$ and $V \sim X^2(r)$, then the random variable $T = \frac{W}{\sqrt{V/r}}$ has t distribution $T \sim t(n-1)$ with p.d.f

$$g(t) = \frac{\Gamma_{(r+1)/2}}{\sqrt{\pi r}\,\Gamma_{r/2}} \cdot \left(1+\frac{t^2}{r}\right)^{\frac{-(r+1)}{2}}, -\infty < t < \infty$$

Now we need to prove that the joint p.d.f of W and V is:

$$\emptyset(w,v) = \frac{1}{\sqrt{2\pi}} e^{\frac{-w^2}{2}} \cdot \frac{1}{\Gamma_{r/2}\, 2^{r/2}} \cdot v^{\left(\frac{r}{2}-1\right)} \cdot e^{\frac{-v}{2}}, -\infty < w < \infty \; and \; 0 < v < \infty$$

Let $t = \frac{w}{\sqrt{v/r}}$ and $u = v$.

Now, define (1-1) transformation mapping from the space

$$\{(w,v), -\infty < w < \infty \ and \ 0 < v < \infty\}$$

Onto the space

$$\{(t,u), -\infty < t < \infty \ and \ 0 < u < \infty\}$$

$$\therefore w = t\frac{\sqrt{u}}{\sqrt{r}} \ and \ v = u$$

$$J = \begin{vmatrix} {dw}/{dt} & {dw}/{du} \\ {dv}/{dt} & {dv}/{du} \end{vmatrix} = \begin{vmatrix} {\sqrt{u}}/{\sqrt{r}} & {1}/{2\sqrt{ur}} \\ 0 & 1 \end{vmatrix} = \frac{\sqrt{u}}{\sqrt{r}}$$

$$g(t,u) = \emptyset\left(\frac{\sqrt{u}}{\sqrt{r}}, u\right) \cdot |J| = \frac{1}{\sqrt{2\pi}} e^{\frac{-ut^2}{2r}} \cdot \frac{1}{\Gamma {r}/{2} \ 2^{r/2}} \cdot u^{\left(\frac{r}{2}-1\right)} \cdot e^{\frac{-u}{2}} \cdot \frac{\sqrt{u}}{\sqrt{r}}$$

$$= \frac{1}{\sqrt{2\pi} \ \Gamma {r}/{2} \ 2^{r/2}} \cdot u^{\left(\frac{r}{2}-1\right)} \cdot e^{\frac{-u}{2}\left(1+\frac{t^2}{r}\right)} \cdot \frac{\sqrt{u}}{\sqrt{r}}$$

$$g(t) = \int_0^\infty g(t,u) \, du = \int_0^\infty \frac{1}{\sqrt{2\pi} \ \Gamma {r}/{2} \ 2^{r/2}} \cdot u^{\left(\frac{r}{2}-1\right)} \cdot e^{\frac{-u}{2}\left(1+\frac{t^2}{r}\right)} \cdot \frac{\sqrt{u}}{\sqrt{r}} \, du$$

Now, assume $z = \frac{u}{2}\left(1 + \frac{t^2}{r}\right)$, then we have $u = \frac{2z}{1+\frac{t^2}{r}}$ and $du = \frac{2}{\left(1+\frac{t^2}{r}\right)} \ dz$

$$g(t) = \int_0^\infty \frac{1}{\sqrt{2\pi}\,\Gamma_{r/2}\,2^{r/2}\,r^{1/2}} \cdot \left(\frac{2z}{1+\frac{t^2}{r}}\right)^{\left(\frac{r}{2}-1\right)} \cdot e^{-z} \cdot dz$$

$$= \frac{1}{\sqrt{2\pi}\,\Gamma_{r/2}\,2^{r/2}\,r^{1/2}} \cdot \frac{2^{\left(\frac{r}{2}-1\right)}}{\left(1+\frac{t^2}{r}\right)^{\left(\frac{r}{2}-1\right)}}$$

$$\cdot \frac{\sqrt{2}}{\sqrt{1+\frac{t^2}{r}}} \frac{2}{\left(1+\frac{t^2}{r}\right)} \int_0^\infty e^{-z} \cdot z^{\left(\frac{r}{2}-1\right)} \cdot z^{1/2}\,dz$$

$$= \frac{1}{\sqrt{\pi}r\,\Gamma_{r/2}\left(1+\frac{t^2}{r}\right)^{\left(\frac{r+1}{2}\right)}} \int_0^\infty z^{\left(\frac{r+1}{2}-1\right)} \cdot e^{-z}\,dz$$

$$= \frac{\Gamma_{\frac{r+1}{2}}}{\sqrt{\pi r}\,\Gamma_{r/2}\left(1+\frac{t^2}{r}\right)^{\left(\frac{r+1}{2}\right)}}, -\infty < t < \infty$$

Properties

1. The mean of students t distribution is:

$$\mu_t = E(t) = E\left(\frac{w}{\sqrt{v/r}}\right) = E(w) \cdot E\left(\frac{1}{\sqrt{v/r}}\right)$$

Since $W \sim N(0,1)$, then $E(w) = 0$, so we get:

$$\mu_t = E(t) = 0$$

2. the variance of t distribution is derived as follows:

$$var(t) = E(t^2) - \left(E(t)\right)^2 = E(t^2) - 0$$

$$\therefore var(t) = E(t^2)$$

$$var(t) = E\left(\left(\frac{w}{\sqrt{v/r}}\right)^2\right) = E\left(\frac{w^2}{v/r}\right) = E(w^2) \cdot E\left(\frac{1}{v/r}\right)$$

Since $W \sim N(0,1)$, then $E(w^2) = var(w) + \left(E(w)\right)^2 = 1 + 0 = 1$, so we get:

$$var(t) = E\left(\frac{1}{v/r}\right) = E\left(\frac{r}{v}\right) = rE\left(\frac{1}{v}\right)$$

Since $V \sim X^2(r)$, then

$$E\left(\frac{1}{v}\right) = \int_0^\infty \frac{1}{v} \cdot \frac{1}{\Gamma r/_2 \, 2^{r/_2}} \cdot v^{\left(\frac{r}{2}-1\right)} \cdot e^{\frac{-v}{2}} \, dv = \frac{1}{\Gamma r/_2 \, 2^{r/_2}} \int_0^\infty v^{\left(\frac{r}{2}-2\right)} \cdot e^{\frac{-v}{2}} \, dv$$

Let $z = \frac{v}{2}$, then $v = 2z$ and $dv = 2dz$, so we can get:

$$E\left(\frac{1}{v}\right) = \frac{1}{\Gamma r/_2 \, 2^{r/_2}} \int_0^\infty (2z)^{\left(\frac{r}{2}-2\right)} \cdot e^{-z} \, 2dz = \frac{2^{\left(\frac{r}{2}-1\right)}}{\Gamma r/_2 \, 2^{r/_2}} \int_0^\infty z^{\left(\frac{r}{2}-1\right)-1} \cdot e^{-z} \, dz$$

$$= \frac{1}{2\,\Gamma r/_2} \Gamma_{\left(\frac{r}{2}-1\right)} = \frac{1}{2\left(\frac{r}{2}-1\right)\Gamma_{\left(\frac{r}{2}-1\right)}} \Gamma_{\left(\frac{r}{2}-1\right)} = \frac{1}{r-2}$$

$$\therefore var(t) = rE\left(\frac{1}{v}\right) = \frac{r}{r-2} \text{ , and } r > 2$$

2.7 <u>The F distribution</u>

Let $X_1 \sim X^2(n_1)$ is independent from $X_2 \sim X^2(n_2)$ then the ration $\left(f = \frac{x_1/_{n_1}}{x_2/_{n_2}}\right)$ is said to have an F distribution with (n_1, n_2) degrees of freedom denoted as $f \sim f(n_1, n_2)$ and the p. d. f of the distribution could be written as follows:

$$g(f) = \frac{\sqrt{\frac{n_1 + n_2}{2}}}{\Gamma_{\frac{n_1}{2}} \Gamma_{\frac{n_2}{2}}} \cdot \left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} \cdot \frac{f^{\frac{n_1}{2}-1}}{\left(1 + \frac{n_1}{n_2} f\right)^{\frac{n_1+n_2}{2}}} \text{ , } f > 0$$

Properties

1. The mean of F distribution is:

$$\mu_f = E(f) = E\left(\frac{x_1/n_1}{x_2/n_2}\right) = E\left(\frac{n_2}{n_1}\cdot\frac{x_1}{x_2}\right) = \frac{n_2}{n_1}\cdot E\left(\frac{x_1}{x_2}\right) = \frac{n_2}{n_1}\cdot E(x_1)\cdot E\left(\frac{1}{x_2}\right)$$

Since $X_1 \sim X^2(n_1)$, then $E(x_1) = n_1$

$$\therefore \mu_f = \frac{n_2}{n_1}\cdot n_1 \cdot E\left(\frac{1}{x_2}\right) = n_2 \cdot E\left(\frac{1}{x_2}\right)$$

Now, we need to find $E\left(\frac{1}{x_2}\right)$, and since $X_2 \sim X^2(n_2)$, so we have

$$E\left(\frac{1}{x_2}\right) = \int_0^\infty \frac{1}{x_2}\cdot\frac{1}{\Gamma n_2/_2\ 2^{n_2/2}} \cdot x_2^{\left(\frac{n_2}{2}-1\right)} \cdot e^{\frac{-x_2}{2}}\, dx_2$$

$$= \frac{1}{\Gamma n_2/_2\ 2^{n_2/2}} \cdot \int_0^\infty x_2^{\left(\frac{n_2}{2}-1\right)-1} \cdot e^{\frac{-x_2}{2}}\, dx_2$$

So, we assume that $z = \frac{x_2}{2}, then\ x_2 = 2z\ and\ dx_2 = 2dz$, that will be given us

$$E\left(\frac{1}{x_2}\right) = \frac{1}{\Gamma n_2/_2\ 2^{n_2/2}} \cdot \int_0^\infty (2z)^{\left(\frac{n_2}{2}-1\right)-1} \cdot e^{-z}\, 2dz$$

$$= \frac{2^{\left(\frac{n_2}{2}-1\right)}}{\Gamma n_2/_2\ 2^{n_2/2}} \cdot \int_0^\infty z^{\left(\frac{n_2}{2}-1\right)-1} \cdot e^{-z}\, dz = \frac{1}{2\,\Gamma n_2/_2}\ \Gamma_{\left(\frac{n_2}{2}-1\right)}$$

$$= \frac{1}{2\left(\frac{n_2}{2}-1\right)\Gamma_{\left(\frac{n_2}{2}-1\right)}}\ \Gamma_{\left(\frac{n_2}{2}-1\right)} = \frac{1}{n_2-2}$$

$$\therefore \mu_f = n_2 \cdot E\left(\frac{1}{x_2}\right) = \frac{n_2}{n_2-2}, and\ n_2 > 2$$

2. The variance of f distribution is given as follows

$$var(f) = \frac{2n_2^2(n_1+n_2-2)}{n_1(n_2-2)^2(n_2-4)}, and\ n_2 > 4$$

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2.8 The Exponential distribution

It is one of the most important distributions to give a useful description of the observation of change and it is called it is called (life time).

The continuous, r.v X is said to have an exponential distribution with parameter λ denoted as $X \sim Exp\ (\lambda)$ if its probability density function (p.d.f) is given as follows:

$$f(x) = \lambda e^{-\lambda x}, \lambda > 0, x \geq 0$$

And the cumulative distribution function is given as follows:

$$F(x) = 1 - e^{-\lambda x}$$

Now, we need to show that the $f(x)$ is actually a probability density function (p.d.f). So we need to satisfy the following two conditions:

1. $f(x) > 0$, which is true since $\lambda > 0, x \geq 0$.

2. $\int_0^\infty f(x)dx = \int_0^\infty \lambda e^{-\lambda x} dx = -e^{-\lambda x} \Big|_0^\infty = 1$

Properties

1. The mean of Exponential distribution is given as follows:

$$\mu_x = E(x) = \int_0^\infty x \cdot f(x)\, dx = \int_0^\infty x \cdot \lambda e^{-\lambda x}\, dx$$

$$\text{let } u = x, \text{then } du = dx$$

$$\text{let } dv = \lambda e^{-\lambda x}\, dx, \text{so} \int dv = \int \lambda e^{-\lambda x}\, dx, \text{then } v = -e^{-\lambda x}$$

$$\int_0^\infty u \cdot dv = u \cdot v \Big|_0^\infty + \int_0^\infty v \cdot du = -xe^{-\lambda x} \Big|_0^\infty + \int_0^\infty e^{-\lambda x}\, dx = \frac{1}{\lambda}$$

$$\therefore \mu_x = \frac{1}{\lambda}$$

2. The variance of Exponential distribution is given as follows:

$$var(x) = \frac{1}{\lambda^2}, (H.W)$$

3. The moment generated (m.g.f) of Exponential distribution is given as follows:

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} \cdot f(x)\, dx = \int_0^\infty e^{tx} \cdot \lambda e^{-\lambda x}\, dx = \lambda \int_0^\infty e^{-x(\lambda - t)}\, dx$$

$$= -\lambda \frac{e^{-x(\lambda-t)}}{\lambda - t} \Big|_0^\infty = \frac{\lambda}{\lambda - t} = \left(1 - \frac{t}{\lambda}\right)^{-1}$$

Lemma: let $x_i, i = 1,2,\cdots,n$ has exponential distribution with parameter (λ), then $(\sum_{i=1}^n x_i)$ has gamma distribution $G\left(n, \frac{1}{\lambda}\right)$.

Proof: let $y = \sum_{i=1}^n x_i$ and since $x_i \sim Exp(\lambda), i = 1,2,\cdots,n$, then,

$$M_{x_i}(t) = \left(1 - \frac{t}{\lambda}\right)^{-1}, i = 1,2,\cdots,n$$

$$M_y(t) = E(e^{ty}) = E\left(e^{t\sum_{i=1}^n x_i}\right) = E\left(e^{t(x_1+x_2+\cdots+x_n)}\right) = E(e^{tx_1+tx_2+\cdots+tx_n})$$

$$= E(e^{tx_1} e^{tx_2} \cdots e^{tx_n}) = E(e^{tx_1})E(e^{tx_2}) \cdots E(e^{tx_n})$$

$$= M_{x_1}(t)\, M_{x_2}(t) \cdots M_{x_n}(t) = \left(1 - \frac{t}{\lambda}\right)^{-1} \left(1 - \frac{t}{\lambda}\right)^{-1} \cdots \left(1 - \frac{t}{\lambda}\right)^{-1}$$

$$= \left(1 - \frac{t}{\lambda}\right)^{-\sum_{i=1}^n 1} = \left(1 - \frac{t}{\lambda}\right)^{-n} = \left(1 - \frac{1}{\lambda}\, t\right)^{-n}$$

$$\therefore y = \sum_{i=1}^n x_i \sim G\left(n, \frac{1}{\lambda}\right)$$

Example: use the moment generated (m.g.f **)** to find μ_x and $var(x)$ of Exponential distribution.

Solution:

$$M_x(t) = \left(1 - \frac{t}{\lambda}\right)^{-1}$$

$$M_x'(t) = \frac{1}{\lambda}\left(1 - \frac{t}{\lambda}\right)^{-2}$$

$$\mu_x = E(x) = M_x'(0) = \frac{1}{\lambda}\left(1 - \frac{0}{\lambda}\right)^{-2} = \frac{1}{\lambda}$$

$$M_x''(t) = \frac{2}{\lambda^2}\left(1 - \frac{t}{\lambda}\right)^{-3}$$

$$E(x^2) = M_x''(0) = \frac{2}{\lambda^2}\left(1 - \frac{0}{\lambda}\right)^{-3} = \frac{2}{\lambda^2}$$

$$var(x) = E(x^2) - \big(E(x)\big)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Example: If the time period for ending customer service in the bank follows an Exponential distribution with a mean of 2 minutes, find:

1. The probability density function expressing the time period for ending customer's service.

2. What is the probability of ending customer service in less than one minute?

Solution:

1.

$$\text{since } E(x) = 2, \text{then } \frac{1}{\lambda} = 2$$

$$\therefore \lambda = \frac{1}{2}$$

$$f(x) = \lambda e^{-\lambda x} = \frac{1}{2} e^{\frac{-1}{2}}$$

2. .

$$Pr(x \leq 1) = F(1) = 1 - e^{-\frac{1}{2}}$$

Example: If the life of a light bulb follows an Exponential distribution with an of 1000 hours, fined the probability that one of the lamps will work for more than 2000 hours and that one of the lamps will burn out within 100 hours .

Solution:

$$E(x) = 1000, \text{then } \frac{1}{\lambda} = 1000$$

$$\therefore \lambda = \frac{1}{1000}$$

$$P\,r(x > 2000) = 1 - Pr(x \leq 2000) = 1 - F(2000) = 1 - \left(1 - e^{\frac{-1}{1000}2000}\right) = -e^{-2}$$

$$Pr(x \leq 100) = F(100) = 1 - e^{\frac{-100}{1000}} = 1 - e^{-0.1}$$

3. Distribution of sample mean and sample variance

3.1 The distribution of sample mean $(\overline{X})$

Let $x_i, i = 1,2, \cdots, n$ be a random samples from $N(\mu, \sigma^2)$, the distribution of $(\overline{X})$ is derived by using the m.g.f as follows:

$$M_{\overline{X}}(t)=E\left(e^{t\overline{X}}\right)=E\left(e^{t\frac{\sum_{i=1}^{n}x_i}{n}}\right)=E\left(e^{\frac{t}{n}(x_1,x_2,\cdots,x_n)}\right)=E\left(e^{\frac{t}{n}x_1}\cdot e^{\frac{t}{n}x}\cdots e^{\frac{t}{n}x_n}\right)$$

$$=E\left(e^{\frac{t}{n}x_1}\right)\cdot E\left(e^{\frac{t}{n}x_2}\right)\cdots E\left(e^{\frac{t}{n}x_n}\right)=M_{x_1}\left(\frac{t}{n}\right)\cdot M_{x_2}\left(\frac{t}{n}\right)\cdots M_{x_n}\left(\frac{t}{n}\right)$$

$$=e^{\frac{t}{n}\mu+\frac{\sigma^2\left(\frac{t}{n}\right)^2}{2}}\cdot e^{\frac{t}{n}\mu+\frac{\sigma^2\left(\frac{t}{n}\right)^2}{2}}\cdots e^{\frac{t}{n}\mu+\frac{\sigma^2\left(\frac{t}{n}\right)^2}{2}}=\left(e^{\frac{t}{n}\mu+\frac{\sigma^2\left(\frac{t}{n}\right)^2}{2}}\right)^n$$

$$=\left(e^{\frac{t}{n}\mu+\frac{\sigma^2\frac{t^2}{n^2}}{2}}\right)^n=e^{\frac{nt}{n}\mu+\frac{\frac{n\sigma^2}{n^2}t^2}{2}}=e^{t\mu+\frac{\frac{\sigma^2}{n}t^2}{2}}$$

Since, $x_i\sim N(\mu,\sigma^2),\ i=1,2,\cdots,n$ then $M_{x_i}(t)=e^{t\mu+\frac{\sigma^2t^2}{2}}$, and that will gives us:

$M_{\overline{X}}(t)=M_x\left(\frac{t}{n}\right)=e^{t\mu+\frac{\frac{\sigma^2}{n}t^2}{2}}$.

This is similar to m.g.f of normal distribution with mean (μ) and variance $\left(\frac{\sigma^2}{n}\right)$.

$$\therefore \overline{X}\sim N\left(\mu,\frac{\sigma^2}{n}\right)$$

$$f(\overline{x})=\frac{1}{\sqrt{2\pi\sigma^2}}\ e^{\frac{-1}{2}\frac{(\overline{x}-\mu)^2}{\sigma^2/n}},-\infty<\overline{x}<\infty$$

Properties

According to theorems (1), (2), and (3), $\overline{X}\sim N\left(\mu,\frac{\sigma^2}{n}\right)$ then we have:

1. $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\sim N(0,1)$
2. $\left(\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right)^2\sim X^2(1)$

3. $\sum_{i=1}^{k}\left(\frac{\bar{X}_i-\mu}{\sigma/\sqrt{n}}\right)^2 \sim X^2(k)$

3.2 Distribution of sample variance (S^2)

The sample variance (S^2) is define as $S^2 = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \bar{X}\right)^2$, consider the following summation:

$$\sum_{i=1}^{n}(x_i-\mu)^2 = \sum_{i=1}^{n}\left(\left(x_i-\bar{X}\right)+\left(\bar{X}-\mu\right)\right)^2$$

$$= \sum_{i=1}^{n}\left(\left(x_i-\bar{X}\right)^2 + 2\left(x_i-\bar{X}\right)\left(\bar{X}-\mu\right)+\left(\bar{X}-\mu\right)^2\right)$$

$$= \sum_{i=1}^{n}\left(x_i-\bar{X}\right)^2 + \sum_{i=1}^{n}\left(2\left(x_i-\bar{X}\right)\left(\bar{X}-\mu\right)\right)+\sum_{i=1}^{n}\left(\bar{X}-\mu\right)^2$$

$$= \sum_{i=1}^{n}\left(x_i-\bar{X}\right)^2 + 2\left(\bar{X}-\mu\right)\sum_{i=1}^{n}\left(x_i-\bar{X}\right)+n\left(\bar{X}-\mu\right)^2$$

$$= \sum_{i=1}^{n}\left(x_i-\bar{X}\right)^2 + 2\left(\bar{X}-\mu\right)\left(\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}\bar{X}\right)+n\left(\bar{X}-\mu\right)^2$$

$$= \sum_{i=1}^{n}\left(x_i-\bar{X}\right)^2 + 2\left(\bar{X}-\mu\right)\left(n\bar{X}-n\bar{X}\right)+n\left(\bar{X}-\mu\right)^2$$

$$= \sum_{i=1}^{n}\left(x_i-\bar{X}\right)^2 + n\left(\bar{X}-\mu\right)^2 = n\cdot S^2 + n\left(\bar{X}-\mu\right)^2$$

Now, dividing both sides by (σ^2), and so we get:

$$\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{\sigma^2} = \frac{n\cdot S^2}{\sigma^2} + \frac{n\left(\bar{X}-\mu\right)^2}{\sigma^2}$$

$$\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2=\frac{n\cdot S^2}{\sigma^2}+\left(\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right)^2$$

$$\therefore \frac{n\cdot S^2}{\sigma^2}=\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2-\left(\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right)^2 \cdots\cdots(*)$$

$$\text{since } \sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2 \sim X^2(n), (\text{by theorem } (3))$$

$$\text{since } \left(\frac{x_i-\mu}{\sigma/\sqrt{n}}\right)^2 \sim X^2(1), (\text{by theorem } (2))$$

According to eq. $(*)$, that will gives us the following:

$$\frac{n\cdot S^2}{\sigma^2}\sim X^2(n-1)$$

Properties

1. The mean

$$E\left(\frac{n\cdot S^2}{\sigma^2}\right)=n-1$$

$$\frac{n}{\sigma^2}E(S^2)=n-1$$

$$\therefore E(S^2)=\frac{\sigma^2(n-1)}{n}$$

2. The variance

$$var\left(\frac{n\cdot S^2}{\sigma^2}\right)=2(n-1)$$

$$\frac{n^2}{\sigma^4}\, var(S^2) = 2(n-1)$$

$$\therefore var(S^2) = \frac{2\sigma^4(n-1)}{n^2}$$

Example: if $(\overline{X})$ is the mean of r. v. of size (n) from $N(\mu, 100)$, find the value of (n) if you know that $Pr(\mu - 5 < \overline{X} < \mu + 5) = 0.954$?

Solution: let $x_1 = \mu - 5 \; and \; x_2 = \mu + 5$

$$Pr(x_1 < \overline{X} < x_2) = 0.954$$

$$Pr\left(\frac{x_1 - \mu}{\sigma/\sqrt{n}} < \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} < \frac{x_2 - \mu}{\sigma/\sqrt{n}}\right) = 0.954$$

$$Pr\left(\frac{-5}{10/\sqrt{n}} < Z < \frac{5}{10/\sqrt{n}}\right) = 0.954$$

$$Pr\left(\frac{-1}{2}\sqrt{n} < Z < \frac{1}{2}\sqrt{n}\right) = 0.954$$

$$Pr\left(Z < \frac{1}{2}\sqrt{n}\right) - Pr\left(Z < \frac{-1}{2}\sqrt{n}\right) = 0.954$$

$$Pr\left(Z < \frac{1}{2}\sqrt{n}\right) - \left(1 - Pr\left(Z < \frac{1}{2}\sqrt{n}\right)\right) = 0.954$$

$$2 \cdot Pr\left(Z < \frac{1}{2}\sqrt{n}\right) - 1 = 0.954$$

$$Pr\left(Z < \frac{1}{2}\sqrt{n}\right) = \frac{0.954 + 1}{2}$$

$$Pr\left(Z < \frac{1}{2}\sqrt{n}\right) = 0.977$$

$$\frac{1}{2}\sqrt{n} = 2, (\text{approximately })\text{from } N(0,1) \text{ table}$$

$$\sqrt{n} = 4 \Rightarrow n = 16$$

Example: let (S^2) be the variance of a r. v. samples of size (6) from $N(\mu, 12)$, find $Pr(2.30 < S^2 < 22.2)$?

Solution: since $\left(\frac{n \cdot S^2}{\sigma^2} \sim X^2(n-1)\right)$, the we have:

$$\frac{6 \cdot S^2}{12} = \sim X^2(6-1) \;\Rightarrow\; \frac{S^2}{2} \sim X^2(5)$$

$$Pr\left(\frac{2.30}{2} < \frac{S^2}{2} < \frac{22.2}{2}\right) = Pr\left(1.1 < \frac{S^2}{2} < 11.1\right) = Pr\left(\frac{S^2}{2} < 11.1\right) - Pr\left(\frac{S^2}{2} < 1.1\right)$$

$$= 0.950 - 0.050 = 0.9 \text{ (from } X^2 \text{ table)}$$

Example: let $x_1, x_2, \cdots, x_n$ be a random samples from $N(\mu, \sigma^2)$**,** let $(\overline{X})$ and (S^2) denoted the mean and the variance of the sample where $(\overline{X})$ and (S^2) are independent such that $E(S^2) = \sigma^2$, show that $t = \frac{\sqrt{n}(\overline{X}-\mu)}{S} \sim t(n-1)$? $(n \to \infty)$ تتوزع تقريبا عندما

Solution: since $x_i \sim N(\mu, \sigma^2), i = 1,2, \cdots, n$, then $\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

$$\text{let } Z_1 = \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \;\Rightarrow\; Z_1 \sim N(0,1)$$

$$\text{let } Z_2 = \frac{S^2(n-1)}{\sigma^2} \;\Rightarrow\; Z_2 \sim X^2(n-1)$$

$$\frac{Z_1}{\sqrt{\frac{Z_2}{n-1}}} \sim t(n-1)$$

$$\frac{Z_1}{\sqrt{\frac{Z_2}{n-1}}} = \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{\frac{S^2(n-1)}{\sigma^2}}{n-1}}} \sim t(n-1) \Rightarrow \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{S^2(n-1)}{\sigma^2(n-1)}}} \sim t(n-1)$$

$$\Rightarrow \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{S^2}{\sigma^2}}} \sim t(n-1) \Rightarrow \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \cdot \frac{\sigma}{S} \sim t(n-1)$$

$$\therefore t = \frac{\sqrt{n}\left(\bar{X}-\mu\right)}{S} \sim t(n-1)$$

4. Distribution of Order Statistics

Let $x_1, x_2, \cdots, x_n$ be random samples from a distribution of continuous type that having p.d.f $f(x)$ which is positive provided $(a < x < b)$ let (y_1) be the smallest of these (x_i), (y_2) the next of (x_i) in order of magnitude ,…, and (y_n) the largest of (x_i), that is $(y_1 < y_2 < \cdots < y_n)$ represent $x_1, x_2, \ldots, x_n$ when the latter are arranged in ascending order of magnitude , then $(y_i, i = 1,2, \cdots, n)$ is called the (i^{th}) order statistic of the r.s $(x_1, x_2, \ldots, x_n)$, it can be shown that the joint p.d.f of $(y_1, y_2, \cdots, y_n)$ is:

$$g(y_1, y_2, \cdots, y_n) = n!\, f(y_1) \cdot f(y_2) \cdot \cdots \cdot f(y_n) \tag{1}$$

The marginal distribution function of (y_k) for $(i = 1,2, \cdots, n)$ is:

$$g(y_k) = \frac{n!}{(k-1)!\,(n-k)!} \left(F(y_k)\right)^{k-1} \cdot \left(1 - F(y_k)\right)^{n-k} \cdot f(y_k) \tag{2}$$

Where $F(\cdot)$ means the distribution function and $F(x) = Pr(X \leq x)$.

The joint p.d.f for any two order statistics $\left(y_i, y_j\right)$ for $(i < j)$ is:

$$g(y_i, y_j) = \frac{n!}{(i-1)!\ (j-i-1)!\,(n-j)!} \left(F(y_i)\right)^{i-1} \cdot \left(F(y_j) - F(y_i)\right)^{j-i-1} \cdot \left(1 - F(y_j)\right)^{n-j} \cdot f(y_i) f(y_j) \qquad (3)$$

Example: let $y_1 < y_2 < y_3 < y_4$ denote the order statistics on of a random samples of size (4) from a distribution having the following p.d.f :

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & O.W \end{cases}$$

Find the p.d.f of (y_3) then find $Pr(\frac{1}{2} < y_3)$?

Solution: Applying formula (2) with $(n = 4)$ and $(k = 3)$.

$$g(y_k) = \frac{n!}{(k-1)!\,(n-k)!} \left(F(y_k)\right)^{k-1} \cdot \left(1 - F(y_k)\right)^{n-k} \cdot f(y_k)$$

$$g(y_3) = \frac{4!}{2!\,1!} \left(F(y_3)\right)^{2} \cdot \left(1 - F(y_3)\right)^{1} \cdot f(y_3)$$

$$F(y_3) = \int_0^{y_3} f(u)du = \int_0^{y_3} 2u\, du = u^2 \Big|_0^{y_3} = y_3^2$$

$$g(y_3) = 12\,(y_3^2)^2 \cdot (1 - y_3^2)^1 \cdot 2y_3 = 24 y_3^5\,(1 - y_3^2)$$

$$\therefore g(y_3) = 24 y_3^5\,(1 - y_3^2), 0 < y_3 < 1$$

$$Pr\left(\frac{1}{2} < y_3\right) = Pr\left(y_3 > \frac{1}{2}\right) = \int_{1/2}^{1} g(y_3)dy_3 = \int_{1/2}^{1} 24 y_3^5\,(1 - y_3^2)dy_3$$

$$= 24 \int_{1/2}^{1} \left(y_3^5 - y_3^7\right)dy_3 = 24\left(\frac{y_3^6}{6} - \frac{y_3^8}{8}\right)\Big|_{1/2}^{1} = ?$$

4.1 The Median Distribution of Order Statistic

If X is ar.v with p.d.f $f(x)$, then the value of the median is the value of x that satisfies the equation $F(x) = \frac{1}{2}$

When the observations are arranged in ascending order of magnitude, then:

1. If n is odd, the median is the observation of orders $\left(\frac{n+1}{2}\right)$.
2. If n is even, the median is the average observation of order $\left(\frac{n}{2}\right)$, $\left(\frac{n+1}{2}\right)$.

Example: let x_1, x_2, x_3 be ar.s from the dist. $f(x) = e^{-x},\ x > 0$

i) Find the p.d.f of the smallest value of the samples?
ii) Find the joint p.d.f of the largest and smallest value of the samples?
iii) Find the p.d.f of the median and the value of the median?

Solution: let y_1 is the smallest value of the sample followed by y_2 and y_3 with $n = 3$.

i)
$$F(y_3) = \int_0^{y_1} f(u)du = \int_0^{y_1} e^{-u}\, du = -e^{-u} \Big|_0^{y_1} = -e^{-y_1} + e^0 = 1 - e^{-y_1}$$

Applying formula (2) with $k = 1$ and $n = 3$

$$g(y_k) = \frac{n!}{(k-1)!\,(n-k)!} \left(F(y_k)\right)^{k-1} \cdot \left(1 - F(y_k)\right)^{n-k} \cdot f(y_k)$$

$$g(y_1) = \frac{3!}{0!\,2!} \left(F(y_1)\right)^0 \cdot \left(1 - F(y_1)\right)^2 \cdot f(y_1) = 3(1 - 1 + e^{-y_1})^2 \cdot e^{-y_1}$$
$$= 3e^{-3y_1}$$

ii) Applying formula (3) with $(i = 1), (j = 3),$ and $(n = 3)$.

$$g(y_i, y_j) = \frac{n!}{(i-1)!\ (j-i-1)!\,(n-j)!}\left(F(y_i)\right)^{i-1}$$
$$\cdot\left(F(y_j) - F(y_i)\right)^{j-i-1}\cdot\left(1 - F(y_j)\right)^{n-j}\cdot f(y_i)f(y_j)$$

$$g(y_1, y_3) = \frac{3!}{0!\ 1!\ 0!}\left(F(y_1)\right)^0\cdot\left(F(y_3) - F(y_1)\right)^1\cdot\left(1 - F(y_3)\right)^0\cdot f(y_1)f(y_3)$$
$$= 6\left((1 - e^{-y_3}) - (1 - e^{-y_1})\right)\cdot e^{-y_1}\cdot e^{-y_3}$$
$$= 6(e^{-y_3} + e^{-y_1})\cdot e^{-(y_1+y_3)}, 0 < y_1 < y_3 < \infty$$

iii) Since $n = 3$ is odd, the median is the observation of orders $\frac{n+1}{2} = \frac{4}{2} = 2$ which is y_2.

$$g(y_k) = \frac{n!}{(k-1)!\,(n-k)!}\left(F(y_k)\right)^{k-1}\cdot\left(1 - F(y_k)\right)^{n-k}\cdot f(y_k)$$

$$g(y_1) = \frac{3!}{1!\,1!}(1 - e^{-y_2})^1\cdot\left(1 - (1 - e^{-y_2})\right)^1\cdot e^{-y_2} = 6(1 - e^{-y_2})\cdot e^{-y_2}\cdot e^{-y_2}$$
$$= 6e^{-2y_2}(1 - e^{-y_2}), 0 < y_2 < \infty$$

To find the value of (y_2)

$$F(y_2) = \frac{1}{2} = 1 - e^{-y_2}$$
$$e^{-y_2} = \frac{1}{2} \Rightarrow ln(e^{-y_2}) = ln\left(\frac{1}{2}\right)$$
$$-y_2 = ln\left(\frac{1}{2}\right) \Rightarrow -y_2 = -ln(2)$$
$$\therefore y_2 = ln(2)$$

4.2 <u>The Range</u>

The range of the sample is the difference between the largest and smallest value of the sample that is $R = y_n - y_1$.

Example: let x_1, x_2, x_3 be a random sample from $\beta(2,1)$, and let $y_1 < y_2 < y_3$ be the order statistics of the sample, find the following:

i) The probability distribution of R?
ii) The mean and variance of the distribution?

Solution: $x_i \sim \beta(2,1) \Rightarrow x_i \sim \beta(\propto, \beta) \Rightarrow \propto = 1 \;\&\; \beta = 2$

$$f(x) = \frac{\Gamma_{\propto+\beta}}{\Gamma_{\propto}\,\Gamma_{\beta}}\, x^{\propto-1}\,(1-x)^{\beta-1}, \qquad 0 < x < 1$$

$$f(x) = \frac{\Gamma_3}{\Gamma_2\,\Gamma_1}\, x^1\,(1-x)^0 = 2x$$

$$F(x) = \int_0^x f(u)du = \int_0^x 2u\,du = u^2 \Big|_0^x = x^2$$

Applying formula (3) we get: $n = 3, i = 1,$ and $j = 3$

$$g(y_1, y_3) = \frac{3!}{0!\;1!\;0!}\left(F(y_1)\right)^0 \cdot \left(F(y_3) - F(y_1)\right)^1 \cdot \left(1 - F(y_3)\right)^0 \cdot f(y_1)f(y_3)$$

$$= 6({y_1}^2 - {y_3}^2) \cdot 2y_1 \cdot 2y_3 = 24y_1y_3({y_3}^2 - {y_1}^2), \qquad 0 < y_1 < y_3 < 1$$

$$\left.\begin{matrix} R = y_3 - y_1 = u_1(y_1, y_3) \\ Z = y_3 = u_2(y_1, y_3) \end{matrix}\right\} \quad (1-1) \text{ transformation from space of } y_1, y_3 \text{ to space of } R, Z$$

$$\left.\begin{matrix} y_1 = Z - R = {u_1}^{-1}(R, Z) \\ y_3 = Z = {u_2}^{-1}(R, Z) \end{matrix}\right\} \quad (1-1) \text{transformation from space of } R, Z \text{ to space of } y_1, y_3$$

$$J = \begin{vmatrix} {dy_1}/{dZ} & {dy_1}/{dR} \\ {dy_3}/{dZ} & {dy_3}/{dR} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = 1$$

$$g(u_1{}^{-1}(R,Z),u_2{}^{-1}(R,Z)) = 24(Z-R)\cdot Z\cdot(Z^2-(Z-R)^2)$$
$$= 24(Z-R)\cdot Z\cdot(Z^2-Z^2+2ZR-R^2) = 24(Z-R)\cdot(2Z^2R-ZR^2)$$
$$= 24(2Z^3R-Z^2R^2-2Z^2R^2+ZR^3) = 24(2Z^3R-3Z^2R^2+ZR^3)$$
$$= 24R(2Z^3-3Z^2R+ZR^2), \qquad 0<R<Z<1$$

$$h(R) = \int_R^1 g(R,Z)dZ = \int_R^1 24R(2Z^3-3Z^2R+ZR^2)\ dZ$$
$$= 24R\left(\frac{2Z^4}{4} - \frac{3Z^3R}{3} + \frac{R^2Z^2}{2}\right)\Big|_R^1$$
$$= 24R\left(\frac{1}{2} - R + \frac{R^2}{2} - \frac{1}{2}R^4 + R^4 - \frac{R^4}{2}\right) = \frac{24}{2}R(1-2R+R^2)$$
$$= 12R(1-R)^2$$

$$\therefore h(R) = 12R(1-R)^2$$

i) Since $h(R) = 12R(R-1)^2$, this is give us that $R \sim \beta(2,3\ )$.

ii) .

$$E(R) = \frac{\propto}{\propto + B} = \frac{2}{5}$$

$$var(R) = \frac{\propto B}{(\propto + B)^2(\propto + B + 1)} = \frac{6}{25(6)} = \frac{1}{25}$$

Example: let $(y_1 < y_2 < y_3 < y_4 < y_5)$ denote the order statistics of a random samples of size 5 from a distribution having p.d.f. $f(x) = e^{-x}, 0 < x < \infty$, show that the statistics $Z_1 = y_2$ and $Z_2 = y_4 - y_2$ are stochastically independent?

Solution: $f(x) = e^{-x}, 0 < x < \infty$, this means that $X \sim Exp(1)$, and $F(x) = 1 - e^{-x}$

Now, according to formula (3)

$$g(y_2, y_4) = \frac{5!}{1!\quad 1!\quad !\,1}(1 - e^{-y_2})\big(1 - e^{-y_4} - (1 - e^{-y_2})\big)\big(1 - (1 - e^{-y_4})\big)e^{-y_2}e^{-y_4}$$

$$= 5!\,(1 - e^{-y_2})(e^{-y_2} - e^{-y_4})e^{-y_4}e^{-(y_2+y_4)}$$

$$= 120(1 - e^{-y_2})(e^{-y_2} - e^{-y_4})e^{-(y_2+2y_4)}, 0 < y_2 < y_4 < \infty$$

Let the space of y_2, y_4 is $A = \{(\,y_2, y_4): 0 < y_2 < y_4 < \infty\,\}$

Let the space of $Z_1, Z_2$is $B = \{(Z_1, Z_2): 0 < Z_1 < \infty, 0 < Z_2 < \infty\}$

$$\left.\begin{array}{r} Z_1 = y_2 = u_1(y_1, y_2) \\ Z_2 = y_4 - y_2 = u_2(y_1, y_2) \end{array}\right\} \qquad (1-1)\text{transformation from space A to space B}$$

$$\left.\begin{array}{r} y_2 = Z_1 = {u_1}^{-1}(Z_1, Z_2) \\ y_4 = Z_2 + Z_1 = {u_2}^{-1}(Z_1, Z_2) \end{array}\right\} \qquad (1-1)\text{transformation from space B to space of A}$$

$$J = \begin{vmatrix} {dy_2}/{dZ_1} & {dy_2}/{dZ_2} \\ {dy_4}/{dZ_1} & {dy_4}/{dZ_2} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

$$g(\mathrm{Z}_1, \mathrm{Z}_2) = g\big(u_1^{-1}(Z_1, Z_2), u_2^{-1}(Z_1, Z_2)\big) \cdot |J|$$

$$= 120(1 - \mathrm{e}^{-\mathrm{Z}_1}) \cdot \left(e^{-Z_1} - e^{-(Z_1+Z_2)}\right) \cdot \mathrm{e}^{-(\mathrm{Z}_1+\mathrm{Z}_2)} \cdot \mathrm{e}^{-(2\mathrm{Z}_1+\mathrm{Z}_2)}$$

$$g(\mathrm{Z}_1, \mathrm{Z}_2) = g\big(u_1^{-1}(Z_1, Z_2), u_2^{-1}(Z_1, Z_2)\big) \cdot |J|$$

$$= 120\,(1 - \mathrm{e}^{-\mathrm{Z}_1})\big(\mathrm{e}^{-\mathrm{Z}_1} - \mathrm{e}^{-(\mathrm{Z}_2+\mathrm{Z}_1)}\big)\mathrm{e}^{-(\mathrm{Z}_2+\mathrm{Z}_1)}e^{-(2Z_1+Z_2)}$$

$$= 120\,(1 - \mathrm{e}^{-\mathrm{Z}_1}) \cdot e^{-Z_1}(1 - e^{-Z_2}) \cdot e^{-Z_1} \cdot \mathrm{e}^{-\mathrm{Z}_2} \cdot e^{-2Z_1} \cdot \mathrm{e}^{-Z_2}$$

$$= 120\,e^{-4Z_1}\,e^{-2Z_2}(1 - e^{-Z_1}) \cdot (1 - e^{-Z_2}), 0 < Z_1 < \infty\,, 0 < Z_2 < \infty$$

The marginal distribution for each of $\mathrm{Z}_1, \mathrm{Z}_2$ are as follows:

$$h(Z_1) = \int_0^\infty g(Z_1, Z_2) dZ_2 = \int_0^\infty 120\, e^{-4Z_1}\, e^{-2Z_2}\, (1 - e^{-Z_1})\, (1 - e^{-Z_2}) dZ_2$$

$$= 120\, e^{-4Z_1}\, (1 - e^{-Z_1}) \int_0^\infty e^{-2Z_2} (1 - e^{-Z_2}) dZ_2$$

$$= 120\, e^{-4Z_1}\, (1 - e^{-Z_1}) \int_0^\infty (e^{-2Z_2} - e^{-3Z_2}) dZ_2$$

$$= 120\, e^{-4Z_1}\, (1 - e^{-Z_1}) \left(\frac{-1}{2}\, e^{-2Z_2} + \frac{1}{3}\, e^{-3Z_2}\right) \Big|_0^\infty$$

$$= 120\, e^{-4Z_1}\, (1 - e^{-Z_1}) \left(\frac{1}{2} - \frac{1}{3}\right) = 20\, e^{-4Z_1}\, (1 - e^{-Z_1})$$

$$h(Z_2) = \int_0^\infty g(Z_1, Z_2)\, dZ_1 = 120 e^{-2Z_2}\, (1 - e^{-Z_2}) \int_0^\infty e^{-4Z_1}\, (1 - e^{-Z_1}) dZ_1$$

$$= 120 e^{-2Z_2}\, (1 - e^{-Z_2}) \int_0^\infty (e^{-4Z_1} - e^{-5Z_1}) dZ_1$$

$$= 120 e^{-2Z_2}\, (1 - e^{-Z_2}) \left(\frac{-1}{4} e^{-4Z_1} + \frac{1}{5} e^{-5Z_1}\right) \Big|_0^\infty$$

$$= 120 e^{-2Z_2}\, (1 - e^{-Z_2}) \left(\frac{1}{4} - \frac{1}{5}\right) = 6 e^{-2Z_2}\, (1 - e^{-Z_2})$$

Since $g(Z_1, Z_2) = h(Z_1) h(Z_2)$ then Z_1, Z_2 are stochastically independent.

Exercises (2)

1. Let $X \sim u(0,1)$, use the transformation method to find the distribution of $y = -2\ln(x)$, then find μ_y and σ_y^2 ?

2. Given that $X_1, X_2, \cdots, X_n$ are independent random variables where $X_i \sim G\ (\propto_i, \beta)$, $i = 1,2,\cdots,n$, show that $Y = \sum_{i=1}^{n} X_i \sim G\ (\sum_{i=1}^{n} \propto_i, \beta)$?

3. Let $X_1 \sim G\ (\propto_1, \beta)$ and $X_2 \sim G\ (\propto_2, \beta)$ are two independent r. vs. find the following:

i) The distribution of $Y = X_1 + X_2$?

ii) The mean and variance of Y?

4. Let $X \sim \beta(\propto, \beta)$ and $Y = ln\left(\frac{X}{1-X}\right)$, find $M_Y(t)$?

5. Let X be a r. v. that follow the Beta distribution, find the constant C in each following cases:

i) $f(x) = C\ x^2(1-x)^5$?

ii) $f(x) = C\ (x - x^2)^{0.5}$?

6. Let $X \sim \beta(\propto, \beta)$, show that $Y = (1 - X) \sim \beta(\propto, \beta)$?

7. Let $X \sim N(0,1)$, find $E(X^{2k}), k \in \mathbb{I}^+$ then find $E(X^2)$ and $E(X^4)$?

8. let $X_1, X_2, \cdots, X_n$ are independent r. vs. where $X_i \sim N\left(\mu_i, \sigma_i^2\right), i = 1,2, \cdots, n$**,** show that $Y = \sum_{i=1}^{n} X_i \sim N\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right)$?

9. let $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$, where X_1 and X_2 are independent r. vs. show that $Y = X_1 - X_2 \sim N(\mu_1 - \mu_2, \sigma_1^2 - \sigma_2^2)$?

10. Let $X \sim X^2(n)$ and $X + Y \sim X^2(n + m)$, where X and Y are independent r. vs. use m.g.f to find the distribution of Y ?

11. Let $X \sim N(0,2)$, find $E\left(X^{k/2}\right)$, where k is even positive number, then find $E(X^2)$?

12. If the m.g.f of the r.v. X is $M_X(t) = e^{3t+8t^2}$, find the following:

i) The distribution of X?

ii) Find the mean and variance of X?

13. Let (t) be a r. v. with p.d.f $f(t) = C\left(1 + \frac{1}{5}t^2\right)^{-3}$, find the value of (C) such that the r. v. follows t distribution?

14. Let $= \frac{w}{\sqrt{v/2}}$, where $W \sim N(0,1)$ and $V \sim X^2(2)$, show that (t^2) has F distribution with parameters $n_1 = 1$ and $n_2 = 2$?

15. Let (f) has F distribution with parameters r_1 and r_2, prove that $\left(\frac{1}{f}\right)$ has F distribution with parameters r_2 and r_1 ?

16. Let (X) be an exponential random variable with parameter (λ). Compute the following probability $Pr(2 < x < 4)$?

17. Suppose that the random variable (X) has an exponential distribution with parameter (λ) Compute the following probability $(x < 2)$?

18. Let $(\overline{X})$ be the mean of random samples of size (5) from $N(0,125)$, find the value of (C) if you know $Pr(\overline{X} < C) = 0.90$, take inform that $(Z_{0.90} = 1.282)$?

19. let $x_1, x_2, \cdots, x_n$ be random samples from $G(\propto, \beta)$, show that $\overline{X} \sim G\left(n \propto, \frac{\beta}{n}\right)$, then show that $E(\overline{X}) = \propto \beta$ and $var(\overline{X}) = \frac{\propto \beta^2}{n}$? use $M_{\overline{X}}(t) = E\left(e^{t\overline{X}}\right) = E\left(e^{\frac{t}{n}(x_1 + x_2 + \cdots + x_n)}\right)$

20. let $x_1, x_2, \cdots, x_n$ be random samples from $G(1,3)$, find the p.d.f of $(\overline{X})$, then find $E(\overline{X})$ and $var(\overline{X})$?

21. Let $(y_1 < y_2 < y_3 < y_4)$ be the order Statistics of r.s of size (4) from the distribution having p.d.f $f(x) = e^{-x}, 0 < x < \infty$, find $Pr(5 \le y_4)$?

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22. Let (x_1, x_2, x_3) be a r. s. from $f(x) = 2x$, $0 < x < 1$, compute the probability that the smallest of these (x_i) exceeds the median of the distribution?

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Chapter Two

Estimation Theory

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Estimation Theory

Let $x_1, x_2, \cdots, x_n$ be a random samples from a distribution having P.d.f $f(x,\theta)$, where $f(x,\theta)$ is of known from with unknown parameter θ, therefore it have to be estimator from the sample data.

Two types of estimation can be done, namely the point estimation and the interval estimation.

Definition: The point estimation of θ is a rule (function) that assigns each element of the sample a value (estimate) of θ denoted as $\hat{\theta}$.

1. Properties of Good Estimator

1.1 Unbiasedness (عدم التحيز)

An estimator $\hat{\theta}$ is said to be unbiased estimator of θ if $E(\hat{\theta}) = \theta$, otherwise, the estimator is said to be biased.

The value of biased denoted by $b(\theta)$ and defined as follows:

$$b(\theta) = E(\hat{\theta} - \theta) = E(\hat{\theta}) - E(\theta) = E(\hat{\theta}) - \theta$$

Example: Let $x_1, x_2, \cdots, x_n$ be an independent random samples from $N(\mu, 1)$, show that $\hat{\theta} = \overline{X}$ is unbiased estimator of μ?

Solution: We have to show that $E(\hat{\theta}) = E(\overline{X}) = \mu$.

$$E(\overline{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n}E\left(\sum_{i=1}^{n} x_i\right) = \frac{1}{n}\left(\sum_{i=1}^{n} E(x_i)\right) = \frac{1}{n}\left(\sum_{i=1}^{n} \mu\right) = \frac{1}{n}\cdot n\mu = \mu$$

$\therefore \hat{\theta} = \overline{X}$ is unbiased estimator of (μ).

Example: Let $x_1, x_2, \cdots, x_n$ be an independent random samples from $N(\mu, \sigma^2)$, show that $\frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \overline{X}\right)^2$ is unbiased estimator of (σ^2)?

Solution: as we know that the variance $S^2 = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \overline{X}\right)^2$, so that:

$$\sum_{i=1}^{n}\left(x_i - \overline{X}\right)^2 = n \cdot S^2$$

$$E\left(\frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \overline{X}\right)^2\right) = \frac{1}{n-1}E\left(\sum_{i=1}^{n}\left(x_i - \overline{X}\right)^2\right) = \frac{1}{n-1}E(n \cdot S^2) = \frac{n}{n-1}E(S^2)$$

$$= \frac{n}{n-1} \cdot \frac{n-1}{n} \cdot \sigma^2 = \sigma^2$$

$$\therefore \frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \overline{X}\right)^2 \text{ is unbiased estimator of } (\sigma^2)$$

1.2 Mean Square Error (متوسط مربعات الخطأ)

The mean square error (MSE) of an estimator $(\hat{\theta})$ is defined as follows:

$$MSE(\hat{\theta}) = E(\hat{\theta} - \theta)^2 = var(\hat{\theta}) - b^2(\theta)$$

If $(\hat{\theta})$ is unbiased of (θ), then $b(\theta) = 0$, and so:

$$MSE(\hat{\theta}) = var(\hat{\theta})$$

Note: The good estimator has MSE as small as possible.

Example: Let $x_1, x_2, \cdots, x_n$ be an independent random samples from following p.d.f:

$$f(x) = \theta^x \cdot (1-\theta)^{1-x}, x = 0,1$$

Use MSE to compare between the two statistics (estimators) $\overline{X}$ and x_i ?

Solution: since $x_i \sim Ber(1, \theta)$, then $E(x_i) = \theta, \forall i = 1,2, \cdots, n$.

$$E(\overline{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n}E\left(\sum_{i=1}^{n} x_i\right) = \frac{1}{n}\left(\sum_{i=1}^{n} E(x_i)\right) = \frac{1}{n}\left(\sum_{i=1}^{n} \theta\right) = \frac{1}{n}\cdot n\theta = \theta$$

Now, both estimators $\overline{X}$ and x_i are unbiased estimator for (θ).

$$MSE(x_i) = var(x_i) = \theta(1-\theta)$$

$$MSE(\overline{X}) = var(\overline{X}) = var\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n^2}var\left(\sum_{i=1}^{n} x_i\right) = \frac{1}{n^2}\left(\sum_{i=1}^{n} var(x_i)\right)$$

$$= \frac{1}{n^2}\left(\sum_{i=1}^{n} \theta(1-\theta)\right) = \frac{1}{n^2}\cdot n\theta(1-\theta) = \frac{\theta(1-\theta)}{n}$$

$MSE(\overline{X}) < MSE(x_i)$, this means that the estimator $\overline{X}$ is better than the estimator x_i.

1.3 Consistency (الاتساق)

The estimator $(\hat{\theta})$ is said to be consistent estimator of (θ) if satisfy the following:

1. $(\hat{\theta})$ is unbiased estimator for (θ).
2. $\lim_{n\to\infty}\left(var(\hat{\theta})\right) = 0$.

Example: Let $x_1, x_2, \cdots, x_n$ be an independent random samples from $P(\theta)$, show that the estimator $\hat{\theta} = \overline{X}$ is consistent estimator of (θ)?

Solution: since $x_i \sim P(\theta), \forall i = 1,2,\cdots,n$, then $f(x) = \frac{e^{-\theta}\theta^x}{x!}$.

$$E(\hat{\theta}) = E(\overline{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n}E\left(\sum_{i=1}^{n} x_i\right) = \frac{1}{n}\left(\sum_{i=1}^{n} E(x_i)\right) = \frac{1}{n}\left(\sum_{i=1}^{n} \theta\right) = \frac{1}{n}\cdot n\theta$$

$$= \theta$$

$\therefore \hat{\theta} = \overline{X}$ is unbiased estimator for (θ).

$$var(\hat{\theta}) = var(\overline{X}) = var\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n^2} var\left(\sum_{i=1}^{n} x_i\right) = \frac{1}{n^2}\left(\sum_{i=1}^{n} var(x_i)\right)$$

$$= \frac{1}{n^2}\left(\sum_{i=1}^{n} \theta\right) = \frac{1}{n^2} \cdot n\theta = \frac{\theta}{n}$$

$$\lim_{n\to\infty}\left(var(\hat{\theta})\right) = \lim_{n\to\infty}\left(\frac{\theta}{n}\right) = \frac{\theta}{\infty} = 0$$

The two conditions consistency are satisfied, then $\hat{\theta} = \overline{X}$ is consistent estimator of (θ).

1.4 Minimum Variance Unbiased Estimate (تقدير الحد الأدنى للتباين غير متحيز)

Let $x_1, x_2, \cdots, x_n$ be an independent random samples from p.d.f $f(x, \theta)$, and let $T = t(x_1, x_2, \cdots, x_n)$ define as a statistic, if T satisfied the following conditions:

1. T is unbiased statistic of (θ).
2. T has smallest variance among all the unbiased statistics of (θ).

Then T is called a minimum variance unbiased estimate (MVUE) of (θ).

Example: Let (y_1) and (y_2) be two stochastically independent unbiased statistics for (θ). Say the variance of (y_1) is twice the variance of (y_2). Find the constants (k_1) and (k_2) such that $(k_1y_1 + k_2y_2)$ is an unbiased statistic with smallest possible variance for such a linear combination.

Solution:

Since each of (y_1), (y_2), and $(k_1y_1 + k_2y_2)$ are unbiased statistic for (θ), then

$E(y_1) = \theta$, $E(y_2) = \theta$, and $E(k_1y_1 + k_2y_2) = \theta$.

Since (y_1) and (y_2) be two stochastically independent unbiased statistics for (θ),then

$$E(k_1y_1 + k_2y_2) = \theta \Rightarrow E(k_1y_1) + E(k_2y_2) = \theta \Rightarrow k_1E(y_1) + k_2E(y_2) = \theta$$

$$\Rightarrow k_1\theta + k_2\theta = \theta \Rightarrow k_1\theta + k_2\theta = \theta \Rightarrow (k_1 + k_2)\theta = \theta \Rightarrow k_1 + k_2 = 1$$

$$\therefore k_2 = 1 - k_1$$

Let $var(y_2) = \sigma^2$, then $var(y_1) = 2\sigma^2$.

Now, assume that $Q = var(k_1 y_1 + k_2 y_2)$, and its follows:

$$Q = var(k_1 y_1 + k_2 y_2) = var(k_1 y_1) + var(k_2 y_2) = k_1^2 var(y_1) + k_2^2 var(y_2)$$
$$= 2\sigma^2 k_1^2 + \sigma^2 k_2^2 = 2\sigma^2 k_1^2 + \sigma^2 (1 - k_1)^2$$

$$\frac{\partial Q}{\partial k_1} = 4\sigma^2 k_1 + 2\sigma^2 (1 - k_1), \text{and let } \frac{\partial Q}{\partial k_1} = 0$$

$$4\sigma^2 k_1 + 2\sigma^2 (1 - k_1) = 0 \Rightarrow 4\sigma^2 k_1 + 2\sigma^2 k_1 - 2\sigma^2 = 0 \Rightarrow 6\sigma^2 k_1 - 2\sigma^2 = 0$$
$$\Rightarrow 2\sigma^2 (3k_1 - 1) = 0 \Rightarrow 3k_1 - 1 = 0$$

$$\therefore k_1 = \frac{1}{3}$$

$$k_2 = 1 - k_1 = 1 - \frac{1}{3}$$

$$\therefore k_2 = \frac{2}{3}$$

1.5 Efficiency (الكفاءة)

Let (T) be unbiased estimator for a parameter (θ), then (T) is called an efficient estimator of (θ) if and only if the variance of (T) attains the Rao-Cramer lower bound denoted by (R.C.L.B) given by:

$$var(T) \geq \frac{1}{nE\left(\left(\frac{\partial ln(f(x,\theta))}{\partial \theta}\right)^2\right)}$$

Note: it can be shown that:

$$E\left(\left(\frac{\partial ln(f(x,\theta))}{\partial\theta}\right)^2\right) = -E\left(\frac{\partial^2 ln(f(x,\theta))}{\partial\theta^2}\right)$$

Example: Let $x_1, x_2, \cdots, x_n$ be an independent random samples from $P(\theta)$, show that the estimator $\overline{X}$ is efficient statistic for (θ)?

Solution: $x_i \sim P(\theta), \forall i = 1,2,\cdots,n$, then $f(x,\theta) = \frac{e^{-\theta}\theta^x}{x!}$, $E(x) = \theta$ & $var(x) = \theta$.

$$ln(f(x,\theta)) = ln\left(\frac{e^{-\theta}\theta^x}{x!}\right) = ln(e^{-\theta}\theta^x) - ln(x!) = ln(e^{-\theta}) + ln(\theta^x) - ln(x!)$$

$$= -\theta + xln(\theta) - ln(x!)$$

$$\frac{\partial ln(f(x,\theta))}{\partial\theta} = -1 + \frac{x}{\theta} = \frac{x-\theta}{\theta}$$

$$E\left(\left(\frac{\partial ln(f(x,\theta))}{\partial\theta}\right)^2\right) = \frac{1}{\theta^2}\,E((x-\theta)^2) = \frac{1}{\theta^2}\,E\left(\left(x - E(\theta)\right)^2\right) = \frac{1}{\theta^2}\,var(x) = \frac{1}{\theta^2}\,\theta$$

$$= \frac{1}{\theta}$$

$$\therefore (R.C.L.B) = \frac{1}{n\,\frac{1}{\theta}} = \frac{\theta}{n}$$

On the other hand we have:

$$var(\overline{X}) = var\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n^2} var\left(\sum_{i=1}^{n} x_i\right) = \frac{1}{n^2}\left(\sum_{i=1}^{n} var(x_i)\right) = \frac{1}{n^2}\left(\sum_{i=1}^{n}\theta\right)$$

$$= \frac{1}{n^2}\cdot n\theta = \frac{\theta}{n}$$

$$\therefore var(\overline{X}) = (R.C.L.B)$$

$\therefore \overline{X}$ is efficient statistic for (θ).

Example: Let $x_1, x_2, \cdots, x_n$ be an independent random samples from $N(0,\theta)$, show that $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} x_i^2$ is:

1. Efficient statistic for (θ) ?
2. Consistent statistic for (θ) ?

Solution:

1. .

$$E(\hat{\theta}) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right) = \frac{1}{n}E\left(\sum_{i=1}^{n} x_i^2\right) = \frac{1}{n}\left(\sum_{i=1}^{n} E(x_i^2)\right)$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n}\left(var(x_i) + \left(E(x_i)\right)^2\right)\right) = \frac{1}{n}\left(\sum_{i=1}^{n}(\theta + 0)\right) = \frac{1}{n}\left(\sum_{i=1}^{n}\theta\right) = \frac{1}{n}\, n\theta$$

$$= \theta$$

$(\hat{\theta})$ is unbiased statistic for (θ).

$$f(x,\theta) = \frac{1}{\sqrt{2\pi\theta}}\, e^{\frac{-x^2}{2\theta}}$$

$$ln\left(f(x,\theta)\right) = ln\left(\frac{1}{\sqrt{2\pi\theta}}\, e^{\frac{-x^2}{2\theta}}\right) = ln\left(\frac{1}{\sqrt{2\pi\theta}}\right) + ln\left(e^{\frac{-x^2}{2\theta}}\right) = \frac{-1}{2}\, ln(2\pi\theta) - \frac{x^2}{2\theta}$$

$$= \frac{-1}{2}\, ln(2\pi) - \frac{1}{2}\, ln(\theta) - \frac{x^2}{2\theta}$$

$$\frac{\partial ln(f(x,\theta))}{\partial\theta} = \frac{-1}{2\theta} + \frac{x^2}{2\theta^2}$$

$$\frac{\partial^2 ln(f(x,\theta))}{\partial\theta^2} = \frac{1}{2\theta^2} - \frac{x^2}{\theta^3}$$

$$E\left(\frac{\partial^2 ln(f(x,\theta))}{\partial\theta^2}\right) = E\left(\frac{1}{2\theta^2} - \frac{x^2}{\theta^3}\right) = E\left(\frac{1}{2\theta^2}\right) - E\left(\frac{x^2}{\theta^3}\right) = \frac{1}{2\theta^2} - \frac{1}{\theta^3}E(x^2)$$

$$= \frac{1}{2\theta^2} - \frac{1}{\theta^3}E\left(var(x) + \left(E(x)\right)^2\right) = \frac{1}{2\theta^2} - \frac{1}{\theta^3}(\theta + 0) = \frac{1}{2\theta^2} - \frac{1}{\theta^2}$$

$$= \frac{-1}{2\theta^2}$$

$$(R.C.L.B) = \frac{1}{-n\,E\left(\frac{\partial^2 ln(f(x,\theta))}{\partial\theta^2}\right)}$$

$$(R.C.L.B) = \frac{1}{-n\,\left(\frac{-1}{2\theta^2}\right)} = \frac{2\theta^2}{n}$$

Now, the derive of $var(\hat{\theta}) = var\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right)$ and since $x_i \sim N(0,\theta)$, then

$$\frac{x_i}{\sqrt{\theta}} \sim N(0,\theta) \Rightarrow \frac{x_i^2}{\theta} \sim X^2(1)$$

$$\therefore \frac{\sum_{i=1}^{n} x_i^2}{\theta} \sim X^2(n), \text{with } E\left(\frac{\sum_{i=1}^{n} x_i^2}{\theta}\right) = n \text{ and } var\left(\frac{\sum_{i=1}^{n} x_i^2}{\theta}\right) = 2n$$

$$var\left(\frac{\sum_{i=1}^{n} x_i^2}{\theta}\right) = 2n \Rightarrow \frac{1}{\theta^2}\, var\left(\sum_{i=1}^{n} x_i^2\right) = 2n \Rightarrow var\left(\sum_{i=1}^{n} x_i^2\right) = 2n\,\theta^2$$

$$\Rightarrow \sum_{i=1}^{n} var(x_i^2) = 2n\,\theta^2 \Rightarrow \frac{1}{n^2}\sum_{i=1}^{n} var(x_i^2) = \frac{1}{n^2}(2n\,\theta^2)$$

$$var(\hat{\theta}) = var\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right) = \frac{2n\,\theta^2}{n^2}$$

$$\therefore var(\hat{\theta}) = \frac{2\,\theta^2}{n}$$

$$\therefore (R.C.L.B) = var(\hat{\theta})$$

$\therefore \hat{\theta}$ is Efficient statistic for (θ).

2. We proved the first condition of consistent that is $(\hat{\theta})$ is unbiased.

$$\lim_{n\to\infty}\left(var(\hat{\theta})\right) = \lim_{n\to\infty}\left(\frac{2\,\theta^2}{n}\right) = \frac{2\,\theta^2}{\infty} = 0$$

$\therefore \hat{\theta}$ is Consistent statistic for (θ).

1.6 Sufficiency (الكفاية)

1.1.1 The Fisher Neyman Theorem

Let $x_1, x_2, \cdots, x_n$ be a random samples from distribution has p.d.f $f(x,\theta)$, and let $y = u(x_1, x_2, \cdots, x_n)$ define as a statistic whose p.d.f $g(y,\theta)$, define the likelihood (L) as follows:

$$L(x_1, x_2, \cdots, x_n, \theta) = \prod_{i=1}^{n} f(x_i,\theta) = f(x_1,\theta)\cdot f(x_2,\theta)\cdots f(x_n,\theta)$$

Then $y = u(x_1, x_2, \cdots, x_n)$ is a sufficient statistic for (θ) if and only if

$$\frac{L(x_1, x_2, \cdots, x_n, \theta)}{g(y,\theta)} = H(x_1, x_2, \cdots, x_n)$$

Where the function (H) dose not depend upon (θ).

Example: Let $x_1, x_2, \cdots, x_n$ be a r. s. from distribution has p.d.f

$$f(x,\theta) = \begin{cases} \theta^x\,(1-\theta)^{1-x} & x = 0,1 \\ 0 & O.W \end{cases}$$

Show that $y = \sum_{i=1}^{n} x_i$ is sufficient statistic for (θ)?

Solution:

$$L(x_1, x_2, \cdots, x_n, \theta) = \prod_{i=1}^{n} f(x_i, \theta) = f(x_1, \theta) \cdot f(x_2, \theta) \cdots f(x_n, \theta)$$

$$= \theta^{x_1} (1-\theta)^{1-x_1} \cdot \theta^{x_2} (1-\theta)^{1-x_2} \cdots \theta^{x_n} (1-\theta)^{1-x_n}$$

$$= \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{n-\sum_{i=1}^{n} x_i}$$

Since $x_i \sim Ber(1, \theta) \Rightarrow y = \sum_{i=1}^{n} x_i \sim b(n, \theta)$, and we have that:

$$M_y(t) = E(e^{ty}) = E\left(e^{t\sum_{i=1}^{n} x_i}\right) = E\left(e^{t(x_1+x_2+\cdots+x_n)}\right) = E(e^{tx_1+tx_2+\cdots+tx_n})$$

$$= E(e^{tx_1}) \cdot E(e^{tx_2}) \cdots E(e^{tx_n}) = M_{x_1}(t) \cdot M_{x_2}(t) \cdots M_{x_n}(t)$$

$$= (1-\theta+\theta e^t) \cdot (1-\theta+\theta e^t) \cdots (1-\theta+\theta e^t) = (1-\theta+\theta e^t)^n$$

$$\therefore g(y, \theta) = \binom{n}{y} \theta^y (1-\theta)^{n-y}, \qquad y = 0,1,\cdots,n$$

$$\frac{L(x_1, x_2, \cdots, x_n, \theta)}{g(y, \theta)} = \frac{\theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{n-\sum_{i=1}^{n} x_i}}{\binom{n}{y} \theta^y (1-\theta)^{n-y}} = \frac{\theta^y (1-\theta)^{n-y}}{\left(\frac{n!}{y!\,(n-y)!}\right) \theta^y (1-\theta)^{n-y}}$$

$$= \frac{y!\,(n-y)!}{n!}$$

$$\therefore H(x_1, x_2, \cdots, x_n) = \frac{(\sum_{i=1}^{n} x_i)!\,(n-\sum_{i=1}^{n} x_i)!}{n!}$$

$H(x_1, x_2, \cdots, x_n)$ does not depend upon (θ).

$\therefore y = \sum_{i=1}^{n} x_i$ is sufficient statistic for (θ).

Example: Let $y_1 < y_2 < \cdots < y_n$ denote the order statistics of r. s. $x_1, x_2, \cdots, x_n$ from distribution has p.d.f

$$f(x, \theta) = e^{-(x-\theta)}, \qquad \theta < x < \infty \ \& -\infty < \theta < \infty$$

Show that (y_1) is sufficient statistic for (θ)?

Solution:

$$L(x_1, x_2, \cdots, x_n, \theta) = \prod_{i=1}^{n} f(x_i, \theta) = f(x_1, \theta) \cdot f(x_2, \theta) \cdots f(x_n, \theta)$$

$$= e^{-(x_1-\theta)} \cdot e^{-(x_2-\theta)} \cdots e^{-(x_n-\theta)} = e^{-\sum_{i=1}^{n} x_i + n\theta}$$

$$g(y_k) = \frac{n!}{(k-1)!\,(n-k)!} \left(F(y_k)\right)^{k-1} \cdot \left(1 - F(y_k)\right)^{n-k} \cdot f(y_k)$$

$$g(y_1) = \frac{n!}{0!\,(n-1)!} \left(F(y_1)\right)^{0} \cdot \left(1 - F(y_1)\right)^{n-1} \cdot f(y_1)$$

$$g(y_1, \theta) = \frac{n!}{(n-1)!} \cdot \left(1 - F(y_1)\right)^{n-1} \cdot f(y_1)$$

$$F(x) = \int_{\theta}^{x} f(u)du = \int_{\theta}^{x} e^{-(u-\theta)}\, du = -e^{-(u-\theta)} \Big|_{\theta}^{x} = -e^{-(x-\theta)} + e^{-(\theta-\theta)}$$

$$= 1 - e^{-(x-\theta)}$$

$$\therefore F(y_1) = 1 - e^{-(y_1-\theta)}$$

$$g(y_1, \theta) = \frac{n!}{(n-1)!} \cdot \left(1 - \left(1 - e^{-(y_1-\theta)}\right)\right)^{n-1} \cdot e^{-(y_1-\theta)}$$

$$= \frac{n(n-1)!}{(n-1)!} \cdot \left(1 - 1 + e^{-(y_1-\theta)}\right)^{n-1} \cdot e^{-(y_1-\theta)}$$

$$= n \left(e^{-(y_1-\theta)}\right)^{n-1} \left(e^{-(y_1-\theta)}\right)^{1} = n \left(e^{-(y_1-\theta)}\right)^{n} = n\, e^{-n(y_1-\theta)}$$

$$\therefore g(y_1, \theta) = n\, e^{-ny_1 + n\theta}, \qquad \theta < y_1 < \infty$$

Since $(y_1 < y_2 < \cdots < y_n)$ denote the order statistics of r. s. $(x_1, x_2, \cdots, x_n)$, then (y_1) is the smallest of (x_i), as a result of that $\left(y_1 = min(x_i)\right), i = 1,2, \cdots, n$.

$$\therefore g(y_1, \theta) = n\, e^{-n\left(min(x_i)\right) + n\theta}$$

$$\frac{L(x_1, x_2, \cdots, x_n, \theta)}{g(y, \theta)} = \frac{e^{-\sum_{i=1}^{n} x_i + n\theta}}{n\, e^{-n\left(min(x_i)\right) + n\theta}} = \frac{e^{-\sum_{i=1}^{n} x_i}\, e^{n\theta}}{n\, e^{-n\left(min(x_i)\right)}\, e^{n\theta}} = \frac{e^{-\sum_{i=1}^{n} x_i}}{n\, e^{-n\left(min(x_i)\right)}}$$

$$H(x_1, x_2, \cdots, x_n) = \frac{e^{-\sum_{i=1}^{n} x_i}}{n\, e^{-n(min(x_i))}}$$

$H(x_1, x_2, \cdots, x_n)$ does not depend upon (θ).

$\therefore y_1 = min(x_i)$ is sufficient statistic for (θ).

1. The Factorization Theorem

Let $x_1, x_2, \cdots, x_n$ be random samples from distribution has p.d.f $f(x, \theta)$, the statistic $T = t(x_1, x_2, \cdots, x_n)$ is sufficient statistic for (θ) if and only if we can find two non-negative functions (K_1) and (K_2) such that:

$$L(x_1, x_2, \cdots, x_n, \theta) = K_1(T, \theta) \cdot K_2(x_1, x_2, \cdots, x_n)$$

Where $K_2(x_1, x_2, \cdots, x_n)$ does not depend on (θ).

Example: Let $x_1, x_2, \cdots, x_n$ be a r. s. from $N(\theta, \sigma^2)$, $-\infty < x < \infty$, where the variance (σ^2) is known, show that $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is sufficient statistic for (θ)?

Solution: since $X \sim N(\theta, \sigma^2)$, then:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\; e^{\frac{-1}{2}\left(\frac{x-\theta}{\sigma}\right)^2}, -\infty < x < \infty$$

$$L(x_1, x_2, \cdots, x_n, \theta) = \prod_{i=1}^{n} f(x_i, \theta) = f(x_1, \theta) \cdot f(x_2, \theta) \cdots f(x_n, \theta)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\; e^{\frac{-1}{2}\left(\frac{x_1-\theta}{\sigma}\right)^2} \cdot \frac{1}{\sqrt{2\pi\sigma^2}}\; e^{\frac{-1}{2}\left(\frac{x_2-\theta}{\sigma}\right)^2} \cdots \frac{1}{\sqrt{2\pi\sigma^2}}\; e^{\frac{-1}{2}\left(\frac{x_n-\theta}{\sigma}\right)^2}$$

$$= (2\pi\sigma^2)^{\frac{-n}{2}}\; e^{\frac{-\sum_{i=1}^{n}(x_i-\theta)^2}{2\sigma^2}}$$

$$\text{let } \sum_{i=1}^{n}(x_i-\theta)^2 = \sum_{i=1}^{n}\left((x_i-\overline{X})+(\overline{X}-\theta)\right)^2$$

$$= \sum_{i=1}^{n}\left((x_i-\overline{X})^2 + 2(x_i-\overline{X})(\overline{X}-\theta) + (\overline{X}-\theta)^2\right)$$

$$= \sum_{i=1}^{n}(x_i-\overline{X})^2 + 2\sum_{i=1}^{n}(x_i-\overline{X})(\overline{X}-\theta) + \sum_{i=1}^{n}(\overline{X}-\theta)^2$$

$$= \sum_{i=1}^{n}(x_i-\overline{X})^2 + n(\overline{X}-\theta)^2$$

$$L(x_1,x_2,\cdots,x_n,\theta) = (2\pi\sigma^2)^{\frac{-n}{2}}\, e^{\frac{-\sum_{i=1}^{n}(x_i-\overline{X})^2}{2\sigma^2} - \frac{n(\overline{X}-\theta)^2}{2\sigma^2}}$$

$$\therefore L(x_1,x_2,\cdots,x_n,\theta) = (2\pi\sigma^2)^{\frac{-n}{2}}\, e^{\frac{-n(\overline{X}-\theta)^2}{2\sigma^2}}\ e^{\frac{-\sum_{i=1}^{n}(x_i-\overline{X})^2}{2\sigma^2}}$$

$$\therefore K_1(T,\theta) = (2\pi\sigma^2)^{\frac{-n}{2}}\, e^{\frac{-n(\overline{X}-\theta)^2}{2\sigma^2}}, \text{and } K_2(x_1,x_2,\cdots,x_n) = e^{\frac{-\sum_{i=1}^{n}(x_i-\overline{X})^2}{2\sigma^2}}$$

$$\therefore L(x_1,x_2,\cdots,x_n,\theta) = K_1(T,\theta)\cdot K_2(x_1,x_2,\cdots,x_n)$$

$\therefore \overline{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is sufficient statistic for (θ).

Example: Let $x_1,x_2,\cdots,x_n$ be a r. s. from distribution has p.d.f.

$$f(x,\theta) = \begin{cases}\theta\, x^{\theta-1} & 0<x<1 \\ 0 & O.W\end{cases}$$

Show that the product $T(x_1,x_2,\cdots,x_n) = x_1\cdot x_2\cdots x_n$ is sufficient statistic for (θ)?

Solution:

$$L(x_1,x_2,\cdots,x_n,\theta) = \theta\, {x_1}^{\theta-1}\cdot\theta\, {x_2}^{\theta-1}\cdots\theta\, {x_n}^{\theta-1} = \theta^n\,(x_1\cdot x_2\cdots x_n)^{\theta-1}$$

$$= \theta^n\,(x_1\cdot x_2\cdots x_n)^{\theta}\,(x_1\cdot x_2\cdots x_n)^{-1}$$

$$\therefore K_1(T,\theta) = \theta^n\,(x_1\cdot x_2\cdots x_n)^{\theta}, \text{and } K_2(x_1,x_2,\cdots,x_n) = (x_1\cdot x_2\cdots x_n)^{-1}$$

$$\therefore L(x_1, x_2, \cdots, x_n, \theta) = K_1(T, \theta) \cdot K_2(x_1, x_2, \cdots, x_n)$$

$\therefore$ The product $T(x_1, x_2, \cdots, x_n) = x_1 \cdot x_2 \cdots x_n$ is sufficient statistic for (θ).

Exercises (3)

1. Let $x_1, x_2, \cdots, x_n$ be a r. s. from distribution has p.d.f.

$$f(x, \theta) = \begin{cases} \frac{1}{\theta} e^{\frac{-x}{\theta}} & 0 < x < \infty, 0 < \theta < \infty \\ 0 & O.W \end{cases}$$

Show that $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is unbiased statistic for (θ)?

2. Let $y_1 < y_2 < y_3$ be the order statistics of a r. s. of size 3 from the uniform distribution having p.d.f. $f(x, \theta) = \frac{1}{\theta}, 0 < x < \theta, 0 < \theta < \infty$. Show that $(4y_1)$, $(2y_2)$ and $\left(\frac{4}{3}y_3\right)$ are all unbiased statistics for (θ) and find the variance of each of these unbiased statistics?

3. Let $x_1, x_2, \cdots, x_n$ be a r. s. from $P(\theta)$. Show that $\sum_{i=1}^{n} x_i$ is a unbiased statistics for θ?

4. Show that the nth order statistic of ar.s of size n from the uniform dist. Having p.d.f. $f(x, \theta) = \frac{1}{\theta}, 0 < x < \theta, 0 < \theta < \infty$, is sufficient statistic for (θ)?

2. Methods of Estimator

2.1 The maximum likelihood Method (MLE)

Definition: Let $x_1, x_2, \cdots, x_n$ be a r. s. from distribution has p.d.f. $f(x, \theta)$ then:

1. The product function $\prod_{i=1}^{n} f(x_i, \theta) = f(x_1, \theta) \cdot f(x_2, \theta) \cdots f(x_n, \theta)$ is called likelihood function and denoted as $L(x_1, x_2, \cdots, x_n, \theta)$.

2. Let $(\hat{\theta})$ be the value of (θ) that maximize (L), thus $(\hat{\theta})$ is the root of the equation $\frac{\partial L}{\partial \theta} = 0$, such that $\frac{\partial^2 L}{\partial \theta^2} = 0$ and it's called maximum likelihood estimate (MLE) for (θ).
3. The value of (θ) that maximize L, maximize $ln(L)$ also, thus $(\hat{\theta})$ may be regard as a solution of $\frac{\partial ln(L)}{\partial \theta} = 0$, such that $\frac{\partial^2 ln(L)}{\partial \theta^2} < 0$.

The following assumptions have to be done:

- The first and second partial derivatives are continuous function of (θ).
- The range of the r. v. X does not depend upon (θ).

2.1.1 Properties of (MLE)

1. MLE are consistent estimators.
2. If MLE exist then it is the most efficient in the class of such estimators.
3. If $\hat{\theta}$ is MLE for θ and $g(\theta)$ is the single valued function of (θ), then $g(\hat{\theta})$ is the MLE for $g(\theta)$, this is called the invariance property.

Example: Let $x_1, x_2, \cdots, x_n$ be a r. s. from distribution has p.d.f.

$$f(x,\theta) = \begin{cases} \theta\, x^{\theta-1} & 0 < x < 1 \\ 0 & O.W \end{cases}$$

Find MLE for (θ)?

Solution:

$$L(x_1, x_2, \cdots, x_n, \theta) = \prod_{i=1}^{n} \theta\, {x_i}^{\theta-1} = \theta^n \prod_{i=1}^{n} {x_i}^{\theta-1}$$

$$ln(L) = ln\left(\theta^n \prod_{i=1}^{n} x_i^{\theta-1}\right) = ln(\theta^n) + ln\left(\prod_{i=1}^{n} x_i^{\theta-1}\right) = n\, ln(\theta) + \sum_{i=1}^{n} ln\left(x_i^{\theta-1}\right)$$

$$= n\, ln(\theta) + \sum_{i=1}^{n}(\theta-1)ln(x_i) = n\, ln(\theta) + (\theta-1)\sum_{i=1}^{n} ln(x_i)$$

$$\frac{\partial ln(L)}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n} ln(x_i)$$

$$\frac{\partial ln(L)}{\partial\theta} = 0, \text{then}$$

$$\frac{n}{\theta} + \sum_{i=1}^{n} ln(x_i) = 0 \;\Rightarrow\; \frac{n}{\theta} = -\sum_{i=1}^{n} ln(x_i) \;\Rightarrow\; \frac{\theta}{n} = \frac{-1}{\sum_{i=1}^{n} ln(x_i)}$$

$$\therefore \hat{\theta} = \frac{-n}{\sum_{i=1}^{n} ln(x_i)}$$

$\therefore \hat{\theta}$ is MLE for (θ).

Example: Let $x_1, x_2, \cdots, x_n$ be a r. s. from $N(\mu, \sigma^2)$, use MLE method to estimate μ and σ^2?

Solution: since $X \sim N(\mu, \sigma^2)$, then:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\; e^{\frac{-1}{2}\left(\frac{x-\pi}{\sigma}\right)^2}, -\infty < x < \infty$$

$$L(x_1, x_2, \cdots, x_n, \mu, \sigma^2) = \prod_{i=1}^{n} f(x_i, \theta) = f(x_1, \mu, \sigma^2)\cdot f(x_2, \mu, \sigma^2)\cdots f(x_n, \mu, \sigma^2)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\; e^{\frac{-1}{2}\left(\frac{x_1-\mu}{\sigma}\right)^2} \cdot \frac{1}{\sqrt{2\pi\sigma^2}}\; e^{\frac{-1}{2}\left(\frac{x_2-\mu}{\sigma}\right)^2} \cdots \frac{1}{\sqrt{2\pi\sigma^2}}\; e^{\frac{-1}{2}\left(\frac{x_n-\mu}{\sigma}\right)^2}$$

$$= (2\pi\sigma^2)^{\frac{-n}{2}}\; e^{\frac{-\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}}$$

$$\ln(L) = ln\left((2\pi\sigma^2)^{\frac{-n}{2}}\ e^{\frac{-\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}}\right) = ln\left((2\pi\sigma^2)^{\frac{-n}{2}}\right) + ln\left(e^{\frac{-\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}}\right)$$

$$= \frac{-n}{2}\ ln(2\pi\sigma^2) - \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}$$

$$= \frac{-n}{2}\ ln(2\pi) - \frac{n}{2}\ ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$= \frac{-n}{2}\ ln(2\pi) - n\ ln(\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\frac{\partial ln(L)}{\partial\mu} = \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)$$

$$\frac{\partial ln(L)}{\partial\mu} = 0 \ \Rightarrow\ \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu) = 0 \ \Rightarrow\ \sum_{i=1}^{n}(x_i-\mu) = 0 \ \Rightarrow\ \sum_{i=1}^{n}(x_i) - \sum_{i=1}^{n}\mu = 0$$

$$\Rightarrow\ \sum_{i=1}^{n}(x_i) - n\,\mu = 0 \ \Rightarrow\ n\,\mu = \sum_{i=1}^{n}(x_i)$$

$$\therefore\ \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n}(x_i) = \overline{X}$$

$\therefore\ \hat{\mu} = \overline{X}$ is MLE for μ.

$$\frac{\partial ln(L)}{\partial\sigma} = \frac{-n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\hat{\mu})^2$$

$$\frac{\partial ln(L)}{\partial\sigma} = 0 \ \Rightarrow\ \frac{-n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\hat{\mu})^2 = 0 \ \Rightarrow\ \frac{n}{2\sigma^2} = \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\hat{\mu})^2$$

$$\Rightarrow\ \frac{2\sigma^4}{2\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\hat{\mu})^2 \ \Rightarrow\ \sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\hat{\mu})^2$$

$$\therefore \hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})^2}$$

$\therefore \hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})^2}$ is MLE for σ.

Example: Let $x_1, x_2, \cdots, x_n$ be a r. s. from distribution has p.d.f.

$$f(x,\theta) = \begin{cases} \frac{1}{\theta} & 0 < x < \theta, 0 < \theta < \infty \\ 0 & O.W \end{cases}$$

Find the MLE for (θ)?

Solution:

$$L(x_1, x_2, \cdots, x_n, \theta) = \prod_{i=1}^{n}\frac{1}{\theta} = \frac{1}{\theta}\cdot\frac{1}{\theta}\cdots\frac{1}{\theta} = \frac{1}{\theta^n}$$

$$\ln(L) = ln\left(\frac{1}{\theta^n}\right) = ln(\theta^{-n}) = -n\ln(\theta)$$

$$\frac{\partial ln(L)}{\partial\theta} = \frac{-n}{\theta}$$

$$\frac{\partial ln(L)}{\partial\theta} = 0, \text{then } \frac{-n}{\theta} = 0$$

We can't use the differentiation method because the range of (x) depend upon (θ), but it is clear that (L) has maximum value at the smallest value of θ, which coincide with the maximum value of (x). Hence, $(\hat{\theta})$ is the largest order statistic of the sample.

$$\therefore \hat{\theta} = max(x_i)$$

$\therefore \hat{\theta} = max(x_i)$ is MLE for (θ).

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Example: Let $x_1, x_2, \cdots, x_n$ be a r. s. from distribution having p.d.f.

$$f(x,\theta) = \begin{cases} \beta\, e^{-\beta(x-\propto)} & \propto < x < 0, 0 < \beta < \infty \\ 0 & O.W \end{cases}$$

Find the MLE for $(\propto)$ and (β)?

Solution:

$$L(x_1, x_2, \cdots, x_n, \propto, \beta) = \prod_{i=1}^{n} \beta\, e^{-\beta(x_i-\propto)} = \beta^n\, e^{-\beta \sum_{i=1}^{n}(x_i-\propto)}$$

The MLE for α can't be found by the method of differentiation since the range of (x) depend upon $(\propto)$.

It is clear that (L) has maximum value at the largest value of $(\propto)$ which coincide with the smallest value of (x). Hence, $\left(\hat{\propto} = min(x_i)\right)$, which is the smallest order statistic of the sample.

$$\ln(L) = ln\left(\beta^n\, e^{-\beta \sum_{i=1}^{n}(x_i-\propto)}\right) = ln(\beta^n) + ln\left(e^{-\beta \sum_{i=1}^{n}(x_i-\propto)}\right)$$

$$= n\, ln(\beta) - \beta \sum_{i=1}^{n}(x_i - \propto)$$

$$\frac{\partial ln(L)}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n}(x_i - \hat{\propto})$$

$$\text{let } \frac{\partial ln(L)}{\partial \beta} = 0$$

$$\frac{n}{\beta} - \sum_{i=1}^{n}(x_i - \hat{\propto}) = 0 \;\Rightarrow\; \frac{n}{\beta} = \sum_{i=1}^{n}(x_i - \hat{\propto}) \;\Rightarrow\; \frac{\beta}{n} = \frac{1}{\sum_{i=1}^{n}(x_i - \hat{\propto})}$$

$$\hat{\beta} = \frac{n}{\sum_{i=1}^{n}\left(x_i - min(x_i)\right)} = \frac{n}{\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} min(x_i)} = \frac{n}{n\,\overline{X} - n\, min(x_i)}$$

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$$\therefore \hat{\beta} = \frac{1}{\overline{X} - min(x_i)}$$

$\therefore \hat{\beta} = \frac{1}{\overline{X} - min(x_i)}$ is the MLE for (β).

Example: Let $x_1, x_2, \cdots, x_n$ be a r. s. from distribution has p.d.f.

$$f(x,\theta) = \begin{cases} \theta^x \, (1-\theta)^{1-x} & x = 0,1 \\ 0 & O.W \end{cases}$$

Find the MLE for $= \frac{\theta}{1-\theta}$?

Solution: first we find MLE for (θ).

$$L(x_1, x_2, \cdots, x_n, \theta) = \prod_{i=1}^{n} \theta^{x_i} \, (1-\theta)^{1-x_i}$$

$$= \left(\theta^{x_1} \, (1-\theta)^{1-x_1}\right) \cdot \left(\theta^{x_2} \, (1-\theta)^{1-x_2}\right) \cdots \left(\theta^{x_n} \, (1-\theta)^{1-x_n}\right)$$

$$= \theta^{\sum_{i=1}^{n} x_i} \, (1-\theta)^{n-\sum_{i=1}^{n} x_i}$$

$$ln(L) = ln\left(\theta^{\sum_{i=1}^{n} x_i} \, (1-\theta)^{n-\sum_{i=1}^{n} x_i}\right) = ln\left(\theta^{\sum_{i=1}^{n} x_i}\right) + ln\left((1-\theta)^{n-\sum_{i=1}^{n} x_i}\right)$$

$$= \left(\sum_{i=1}^{n} x_i\right) \cdot ln(\theta) + \left(n - \sum_{i=1}^{n} x_i\right) \cdot ln(1-\theta)$$

$$\frac{\partial ln(L)}{\partial \theta} = \frac{\sum_{i=1}^{n} x_i}{\theta} - \frac{n - \sum_{i=1}^{n} x_i}{1-\theta}$$

$$\frac{\partial ln(L)}{\partial \theta} = 0 \;\Rightarrow\; \frac{\sum_{i=1}^{n} x_i}{\theta} - \frac{n - \sum_{i=1}^{n} x_i}{1-\theta} = 0 \;\Rightarrow\; \frac{\sum_{i=1}^{n} x_i}{\theta} = \frac{n - \sum_{i=1}^{n} x_i}{1-\theta}$$

$$\Rightarrow\; \frac{1-\theta}{\theta} = \frac{n - \sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i} \;\Rightarrow\; \frac{1}{\theta} - 1 = \frac{n}{\sum_{i=1}^{n} x_i} - 1 \;\Rightarrow\; \frac{1}{\theta} = \frac{n}{\sum_{i=1}^{n} x_i}$$

$$\hat{\theta} = \frac{\sum_{i=1}^{n} x_i}{n} \;\Rightarrow\; \hat{\theta} = \overline{X}$$

$$\therefore \hat{w} = \frac{\hat{\theta}}{1-\hat{\theta}} = \frac{\overline{X}}{1-\overline{X}}$$

$\therefore \hat{w} = \frac{\overline{X}}{1-\overline{X}}$ is the MLE for $w = \frac{\theta}{1-\theta}$.

Example: Eight trials are conducted of a given system with the following results (S, F, S, F, S, S, S, S) where (S) denote success and (F) denote failure. Find the MLE (p) of the probability of the successful events?

Solution: Let the r. v. X denote the success event, then

$$x = \begin{cases} 1 & \text{if the event } S \text{ occur} \\ 0 & \text{if the event } S \text{ does not occur} \end{cases}$$

$X \sim Ber(1, p)$, then $f(x) = p^x\,(1-p)^{1-x}, x = 0,1$

$$L(x_1, x_2, \cdots, x_n, \theta) = \prod_{i=1}^{n} f(x_i, p) = \prod_{i=1}^{n} p^{x_i}\,(1-p)^{1-x_i}$$

$$= p^{x_1}\,(1-p)^{1-x_1} \cdot p^{x_2}\,(1-p)^{1-x_2} \cdots p^{x_n}\,(1-p)^{1-x_n}$$

$$= p^{\sum_{i=1}^{n} x_i}\,(1-p)^{n-\sum_{i=1}^{n} x_i} = p^6\,(1-p)^{8-6} = p^6\,(1-p)^2$$

$$ln(L) = ln(p^6\,(1-p)^2) = 6\,ln(p) + 2ln(1-p)$$

$$\frac{\partial ln(L)}{\partial p} = \frac{6}{p} - \frac{2}{1-p}$$

$$\frac{\partial ln(L)}{\partial p} = 0 \;\Rightarrow\; \frac{6}{p} - \frac{2}{1-p} = 0 \;\Rightarrow\; \frac{6}{p} = \frac{2}{1-p} \;\Rightarrow\; \frac{1-p}{p} = \frac{2}{6} \;\Rightarrow\; \frac{1}{p} - 1 = \frac{1}{3}$$

$$\Rightarrow\; \frac{1}{p} = \frac{1}{3} - 1 \;\Rightarrow\; \frac{1}{p} = \frac{4}{3} \;\Rightarrow \hat{p} = \frac{3}{4}$$

$\therefore \hat{p} = \frac{3}{4}$ is the MLE for (p).

Example: Let $x_1, x_2, \cdots, x_n$ be a r. s. from Exponential distribution having p.d.f.

$$f(x,\lambda)=\begin{cases}\lambda e^{-\lambda x} & \lambda>0, x\geq 0\\ 0 & O.W\end{cases}$$

Find the MLE for (λ)?

Solution:

$$L(x_1,x_2,\cdots,x_n,\lambda)=\prod_{i=1}^{n} f(x_i,\lambda)=\prod_{i=1}^{n}\lambda e^{-\lambda x_i}=\lambda e^{-\lambda x_1}\cdot\lambda e^{-\lambda x_2}\cdots\lambda e^{-\lambda x_n}$$

$$=\lambda^n\, e^{-\lambda\sum_{i=1}^{n}x_i}$$

$$ln(L)=ln\left(\lambda^n\, e^{-\lambda\sum_{i=1}^{n}x_i}\right)=ln(\lambda^n)+l\,n\left(e^{-\lambda\sum_{i=1}^{n}x_i}\right)=n\,ln(\lambda)-\lambda\sum_{i=1}^{n}x_i$$

$$\frac{\partial ln(L)}{\partial\lambda}=\frac{n}{\lambda}-\sum_{i=1}^{n}x_i$$

$$\frac{\partial ln(L)}{\partial\lambda}=0\ \Rightarrow\ \frac{n}{\lambda}-\sum_{i=1}^{n}x_i=0\ \Rightarrow\ \frac{n}{\lambda}=\sum_{i=1}^{n}x_i\ \Rightarrow\ \frac{\lambda}{n}=\frac{1}{\sum_{i=1}^{n}x_i}\ \Rightarrow\ \lambda=\frac{n}{\sum_{i=1}^{n}x_i}$$

$$\Rightarrow\ \lambda=\frac{1}{\frac{\sum_{i=1}^{n}x_i}{n}}$$

$$\therefore\hat{\lambda}=\frac{1}{\overline{X}}$$

$\therefore\hat{\lambda}=\frac{1}{\overline{X}}$ is the MLE for (λ) and since $\theta=\frac{1}{\lambda}$, then $\hat{\theta}=\overline{X}$ is the MLE for (θ).

2.2 The Moments Method

Let $f(x,\theta_1,\theta_2,\cdots,\theta_n)$be the p.d.f of the population with k parameters $(\theta_1,\theta_2,\cdots,\theta_n)$. By this method, we equate the population moments $M_r=E(\,x^r)$ with the sample moments $m_r=\frac{1}{n}\sum_{i=1}^{n}x_i^r\,, r=1,2,\cdots,n$. Then solving for the unknown parameters.

Note: $m_1=\frac{1}{n}\sum_{i=1}^{n}x_i$.

Example: Let $x_1, x_2, \cdots, x_n$ be a r. s. from $P(\theta)$. Find the moment estimator for (θ).

Solution: $f(x) = \frac{e^{-\theta}\,\theta^x}{x!} \quad x = 0,1,2,\cdots,\theta > 0, \& \ E(x) = var(x) = \theta$

We have the population moment $M_1 = E(x^1) = E(x) = \theta$.

And the sample moment $m_1 = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{X}$.

$$M_1 = m_1 \Rightarrow \overline{X} = \theta$$

$\therefore \hat{\theta} = \overline{X}$ is the moment estimator for (θ).

Example: Let $x_1, x_2, \cdots, x_n$ be a r. s. from $f(x,\theta) = \frac{1}{\theta},\ 0 < x < \theta, 0 < \theta < \infty$. Find the moment estimator for (θ).

Solution:

$$M_1 = E(x^1) = E(x) = \int_0^\theta x\,\frac{1}{\theta}\,dx = \frac{1}{2\theta}\,x^2\Big|_0^\theta = \frac{1}{2\theta}(\theta^2 - 0) = \frac{\theta}{2}$$

$$m_1 = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{X}$$

$$M_1 = m_1 \Rightarrow \overline{X} = \frac{\theta}{2} \Rightarrow \hat{\theta} = 2\overline{X}$$

$\therefore \hat{\theta} = 2\overline{X}$ is the moment estimator for (θ).

Example: Let $x_1, x_2, \cdots, x_n$ be a r. s. from $u(\propto, \beta)$. Find the moment estimator for $(\propto)$ and (β).

Solution: $f(x,\propto,\beta) = \frac{1}{\propto+\beta},\ \propto < x < \beta$

$$M_1 = E(x^1) = E(x) = \frac{\beta+\propto}{2}$$

$$m_1 = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{X}$$

$$M_1 = m_1 \Rightarrow \overline{X} = \frac{\beta+\propto}{2} \cdots (1)$$

$$M_2 = E(x^2) = var(x) + \left(E(x)\right)^2 = \frac{(\beta-\propto)^2}{12} + \left(\frac{\beta+\propto}{2}\right)^2$$

$$m_1 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

$$M_2 = m_2 \Rightarrow \frac{(\propto-\beta)^2}{12} + \left(\frac{\beta+\propto}{2}\right)^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 \cdots (2)$$

From eq. (1) we have: $\beta = 2\overline{X} - \propto$ and putting that in eq. (2) we get:

$$\frac{\left(2\overline{X}-\propto-\propto\right)^2}{12} + \left(\overline{X}\right)^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 \Rightarrow \frac{\left(2\overline{X}-2\propto\right)^2}{12} + \left(\overline{X}\right)^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

$$\Rightarrow \frac{4\left(\overline{X}-\propto\right)^2}{12} + \left(\overline{X}\right)^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 \Rightarrow \frac{\left(\overline{X}-\propto\right)^2}{3} + \left(\overline{X}\right)^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

$$\Rightarrow \left(\overline{X}-\propto\right)^2 + 3\left(\overline{X}\right)^2 = \frac{3}{n}\sum_{i=1}^{n} x_i^2 \Rightarrow \left(\overline{X}-\propto\right)^2 = \frac{3}{n}\sum_{i=1}^{n} x_i^2 - 3\left(\overline{X}\right)^2$$

$$\overline{X}-\propto = \sqrt{\frac{3}{n}\sum_{i=1}^{n} x_i^2 - 3\left(\overline{X}\right)^2}$$

$$\therefore \hat{\propto} = \overline{X} - \sqrt{\frac{3}{n}\sum_{i=1}^{n} x_i^2 - 3\left(\overline{X}\right)^2}$$

Since $\beta = 2\overline{X} - \propto$, the we have the following:

$$\beta = 2\overline{X} - \left(\overline{X} - \sqrt{\frac{3}{n}\sum_{i=1}^{n} x_i^2 - 3\left(\overline{X}\right)^2} \right)$$

$$\therefore \hat{\beta} = \overline{X} + \sqrt{\frac{3}{n}\sum_{i=1}^{n} x_i^2 - 3\left(\overline{X}\right)^2}$$

$\therefore \hat{\propto}$ and $\hat{\beta}$ are moment estimators for $(\propto)$ and (β) respectively.

Exercises (4)

1. Let $x_1, x_2, \cdots, x_n$ be a r. s. from $P(\theta)$, find MLE estimator for $Pr(x > 0)$?
2. Let $x_1, x_2, \cdots, x_n$ be a r. s. from the following p.d.f.

$$f(x, \theta_1, \theta_2) = \frac{1}{\theta_1}\, e^{\frac{-(x-\theta_1)}{\theta_2}}, \theta_1 < x < \infty, -\infty < \theta_1 < \infty, \&\ 0 < \theta_2 < \infty$$

   Find the MLE for θ_1 and θ_2?

3. Let $x_1, x_2, \cdots, x_n$ be a r. s. from $N(\mu, \sigma^2)$, find the moment estimator for μ & σ^2?
4. Let $x_1, x_2, \cdots, x_n$ be a r. s. from $G(\propto, \beta)$, find the moment estimator for $\propto$ & β?

2.3 The Method of Least Squares

Suppose that we can write the observations in the form:

$$y_1 = g_1(\theta_1, \theta_2, \cdots, \theta_n) + \varepsilon_i, i = 1,2, \cdots, n$$

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Where (g_i) are known functions and the real numbers $(\theta_1, \theta_2, \cdots, \theta_k)$ are the unknown parameters of interest, suppose that (ε_i) satisfy the conditions:

$$E(\varepsilon_i) = 0, var(\varepsilon_i) = \sigma^2 \;\&\; cov(\varepsilon_i, \varepsilon_j) = 0, i = 1,2,\cdots,n, j = 1,2,\cdots,n$$

The method of least squares says that we should find the point

$$\theta^n = (\theta_1^n, \theta_2^n, \cdots, \theta_k^n)$$

Which makes the expected value vector as close as possible to observe value that is we should minimize the following value:

$$\sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} \big(y_i - E(y_i)\big)^2$$

Example: let$y_i = \theta_1 + \varepsilon_i, i = 1,2,\cdots,n$. Estimate θ_1 using LS method.

Solution: we have $E(y_i) = E(\theta_1 + \varepsilon_i) = E(\theta_1) + E(\varepsilon_i) = \theta_1$, and let:

$$Q = \sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} \big(y_i - E(y_i)\big)^2 = \sum_{i=1}^{n} (y_i - \theta_1)^2$$

$$\frac{\partial Q}{\partial \theta_1} = -2 \sum_{i=1}^{n} (y_i - \theta_1)$$

Now, let $\frac{\partial Q}{\partial \theta_1} = 0$ and then we have:

$$-2 \sum_{i=1}^{n} (y_i - \theta_1) = 0 \Rightarrow \sum_{i=1}^{n} (y_i - \theta_1) = 0 \Rightarrow \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} \theta_1 = 0$$

$$\Rightarrow \sum_{i=1}^{n} y_i - n\theta_1 = 0 \Rightarrow n\theta_1 = \sum_{i=1}^{n} y_i \Rightarrow \hat{\theta}_1 = \frac{1}{n} \sum_{i=1}^{n} y_i$$

$\therefore \hat{\theta}_1 = \overline{X}$ is the LS method for θ_1.

Example: let x_1, x_2, x_3 are three random variables with the same variance (σ^2), and let. $E(x_1) = \theta_1, E(x_2) = \theta_1 + \theta_2,$ and $E(x_3) = 2\theta_1 + \theta_2$, find the LS estimators for (θ_1) and (θ_1), then find the mean and variance for each estimator.

Solution: let

$$Q = \sum_{i=1}^{3} \varepsilon_i^2 = \sum_{i=1}^{n} \left(x_i - E(x_i)\right)^2 = \left(x_1 - E(x_1)\right)^2 + \left(x_2 - E(x_2)\right)^2 + \left(x_3 - E(x_3)\right)^2$$

$$= (x_1 - \theta_1)^2 + \left(x_2 - (\theta_1 + \theta_2)\right)^2 + \left(x_3 - (2\theta_1 + \theta_2)\right)^2$$

$$= (x_1 - \theta_1)^2 + (x_2 - \theta_1 - \theta_2)^2 + (x_3 - 2\theta_1 - \theta_2)^2$$

$$\frac{\partial Q}{\partial \theta_1} = -2(x_1 - \theta_1) - 2(x_2 - \theta_1 - \theta_2) - 4(x_3 - 2\theta_1 - \theta_2)$$

$$= -2x_1 - 2x_2 - 4x_3 + 12\theta_1 + 6\theta_2$$

$$\frac{\partial Q}{\partial \theta_1} = 0 \ \Rightarrow \ -2x_1 - 2x_2 - 4x_3 + 12\theta_1 + 6\theta_2 = 0$$

$$12\theta_1 + 6\theta_2 = 2x_1 + 2x_2 + 4x_3$$

$$\therefore 6\theta_1 + 3\theta_2 = x_1 + x_2 + 2x_3 \quad \cdots (1)$$

$$\frac{\partial Q}{\partial \theta_2} = -2(x_2 - \theta_1 - \theta_2) - 2(x_3 - 2\theta_1 - \theta_2) = -2x_2 - 2x_3 + 6\theta_1 + 4\theta_2$$

$$\frac{\partial Q}{\partial \theta_2} = 0 \ \Rightarrow = -2x_2 - 2x_3 + 6\theta_1 + 4\theta_2 = 0 \Rightarrow 6\theta_1 + 4\theta_2 = 2x_2 + 2x_3$$

$$\Rightarrow 3\theta_1 + 2\theta_2 = x_2 + x_3$$

$$\theta_1 = \frac{x_2 + x_3 - 2\theta_2}{3} \cdots (2)$$

Now, Substitute eq. (2), eq. (1) we get:

$$6\left(\frac{x_2 + x_3 - 2\theta_2}{3}\right) + 3\theta_2 = x_1 + x_2 + 2x_3$$

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$$2x_2 + 2x_3 - 4\theta_2 + 3\theta_2 = x_1 + x_2 + 2x_3$$

$$\therefore \hat{\theta}_2 = x_2 - x_1$$

Substitute $(\hat{\theta}_2)$ in eq. (1), and we get:

$$\theta_1 = \frac{x_2 + x_3 - 2(x_2 - x_1)}{3} \Rightarrow \theta_1 = \frac{x_2 + x_3 - 2x_2 + 2x_1}{3}$$

$$\therefore \hat{\theta}_1 = \frac{2x_1 - x_2 + x_3}{3}$$

$\therefore \hat{\theta}_1$ & $\hat{\theta}_2$ are LS estimators for (θ_1) and (θ_1).

The mean of $\hat{\theta}_1$ & $\hat{\theta}_2$could be found as follows:

$$\mu_{\hat{\theta}_1} = E(\hat{\theta}_1) = E\left(\frac{2x_1 - x_2 + x_3}{3}\right) = \frac{1}{3}E(2x_1 - x_2 + x_3)$$
$$= \frac{2}{3}E(x_1) - \frac{1}{3}E(x_2) + \frac{1}{3}E(x_3) = \frac{2}{3}(\theta_1) - \frac{1}{3}(\theta_1 + \theta_2) + \frac{1}{3}(2\theta_1 + \theta_2)$$
$$= \frac{2\theta_1}{3} - \frac{\theta_1}{3} - \frac{\theta_2}{3} + \frac{2\theta_1}{3} + \frac{\theta_2}{3} = \theta_1$$

$$\mu_{\hat{\theta}_2} = E(\hat{\theta}_2) = E(x_2 - x_1) = E(x_2) - E(x_1) = 2\theta_1 + \theta_2 - \theta_1 = 2\theta_1$$

The variance of $\hat{\theta}_1$ & $\hat{\theta}_2$ are H. W.?

Example: let $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i, i = 1,2,\cdots,n$ (simple linear regression model). Estimate β_0 & β_1 using LS method.

Solution: let

$$Q = \sum_{i=1}^{3} \varepsilon_i^2 = \sum_{i=1}^{n} \left(y_i - E(y_i)\right)^2 = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2$$

$$\frac{\partial Q}{\partial \beta_0} = -2\sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)$$

$$\frac{\partial Q}{\partial \beta_0} = 0 \Rightarrow -2\sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i) = 0$$

$$\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i) = 0 \Rightarrow \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} \beta_0 - \sum_{i=1}^{n} \beta_1 x_i = 0$$

$$\sum_{i=1}^{n} y_i - n\beta_0 - \beta_1 \sum_{i=1}^{n} x_i = 0 \Rightarrow n\beta_0 = \sum_{i=1}^{n} y_i - \beta_1 \sum_{i=1}^{n} x_i$$

$$\beta_0 = \frac{1}{n}\left(\sum_{i=1}^{n} y_i - \beta_1 \sum_{i=1}^{n} x_i\right) \cdots (1)$$

$$\frac{\partial Q}{\partial \beta_1} = -2\sum_{i=1}^{n} x_i(y_i - \beta_0 - \beta_1 x_i)$$

$$\frac{\partial Q}{\partial \beta_1} = 0 \Rightarrow -2\sum_{i=1}^{n} x_i(y_i - \beta_0 - \beta_1 x_i) = 0$$

$$\sum_{i=1}^{n} x_i(y_i - \beta_0 - \beta_1 x_i) = 0 \Rightarrow \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} \beta_0 x_i - \sum_{i=1}^{n} \beta_1 x_i^2 = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i y_i - \beta_0 \sum_{i=1}^{n} x_i - \beta_1 \sum_{i=1}^{n} x_i^2 = 0$$

$$\sum_{i=1}^{n} x_i y_i = \beta_0 \sum_{i=1}^{n} x_i + \beta_1 \sum_{i=1}^{n} x_i^2 \cdots (2)$$

Substitute eq. (1) in eq. (2), and we get:

$$\sum_{i=1}^{n} x_i y_i = \frac{1}{n}\sum_{i=1}^{n} x_i \left(\sum_{i=1}^{n} y_i - \beta_1 \sum_{i=1}^{n} x_i\right) + \beta_1 \sum_{i=1}^{n} x_i^2$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i - \beta_1 \left(\sum_{i=1}^{n} x_i\right)^2\right) + \beta_1 \sum_{i=1}^{n} x_i^2$$

$$n\sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i - \beta_1 \left(\sum_{i=1}^{n} x_i\right)^2 + n\beta_1 \sum_{i=1}^{n} x_i^2$$

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$$n\beta_1 \sum_{i=1}^{n} x_i^2 - \beta_1 \left(\sum_{i=1}^{n} x_i\right)^2 = n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i$$

$$\beta_1 \left(n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2\right) = n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i$$

$$\therefore \hat{\beta}_1 = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

Substitute $(\hat{\beta}_1)$in eq. (1), and we get:

$$n\beta_0 = \sum_{i=1}^{n} y_i - \left(\frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}\right)\sum_{i=1}^{n} x_i$$

$$n\beta_0 = \sum_{i=1}^{n} y_i - \frac{n\sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i y_i - (\sum_{i=1}^{n} x_i)^2 \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$n\beta_0 = \frac{n\sum_{i=1}^{n} y_i \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} y_i \,(\sum_{i=1}^{n} x_i)^2 - n\sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i y_i + (\sum_{i=1}^{n} x_i)^2 \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$n\beta_0 = \frac{n\sum_{i=1}^{n} y_i \sum_{i=1}^{n} x_i^2 - n\sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$\therefore \hat{\beta}_0 = \frac{\sum_{i=1}^{n} y_i \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$\therefore \hat{\beta}_0$ & $\hat{\beta}_1$are the LS estimators for β_0 & β_1.

Example: for the simple linear regression model (in above example) show that $(\hat{\beta}_1)$ can be written as follows:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \overline{X})\,(y_i - \overline{Y})}{\sum_{i=1}^{n}(x_i - \overline{X})^2} = \frac{\sum_{i=1}^{n}(x_i - \overline{X})\, y_i}{\sum_{i=1}^{n}(x_i - \overline{X})^2}$$

Then show that its unbiased estimator for (β_1).

Solution: from numerators, we have the following:

$$\sum_{i=1}^{n}(x_i-\overline{X})\,(y_i-\overline{Y})=\sum_{i=1}^{n}(x_iy_i-x_i\overline{Y}-\overline{X}y_i+\overline{XY})$$
$$=\sum_{i=1}^{n}x_iy_i-\overline{Y}\sum_{i=1}^{n}x_i-\overline{X}\sum_{i=1}^{n}y_i+\sum_{i=1}^{n}\overline{X}\,\overline{Y}$$
$$=\sum_{i=1}^{n}x_iy_i-n\,\overline{Y}\,\overline{X}-n\,\overline{Y}\,\overline{X}+n\,\overline{X}\,\overline{Y}=\sum_{i=1}^{n}x_iy_i-n\,\overline{Y}\,\overline{X}$$
$$=\sum_{i=1}^{n}x_iy_i-n\left(\frac{\sum_{i=1}^{n}x_i}{n}\right)\left(\frac{\sum_{i=1}^{n}y_i}{n}\right)=\sum_{i=1}^{n}x_iy_i-\frac{1}{n}\sum_{i=1}^{n}x_i\sum_{i=1}^{n}y_i$$

Also, we have the following:

$$\sum_{i=1}^{n}(x_i-\overline{X})\,y_i=\sum_{i=1}^{n}x_iy_i-\overline{X}\sum_{i=1}^{n}y_i=\sum_{i=1}^{n}x_iy_i-\frac{1}{n}\sum_{i=1}^{n}x_i\sum_{i=1}^{n}y_i$$

$$\therefore\ \hat{\beta}_1=\frac{\sum_{i=1}^{n}(x_i-\overline{X})\,y_i}{\sum_{i=1}^{n}(x_i-\overline{X})^2}$$

To show that $(\hat{\beta}_1)$ is unbiased estimator for (β_1)

$$E(\hat{\beta}_1)=E\left(\frac{\sum_{i=1}^{n}(x_i-\overline{X})\,y_i}{\sum_{i=1}^{n}(x_i-\overline{X})^2}\right)=\frac{E\left(\sum_{i=1}^{n}(x_i-\overline{X})\,y_i\right)}{E\left(\sum_{i=1}^{n}(x_i-\overline{X})^2\right)}=\frac{\sum_{i=1}^{n}(x_i-\overline{X})\,E(y_i)}{\sum_{i=1}^{n}(x_i-\overline{X})^2}$$
$$=\frac{\sum_{i=1}^{n}(x_i-\overline{X})\,E(\beta_0+\beta_1x_i+\varepsilon_i)}{\sum_{i=1}^{n}(x_i-\overline{X})^2}$$
$$=\frac{\sum_{i=1}^{n}(x_i-\overline{X})\left(E(\beta_0)+E(\beta_1x_i)+E(\varepsilon_i)\right)}{\sum_{i=1}^{n}(x_i-\overline{X})^2}=\frac{\sum_{i=1}^{n}(x_i-\overline{X})\,(\beta_0+\beta_1x_i)}{\sum_{i=1}^{n}(x_i-\overline{X})^2}$$

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$$E(\hat{\beta}_1) = \frac{\beta_0 \sum_{i=1}^{n}(x_i - \overline{X}) + \beta_1 \sum_{i=1}^{n} x_i(x_i - \overline{X})}{\sum_{i=1}^{n}(x_i - \overline{X})^2}$$

$$= \beta_0 \left(\frac{\sum_{i=1}^{n}(x_i - \overline{X})}{\sum_{i=1}^{n}(x_i - \overline{X})^2}\right) + \beta_1 \left(\frac{\sum_{i=1}^{n} x_i(x_i - \overline{X})}{\sum_{i=1}^{n}(x_i - \overline{X})^2}\right)$$

$$= \beta_0 \left(\frac{\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \overline{X}}{\sum_{i=1}^{n}(x_i - \overline{X})^2}\right) + \beta_1 \left(\frac{\sum_{i=1}^{n}(x_i^2 - x_i\overline{X})}{\sum_{i=1}^{n}(x_i - \overline{X})^2}\right)$$

$$= \beta_0 \left(\frac{n\overline{X} - n\overline{X}}{\sum_{i=1}^{n}(x_i - \overline{X})^2}\right) + \beta_1 \left(\frac{\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i\overline{X}}{\sum_{i=1}^{n}(x_i - \overline{X})^2}\right)$$

$$= \beta_1 \left(\frac{\sum_{i=1}^{n} x_i^2 - \overline{X}\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n}\left(x_i^2 - 2x_i\overline{X} + (\overline{X})^2\right)^2}\right) = \beta_1 \left(\frac{\sum_{i=1}^{n} x_i^2 - n(\overline{X})^2}{\sum_{i=1}^{n} x_i^2 - 2n(\overline{X})^2 + n(\overline{X})^2}\right)$$

$$= \beta_1 \left(\frac{\sum_{i=1}^{n} x_i^2 - n(\overline{X})^2}{\sum_{i=1}^{n} x_i^2 - n(\overline{X})^2}\right) = \beta_1$$

2.3 Interval Estimation

Definition: a $(1-\alpha)$ confidence interval (C. I.) estimator is an interval whose end points are functions of the sample statistics such that if we could generate indefinitely samples, the interval should contain the true parameters $(1-\alpha)$ % of the times.

Constructing of C. I.:

The following steps are necessary to construct the C.I.

Step (1): obtain the probability distribution of the point estimator for the unknown parameter.

Step (2): Standardize the estimator such that we get a r. v. with completely known distribution.

Step (3): Construct C.I. for standardized r. v. then Solve for the unknown parameter.

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2.2.1 C.I. for Means of Normal Population

1. If σ^2 is known

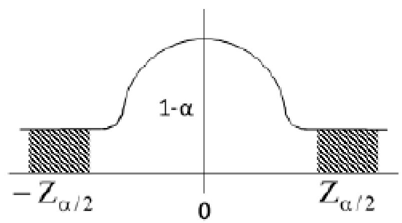
Let $x_1, x_2, \cdots, x_n$ be a r. s. from normal population with unknown mean μ and known variance of σ^2 Applying the above steps as follows:

Step (1): the sample mean $\overline{X}$ is a point estimate of μ with probability distribution

$$N\left(\mu, \frac{\sigma^2}{n}\right).$$

Step (2): Standardize $Z = \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$.

Step (3): The values $-Z_{\frac{\alpha}{2}}$ & $Z_{\frac{\alpha}{2}}$ place $\left(\frac{\alpha}{2}\right)$ in each tail of normal distribution, therefor:



$$Pr\left(-Z_{\frac{\alpha}{2}} < \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} < Z_{\frac{\alpha}{2}}\right) = 1-\alpha$$

$$Pr\left(-Z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}} < \overline{X}-\mu < Z_{\frac{\alpha}{2}}\right) = 1-\alpha$$

$$\therefore Pr\left(\overline{X} - Z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + Z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$$

Where $0 < \alpha < 1$ and selected often to be $0.1, 0.01, or\ 0.05$.

Example: find 95% C.I. for the mean of normal population $N(\mu, 25)$ if it is known that $\overline{X} = 10$ and $n = 100$.

Solution: we have $1-\alpha = 0.95 \Rightarrow \alpha = 0.05 \Rightarrow \frac{\alpha}{2} = 0.025$

From tables of standard distribution, we get $Z_{\frac{\alpha}{2}} = Z_{0.025} = 1.96$

$$Pr\left(-Z_{\frac{\alpha}{2}} < \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} < Z_{\frac{\alpha}{2}}\right) = 1-\alpha \Rightarrow Pr\left(\overline{X} - Z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + Z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$$

$$Pr\left(10 - 1.96 \cdot \frac{5}{\sqrt{100}} < \mu < 10 + 1.96 \cdot \frac{5}{\sqrt{100}}\right) = 0.95$$

$$Pr(9.02 < \mu < 10.98) = 0.95$$

Lower bound ($CL = 9.02$) and upper bound ($CU = 10.98$).

2. <u>If σ^2 is unknown</u>

Let $x_1, x_2, \cdots, x_n$ be a r. s. from normal population with unknown mean μ and unknown variance of σ^2, the we have two options:

a) <u>For Small Samples ($n < 30$)</u>

In the case of ($n < 30$) and unknown (σ^2), then the C.I. for μ can be found as follows:

Since (σ^2) is unknown, so we depend on the sample variance

$$S^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n}\left(x_i - \overline{X}\right)^2\right).$$

$$W = \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1) \text{ and } V = \frac{n\,S^2}{\sigma^2} \sim X^2(n-1), \text{then } T = \frac{W}{\sqrt{V/r}} \sim t(n-1)$$

$$T = \frac{W}{\sqrt{V/r}} = \frac{\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{\frac{n\,S^2}{\sigma^2}}{n-1}}} = \frac{\sqrt{n}(\overline{X}-\mu)}{\sigma} \cdot \frac{\sigma\sqrt{n-1}}{\sqrt{n}\;S} = \frac{(\overline{X}-\mu)}{S/\sqrt{n-1}}$$

$$\therefore T = \frac{\overline{X}-\mu}{S/\sqrt{n}} \sim t(n-1)$$

Applying the steps stated earlier we get:

$$Pr\left(-t_{\frac{\alpha}{2}} < \frac{\overline{X}-\mu}{S/\sqrt{n-1}} < t_{\frac{\alpha}{2}}\right) = 1-\alpha$$

$$Pr\left(-t_{\frac{\alpha}{2}}\cdot\frac{S}{\sqrt{n-1}} < \overline{X}-\mu < t_{\frac{\alpha}{2}}\cdot\frac{S}{\sqrt{n-1}}\right) = 1-\alpha$$

$$\therefore Pr\left(\overline{X}-t_{\frac{\alpha}{2}}\cdot\frac{S}{\sqrt{n-1}} < \mu < \overline{X}+t_{\frac{\alpha}{2}}\cdot\frac{S}{\sqrt{n-1}}\right) = 1-\alpha$$

Example: let $\overline{X} = 20$ & $S^2 = 9$ denote the means and variance of a r. s. of size (16) is from $N(\mu,\sigma^2)$. Find 95% C.I. form (μ).

Solution: we have $1-\alpha = 0.95 \Rightarrow \alpha = 0.05 \Rightarrow \frac{\alpha}{2} = 0.025.$

From tables of T distribution $t_{\frac{\alpha}{2}}(n-1) = t_{0.025}(15) = 2.131$

$$Pr\left(\overline{X}-t_{\frac{\alpha}{2}}\cdot\frac{S}{\sqrt{n-1}} < \mu < \overline{X}+t_{\frac{\alpha}{2}}\cdot\frac{S}{\sqrt{n-1}}\right) = 1-\alpha$$

$$\Rightarrow Pr\left(20-2.131\cdot\frac{3}{\sqrt{15}} < \mu < 20+2.131\cdot\frac{3}{\sqrt{15}}\right) = 0.95$$

$$\Rightarrow Pr(18.349 < \mu < 21.651) = 0.95$$

Another way to represent C.I.

$$C.I. = \overline{X} \mp t_{\frac{\alpha}{2}}\cdot\frac{S}{\sqrt{n-1}} = 20 \mp 2.131\cdot\frac{3}{\sqrt{15}}$$

Lower bound $(CL = 18.349)$ and upper bound $(CU = 21.651)$.

b) <u>For Large Samples $(n \geq 30)$</u>

In this case and form statistical inference theory the distribution of the r. v. $t = \frac{\sqrt{n}(\overline{X}-\mu)}{S}$ will converge to $N(0,1)$.

Which means that we can use the standard normal tables instead of t distribution table and hence C. I. for (μ) can be found as follows:

$$Pr\left(-Z_{\frac{\alpha}{2}} < \frac{\overline{X}-\mu}{S/\sqrt{n}} < Z_{\frac{\alpha}{2}}\right) = 1-\alpha$$

$$Pr\left(-Z_{\frac{\alpha}{2}} \cdot \frac{S}{\sqrt{n}} < \overline{X}-\mu < Z_{\frac{\alpha}{2}} \cdot \frac{S}{\sqrt{n}}\right) = 1-\alpha$$

$$\therefore Pr\left(\overline{X}-Z_{\frac{\alpha}{2}} \cdot \frac{S}{\sqrt{n}} < \mu < \overline{X}+Z_{\frac{\alpha}{2}} \cdot \frac{S}{\sqrt{n}}\right) = 1-\alpha$$

Or we can find C. I. for (μ) in another form as follows:

$$C.I. = \overline{X} \mp Z_{\frac{\alpha}{2}} \cdot \frac{S}{\sqrt{n}}$$

Example: let $\overline{X} = 20$ & $S^2 = 16$ denote the means and variance of a r. s. of size (100) is from $N(\mu, \sigma^2)$. Find 99% C.I. form (μ).

Solution: we have $1 - \alpha = 0.99 \Rightarrow \alpha = 0.01 \Rightarrow \frac{\alpha}{2} = 0.005$

From tables of stander Normal distribution $Z_{\frac{\alpha}{2}} = Z_{0.005} = 2.58$

$$C.I. = \overline{X} \mp Z_{\frac{\alpha}{2}} \cdot \frac{S}{\sqrt{n}} \Rightarrow C.I. = 20 \mp 2.58 \cdot \frac{4}{\sqrt{100}} \Rightarrow C.I. = (18.968\,, 21.032)$$

Lower bound $(CL = 18.968)$ and upper bound $(CU = 21.032)$.

2.2.2 C.I. for Difference Between Two Means

1. If σ_1^2 & σ_2^2 are known

Let $\overline{X}_1$& $\overline{X}_2$ denote the means of two independent random samples of size n_1 & n_2 from normal population with known variances σ_1^2 & σ_2^2 respectively. A $(1-\alpha)$ C.I. for $\mu_1 - \mu_2$ is defining as given:

$$Pr\left(\left(\overline{X}_1-\ \overline{X}_2\right)-Z_{\frac{\alpha}{2}}\cdot\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}<\mu_1-\mu_2<\ \left(\overline{X}_1-\ \overline{X}_2\right)+Z_{\frac{\alpha}{2}}\cdot\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}\right)=1-\alpha$$

$$\text{C.I. for }\ (\mu_1-\mu_2)=\left(\overline{X}_1-\ \overline{X}_2\right)\mp\ Z_{\frac{\alpha}{2}}\ \cdot\ \sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}$$

2. If σ_1^2 & σ_2^2 are unknown

a. For large samples $(n_1\ ,n_2\geq 30)$

Let $\overline{X}_1$& $\overline{X}_2$ denote the means of two independent random samples of size n_1 & n_2 from normal population with unknown variances σ_1^2 & σ_2^2 respectively. A $(1-\alpha)$ C.I. for $\mu_1-\mu_2$ is defining as given:

$$Pr\left(\left(\overline{X}_1-\ \overline{X}_2\right)-Z_{\frac{\alpha}{2}}\cdot\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}}<\mu_1-\mu_2<\ \left(\overline{X}_1-\ \overline{X}_2\right)+Z_{\frac{\alpha}{2}}\cdot\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}}\right)=1-\alpha$$

$$\text{C.I. for }\ (\mu_1-\mu_2)=\left(\overline{X}_1-\ \overline{X}_2\right)\mp\ Z_{\frac{\alpha}{2}}\ \cdot\ \sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}}$$

Where S_1^2 & S_2^2 denoted the sample variance of two samples.

Example: construct 96% C.I. for $(\mu_1-\mu_2)$, if its known that:

$n_1=75, n_2=50, S_1=8\ , S_2=6, \overline{X}_1=82, and\ \overline{X}_1=76.$

Solution: we have $1-\alpha=0.96\Rightarrow\alpha=0.04\Rightarrow\frac{\alpha}{2}=0.02.$

From tables of standard normal distribution $Z_{\frac{\alpha}{2}}=Z_{0.02}=2.054.$

$$\text{C.I. for }\ (\mu_1-\mu_2)=\left(\overline{X}_1-\ \overline{X}_2\right)\mp\ Z_{\frac{\alpha}{2}}\ \cdot\ \sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}}=(82-\ 76)\mp\ 2.054\cdot\sqrt{\frac{64}{75}+\frac{36}{50}}$$

$$=6\mp\ 2.054\cdot\sqrt{\frac{64}{75}+\frac{36}{50}}\Rightarrow C.I.=(3.424\ ,8.576)$$

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Lower bound ($CL = 3.424$) and upper bound ($CU = 8.576$)

b. For small samples ($n_1, n_2 < 30$)

Let $\overline{X}_1 \& \overline{X}_2$ denote the means of two independent random samples of size n_1 & n_2 from normal population with unknown variances σ_1^2 & σ_2^2 respectively. A $(1-\alpha)$ C.I. for $\mu_1 - \mu_2$ is defining as given:

$$Pr\left(\left(\overline{X}_1 - \overline{X}_2\right) - t_{\frac{\alpha}{2}} \cdot S_p \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} < \mu_1 - \mu_2 < \left(\overline{X}_1 - \overline{X}_2\right) - t_{\frac{\alpha}{2}} \cdot S_p \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right) = 1 - \alpha$$

$$\text{C.I. for } (\mu_1 - \mu_2) = \left(\overline{X}_1 - \overline{X}_2\right) \mp t_{\frac{\alpha}{2}} \cdot S_p \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$\left(t_{\frac{\alpha}{2}}\right)$ could be found from t distribution table with degree of freedom $(n_1 + n_2 - 2)$.

Where $\left(S_p^2\right)$ is the pooled variance obtained from the sample variances S_1^2 & S_2^2 and define as follows:

$$S_p^2 = \frac{(n_1 - 1)\, S_1^2 + (n_2 - 1)\, S_2^2}{n_1 + n_2 - 2}$$

Example: construct 90% C.I. for $(\mu_1 - \mu_2)$, if its known that:

$n_1 = 12, n_2 = 10, S_1 = 4\ , S_2 = 5, \overline{X}_1 = 85, and\ \overline{X}_1 = 81.$

Solution: we have $1 - \alpha = 0.90 \Rightarrow \alpha = 0.1 \Rightarrow \frac{\alpha}{2} = 0.05.$

From tables of t distribution $t_{\frac{\alpha}{2}}(n_1 + n_2 - 2) = t_{0.05}(20) = 1.725$

$$S_p^2 = \frac{(n_1 - 1)\, S_1^2 + (n_2 - 1)\, S_2^2}{n_1 + n_2 - 2} = \frac{(12-1)16 + (10-1)25}{12 + 10 - 2} = 20.05$$

$$S_p = 4.478$$

$$\text{C.I. for } (\mu_1 - \mu_2) = (\overline{X}_1 - \overline{X}_2) \mp t_{\frac{\alpha}{2}} \cdot S_p \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$$= (85 - 81) \mp (1.725) \cdot (4.478) \cdot \sqrt{\frac{1}{12} + \frac{1}{10}}$$

$$= 4 \mp (1.725) \cdot (4.478) \cdot (0.428) = 4 \mp 3.307 \Rightarrow C.I. = (0.693\,, 7.307)$$

Lower bound ($CL = 0.693$) and upper bound ($CU = 7.307$)

2.2.3 C.I. for Variance σ^2

a. If mean μ is known

$(1-\alpha)\%$ C.I for (σ^2) is given by:

$$Pr\left(\frac{n\,S^2}{X^2_{1-\frac{\alpha}{2}}(n)} < \sigma^2 < \frac{n\,S^2}{X^2_{\frac{\alpha}{2}}(n)}\right) = 1 - \alpha$$

$$C.I \text{ for } \sigma^2 = \left(\frac{n\,S^2}{X^2_{1-\frac{\alpha}{2}}(n)}\,, \frac{n\,S^2}{X^2_{\frac{\alpha}{2}}(n)}\right)$$

Where $X^2_{1-\frac{\alpha}{2}}$ & $X^2_{\frac{\alpha}{2}}$ are the Chi Square values are obtained from Chi Square distribution table with (n) degrees of freedom and level of significant $\left(1-\frac{\alpha}{2}\right)$ & $\left(\frac{\alpha}{2}\right)$.

Example: let $S^2 = 9$ denoted the variance of a r. s. of size 25 from $N(10, \sigma^2)$, find 95% C.I for (σ^2).

Solution: we have $1 - \alpha = 0.95 \Rightarrow \alpha = 0.05 \Rightarrow \frac{\alpha}{2} = 0.025 \Rightarrow 1 - \frac{\alpha}{2} = 0.975$

$$X^2_{1-\frac{\alpha}{2}}(n) = X^2_{0.975}(25) = 40.6465$$

$$X^2_{1-\frac{\alpha}{2}}(n) = X^2_{0.025}(25) = 13.1197$$

$$Pr\left(\frac{n\,S^2}{X^2_{1-\frac{\alpha}{2}}(n)} < \sigma^2 < \frac{n\,S^2}{X^2_{\frac{\alpha}{2}}(n)}\right) = 1-\alpha \Rightarrow Pr\left(\frac{25\cdot 9}{40.6465} < \sigma^2 < \frac{25\cdot 9}{13.1197}\right) = 0.95$$

$$\Rightarrow Pr(5.5355 < \sigma^2 < 17.1498) = 0.95 \Rightarrow C.I. = (5.5355\,, 17.1498)$$

Lower bound ($CL = 5.5355$) and upper bound ($CU = 17.1498$)

b. <u>If mean μ is unknown</u>

$(1-\alpha)$% C.I for (σ^2) is given by:

$$Pr\left(\frac{(n-1)\,S^2}{X^2_{1-\frac{\alpha}{2}}(n-1)} < \sigma^2 < \frac{(n-1)\,S^2}{X^2_{\frac{\alpha}{2}}(n-1)}\right) = 1-\alpha$$

$$C.I \text{ for } \sigma^2 = \left(\frac{(n-1)\,S^2}{X^2_{1-\frac{\alpha}{2}}(n-1)}\,, \frac{(n-1)\,S^2}{X^2_{\frac{\alpha}{2}}(n-1)}\right)$$

Where $X^2_{1-\frac{\alpha}{2}}$ & $X^2_{\frac{\alpha}{2}}$ are the Chi Square values are obtained from Chi Square distribution table with $(n-1)$ degrees of freedom and level of significant $\left(1-\frac{\alpha}{2}\right)$ & $\left(\frac{\alpha}{2}\right)$.

Example: let $x_1, x_2, \cdots, x_{10}$ be a r. s. from normal population from $N(\mu, \sigma^2)$, where μ & σ^2 are unknown, suppose that $\sum_{i=1}^{10} x_i = 159$ & $\sum_{i=1}^{10} x_i^2 = 2531$, compute 95% C.I. for ($\sigma^2$) if it's known that

$X^2_{0.025}(9) = 2.7$ & $X^2_{0.975}(9) = 19$.

Solution: we have $1-\alpha = 0.95 \Rightarrow \alpha = 0.05 \Rightarrow \frac{\alpha}{2} = 0.025 \Rightarrow 1-\frac{\alpha}{2} = 0.975$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \overline{X}\right)^2 \Rightarrow (n-1)S^2 = \sum_{i=1}^{n}\left(x_i - \overline{X}\right)^2$$

$$9 \cdot S^2 = \sum_{i=1}^{10} \left(x_{i_i}^2 - 2x_i\overline{X} + \left(\overline{X}\right)^2 \right) = \sum_{i=1}^{10} x_i^2 - 2\overline{X} \sum_{i=1}^{10} x_i + n\left(\overline{X}\right)^2$$

$$= 2531 - 2\,\overline{X} \cdot n\overline{X} + n\left(\overline{X}\right)^2 = 2531 - 2\,n\left(\overline{X}\right)^2 + n\left(\overline{X}\right)^2$$

$$= 2531 - n\left(\overline{X}\right)^2 = 2531 - 10\left(\frac{\sum_{i=1}^{10} x_i}{10}\right)^2 2531 - 10\left(\frac{159}{10}\right)^2$$

$$= 2531 - 2528.1 = 2.9$$

$$\therefore 9 \cdot S^2 = 2.9$$

$$Pr\left(\frac{(n-1)\,S^2}{X^2_{1-\frac{\alpha}{2}}(9)} < \sigma^2 < \frac{(n-1)\,S^2}{X^2_{\frac{\alpha}{2}}(9)} \right) = 1 - \alpha$$

$$Pr\left(\frac{2.9}{X^2_{0.975}(9)} < \sigma^2 < \frac{2.9}{X^2_{0.025}(9)} \right) = 0.95 \Rightarrow Pr\left(\frac{2.9}{19} < \sigma^2 < \frac{2.9}{2.7} \right) = 0.95$$

$$\Rightarrow Pr(0.1526 < \sigma^2 < 1.0740) = 0.95$$

$$\therefore C.I. = (0.1526\,, 1.0740)$$

Lower bound ($CL = 0.1526$) and upper bound ($CU = 1.0740$).

c. <u>C.I. for the Ration of Two Variances</u>

Where S_1^2 & S_2^2 denoted the sample variance of two independent random samples of size n_1 & n_2 respectively

Let $v_1 = n_1 - 1$ & $v_2 = n_2 - 1$ be the degree of freedoms, then

. A $(1-\alpha)$ C.I. for $\left(\frac{\sigma_1^2}{\sigma_2^2}\right)$ is defining as given:

$$Pr\left(\frac{S_1^2}{S_2^2 \cdot f_{\frac{\alpha}{2}}(v_1, v_2)} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{S_1^2 \cdot f_{\frac{\alpha}{2}}(v_2, v_1)}{S_2^2} \right) = 1 - \alpha$$

The values of $f_{\frac{\alpha}{2}}(v_1, v_2)$ and $f_{\frac{\alpha}{2}}(v_2, v_1)$ are obtained from the F distribution table.

$$\text{C.I. for } (\mu_1 - \mu_2) = (\overline{X}_1 - \overline{X}_2) \mp Z_{\frac{\alpha}{2}} \cdot \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}$$

Example: find 98% C.I. for $\left(\frac{\sigma_1^2}{\sigma_2^2}\right)$ if it's known that $n_1 = 25, n_2 = 16, S_1 = 8, \& S_2 = 7.$

Solution: we have $1 - \alpha = 0.98 \Rightarrow \alpha = 0.02 \Rightarrow \frac{\alpha}{2} = 0.01$ & $v_1 = 24, v_2 = 15$

$$f_{\frac{\alpha}{2}}(v_1, v_2) = f_{0.01}(24{,}15) = 3.29$$

$$f_{\frac{\alpha}{2}}(v_2, v_2) = f_{0.01}(15{,}24) = 2.89$$

$$Pr\left(\frac{S_1^2}{S_2^2 \cdot f_{\frac{\alpha}{2}}(v_1, v_2)} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{S_1^2 \cdot f_{\frac{\alpha}{2}}(v_2, v_1)}{S_2^2}\right) = 1 - \alpha$$

$$\Rightarrow Pr\left(\frac{64}{49 \cdot 3.29} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{64 \cdot 2.89}{49}\right) = 0.98$$

$$\Rightarrow Pr\left(0.397 < \frac{\sigma_1^2}{\sigma_2^2} < 3.775\right) = 0.98$$

$$\therefore C.I. = (0.397, 3.775)$$

Lower bound ($CL = 0.397$) and upper bound ($CU = 3.775$).

Exercises (5)

1. If it is known that $n = 17$ is the size of r.s. from $N(\mu, \sigma^2)$ with $\overline{X} = 5.3$, $S^2 = 6.2$, Find 95% C.I. for both μ & σ^2. The tabulated values are:
   $t_{0.025}(16) = 2.120, X_{0.975}^2(16) = 28.8, X_{0.025}^2(16) = 6.91.$
2. Given $\overline{X} = 18$, is the mean of a r. s. of size 20 from $N(\mu, \sigma^2)$. Find 99% C.I. for μ if it is known that $Z_{0.005} = 2.58$.
3. A r. s. of size 10 is drawn from $N(\mu, \sigma^2)$. The values of individuals are 10.7, 12.6, 9.3, 9.5, 11.3, 12.2, 11.5, 11.1, 10.4 and 10.2. Find 95% C.I. for both μ & σ^2, $t_{0.025}(9) = 2.262$.

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4. Two random samples each of size 10 from $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$ yield $\overline{X}_1 = 4.8$, $\overline{X}_2 = 5.6$, $S_1^2 = 8.64$, $S_2^2 = 7.88$, find 95% for C.I. $(\mu_1 - \mu_2)$ if it's known $t_{0.025}(18) = 2.101$.
5. let $x_1, x_2, \cdots, x_{10}$ be a r. s. from normal population from $N(\mu, \sigma^2)$, let $0 < a < b$, show that the mathematical expectation of the length of random interval

$$\left[\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{b}, \frac{\sum_{i=1}^{n}(x_i - \mu)^2}{a}\right] \text{ is } (b - a)\left(\frac{n\,\sigma^2}{ab}\right)$$

Chapter Three

Testing of Hypothesis

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1. مقدمة عن اختبار الفرضيات Introduction to Test of Hypothesis

الفرضية الاحصائية: الفرضية الإحصائية هي عبارة عن تخمين أو تأكيد عن التوزيعات التي تحتوي على متغير عشوائي واحد أو أكثر. بعبارة أخرى يمكن القول بأن الفرضية الإحصائية هي تصريح أو ادعاء (قد يكون صائباً أو خاطئا) يتعلق بالتوزيع الاحتمالي (المجتمع الإحصائي) لمتغير عشوائي.

يقال عن الفرضية الاحصائية H بانها بسيطة (simple hypothesis) اذا وصفت التوزيع الاحتمالي بشكل متكامل, بعبارة اخرى اذا كانت H تصف او تعين التوزيع الاحتمالي باكمله, وبخلاف ذلك يقال عن H بانها فرضية مركبة (composite hypothesis).

ان اختبار الفرضية الاحصائية H هو عبارة عن طريقة او قاعدة لاتخاذ القرار بشان H وهذا القرار ذو حدين بمعنى اخر قبول الفرضية (acceptance) او رفض الفرضية (reject).

ان قبول الفرضية لايعني بالضرورة انها صحيحة دائما وانما لاتوجد ادلة (من بيانات العينة) كافية لرفضها. اما اذا تم رفض الفرضية الاحصائية بناءا على المعلومات المتوفرة في العينة موضوع الدراسة فهذا يعني انها خاطئة, لذلك فان الباحث يحاول دائما ان يضع الفرضيات بشكل معين على امل ان يتم رفضها.

مثال على ذلك اذا أراد الباحث أن يبرهن أن طريقة جديدة في طرق التدريس أحسن من طرق التدريس السابقة فأن يضع فرضية تقول لايوجد فرق بين طرق التدريس.

ان الفرضية الاحصائية التي يضعها الباحث على امل رفضها تسمى بفرضية العدم (null hypothesis) ورمز لها بالرمز (H_0) , ان الرفض لفرضية العدم يقود الى قبول فرضية اخرى تسمى بالفرضية البديلة (alternative hypothesis) ورمز لها بالرمز (H_1).

بشكل عام لاي اختبار احصائي توجد فرضيتان (H_0, H_1) وان الاهتمام يكون منصب على اختبار (H_0) ضد (H_1), وعليه من الناحية العملية يتوجب ملاحظة من هي (H_0) ومن هي (H_1) على ضوء المشكلة موضوع الدراسة.

مثال: الفرضية الاحصائية $(H_0: \mu = \mu_0)$ يمكن ان تختبر ضد احد البدائل المثالية:

1. $H_1: \mu < \mu_0$
2. $H_1: \mu > \mu_0$
3. $H_1: \mu \neq \mu_0$

يقال عن الاختبار اعلاه بانه من طرف واحد (one-tailed test) اذا كانت (H_1) تاخذ احد الشكلين (1) او (2).

اختبار من الطرف الايسر (left tailed test) اذا كانت $H_1: \mu < \mu_0$.

اختبار من الطرف الايمن (right tailed test) اذا كانت $H_1: \mu < \mu_0$.

ويقال عنه اختبار من طرفين (two-tailed test) اذا كانت (H_1) تاخذ الشكل (3), اي ان $H_1: \mu \neq \mu_0$.

يراعى قدر الامكان عند صياغة (H_0) ان تكون فرضية بسيطة و الابتعاد قدر الامكان عن صياغتها بشكل فرضية مركبة وحسب المشكلة قيد الدراسة.

1.1 الخطأ من النوع الاول Type I Error

يعرف الخطا من النوع الاول (type I error) بانه الخطا الحاصل من رفض (H_0) عندما تكون (H_0) صحيحة.

1.2 الخطأ من النوع الثاني Type II Error

يعرف الخطا من النوع الثاني (type II error) بسبب قبول (H_0) عندما تكون (H_0) خاطئة.

1.4 إحصاءة الاختبار Test Statistic

عبارة عن متغير عشوائي بدلالة مفردات العينة وله توزيع احتمالي معلوم ويصف العلاقة بين القيم النظرية للمجتمع والقيم المحسوبة عن العينة, مثلا $\left(\sum_{i=1}^{n} x_i , \overline{X}\right)$

1.4 منطقة الرفض أو المنطقة الحرجة Critical Region

هي تلك المنطقة التي إذا وقعت فيها قيمة إحصاءة الاختبار وأدت إلى رفض فرضية العدم(H_0) ويرمز لها بالرمز C.

1.5 منطقه القبول Acceptance Region

هي تلك المنطقة التي إذا وقعت فيها قيمة إحصاءة الاختبار وأدت إلى عدم رفض فرضية العدم(H_0) ويرمز لها بالرمز R.

1.6 حجم الاختبار أو مستوى المعنوية The Size of the Test or Level of Significant

وهو احتمال الوقوع في الخطأ من النوع الأول ويرمز له (α) اي انه:

$$\alpha = Pr(Type\ I\ Error) = Pr(reject\ H_0 \mid H_0\ is\ true) = Pr(x_1, x_2, \cdots, x_n \in C \mid H_0)$$

1.7 حجم الخطأ من النوع الثاني The Size of Type II Error

ويعرف كالآتي:

$$\beta = Pr(Type\ II\ Error) = Pr(accept\ H_0 \mid H_1\ is\ true) = Pr(x_1, x_2, \cdots, x_n \in C^c \mid H_1)$$

حيث ان (C^c) تمثل المنطقة المكملة لمنطقة الرفض (C) وهي منطقة قبول فرضية العدم $(\boldsymbol{H_0})$.

1.8 قوة الاختبار The Power of the Test

وهو احتمال رفض فرضية العدم $(\boldsymbol{H_0})$ عندما تكون خاطئة (اي ان الفرضية البديلة $\boldsymbol{H_1}$ هي صحيحة) ويرمز له بالرمز $\boldsymbol{K(\theta)}$ **اي ان:**

$$K(\theta) = Pr(reject\ H_0 \mid H_1\ is\ true) = Pr(x_1, x_2, \cdots, x_n \in C \mid H_1)$$
$$= 1 - Pr(x_1, x_2, \cdots, x_n \in C^c \mid H_1) = 1 - \beta$$

Example: let X be a r. v. having the following p.d.f.:

$$f(x) = \begin{cases} \theta^x (1-\theta)^{1-x} & x = 0,1 \\ 0 & O.W. \end{cases}$$

To test the simple hypothesis $\left(H_0: \theta = \frac{1}{4}\right)$ against the alternative hypothesis $\left(H_1: \theta < \frac{1}{4}\right)$.

Suppose that critical region is $C = \left\{x_1, x_2, \cdots, x_{10}, \sum_{i=1}^{10} x_i \leq 1\right\}$, find the following:

1. The power of the test $K(\theta)$
2. The power of the test at $\left(\theta = \frac{1}{16}\right)$.
3. $Pr(Type\ II\ Error)$ at $\left(\theta = \frac{1}{16}\right)$.
4. The significant level (α).

Solution: since $X \sim Ber(1, \theta)$, then $y = \sum_{i=1}^{10} x_i \sim b(10, \theta)$.

$$f(y) = \begin{cases} \binom{10}{y} \theta^{y} (1-\theta)^{10-y} & x = 0,1,\cdots,10 \\ 0 & O.W. \end{cases}$$

1.

$$K(\theta) = Pr(reject\ H_0 \mid H_1\ is\ true) = Pr(x_1, x_2, \cdots, x_{10} \in C \mid H_1)$$
$$= Pr\left(y \le 1 \mid \theta < \frac{1}{4}\right) = Pr(y=0) + Pr(y=1)$$
$$= \binom{10}{0} \theta^0 (1-\theta)^{10-0} + \binom{10}{1} \theta^1 (1-\theta)^{10-1}$$
$$= (1-\theta)^{10} + 10\,\theta\,(1-\theta)^9 = (1-\theta)^9 (1 - \theta + 10\theta)$$
$$= (1-\theta)^9 (1+9\theta), \left(0 < \ \theta < \frac{1}{4}\right)$$

2.

$$K\left(\frac{1}{16}\right) = \left(1 - \frac{1}{16}\right)^9 \left(1 + \frac{9}{16}\right) = \left(\frac{15}{16}\right)^9 \left(\frac{25}{16}\right)$$

3.

$$\beta\left(\frac{1}{16}\right) = 1 - K\left(\frac{1}{16}\right) = 1 - \left(\frac{15}{16}\right)^9 \left(\frac{25}{16}\right)$$

4.

$$\alpha = Pr(Type\ I\ Error) = Pr(reject\ H_0 \mid H_0\ is\ true)$$
$$= Pr\left(x_1, x_2, \cdots, x_{10} \in C \mid \theta = \frac{1}{4}\right) = Pr\left(y \le 1 \mid \theta = \frac{1}{4}\right)$$
$$= Pr(y=0) + Pr(y=1) = \binom{10}{0} \theta^0 (1-\theta)^{10-0} + \binom{10}{1} \theta^1 (1-\theta)^{10-1}$$
$$= (1-\theta)^9 (1+9\theta) = \left(1 - \frac{1}{4}\right)^9 \left(1 + \frac{9}{4}\right) = \left(\frac{3}{4}\right)^9 \left(\frac{13}{4}\right)$$

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Example: Suppose that $x_1, x_2, \cdots, x_n$ are a random sample from $N(\theta, 1)$, to test the simple hypothesis ($H_0: \theta = 0$) against the simple alternative hypothesis ($H_1: \theta = 1$), the critical region is:

$C = \{x_1, x_2, \cdots, x_n: \overline{X} \geq k\}$, find n and k such that $\alpha = \beta = 0.01$, if $Z_{0.99} = 2.33$.

Solution:

$$X \sim N(\theta, 1) \Rightarrow \overline{X} \sim N\left(\theta, \frac{1}{n}\right) \quad \text{Since } X \sim N(\mu, \sigma^2) \Rightarrow \overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$\alpha = Pr(Type\ I\ Error) = Pr(reject\ H_0 \mid H_0\ is\ true) = Pr\big(\overline{X} \geq k, \theta = 0\big)$$

$$= Pr\left(\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \geq \frac{k - \mu}{\sigma/\sqrt{n}}\right) = Pr\left(Z \geq \frac{k - 0}{1/\sqrt{n}}, \theta = 0\right) = Pr\big(Z \geq k\sqrt{n}\big)$$

$$= 1 - Pr\big(Z < k\sqrt{n}\big)$$

And since ($\alpha = 0.01$), then we have:

$$1 - Pr\big(Z < k\sqrt{n}\big) = 0.01 \Rightarrow Pr\big(Z < k\sqrt{n}\big) = 0.99$$

$$\therefore k\sqrt{n} = 2.33 \ \cdots (1)$$

Now, we have:

$$\beta = Pr(Type\ II\ Error) = Pr(accept\ H_0 \mid H_1\ is\ true) = Pr(x_1, x_2, \cdots, x_n \in C^c \mid H_1)$$

$$= Pr\big(\overline{X} < k, \theta = 1\big) = Pr\left(\frac{\overline{X} - \mu}{\sigma^2/n} < \frac{k - \mu}{\sigma^2/\sqrt{n}},\right)$$

$$= Pr\left(Z < \frac{k - 1}{1/\sqrt{n}}, \theta = 1\right) = Pr\big(Z < (k - 1)\sqrt{n}\big)$$

And since ($\beta = 0.01$), then we have:

$$Pr\big(Z < (k - 1)\sqrt{n}\big) = 0.01$$

$$\therefore (k-1)\sqrt{n} = -2.33 \cdots (2)$$

From eq. (2), we can get:

$$\sqrt{n} = \frac{-2.33}{k-1} \cdots (3)$$

And by substitute eq. (3) in eq. (1) we can get:

$$k\left(\frac{-2.33}{k-1}\right) = 2.33 \Rightarrow -2.33k = 2.33(k-1) \Rightarrow -2.33k = 2.33k - 2.33$$

$$\Rightarrow 4.66k = 2.33$$

$$\therefore k = 0.5$$

By substitute the value of (k) in eq. (1) we can get:

$$0.5\sqrt{n} = 2.33 \Rightarrow \sqrt{n} = \frac{2.33}{0.5} \Rightarrow \sqrt{n} = 4.66$$

$$n = 21.715 \Rightarrow \therefore n \cong 22$$

Example: The consumption of electricity in a small township is assumed to be Exponentially distributed with parameter (θ). Determine the size of type I and type II errors if $(H_0: \theta = 1000\ K.W)$ is tested against $(H_1: \theta = 2000\ K.W)$ and if the test criterion is as follows:

Select any day at random. If the consumption of that day is 4000 K.W. or more, reject (H_0), otherwise accept (H_0).

Solution: The critical region is $C = \{x_1, x_2, \cdots, x_n : x_i \geq 4000\}$

$$\text{Since } X \sim Exp(\theta), \text{then } F(x) = \frac{1}{\theta} e^{\frac{-x}{\theta}}, x \geq 0 \ \&\ \theta > 0$$

$$\alpha = Pr(Type\ I\ Error) = Pr(reject\ H_0 \mid H_0\ is\ true) = Pr(x \geq 4000, \theta = 1000)$$

$$\alpha = \int_{4000}^{\infty} \frac{1}{1000} e^{\frac{-x}{1000}}\, dx = -e^{\frac{-x}{1000}} \Big|_{4000}^{\infty} = -e^{-\infty} + -e^{\frac{-4000}{1000}} = e^{-4}$$

$$\beta = Pr(Type\ II\ Error) = Pr(accept\ H_0 \mid H_1\ is\ true) = Pr(x_1, x_2, \cdots, x_n \in C^c \mid H_1)$$

$$= Pr(x < 4000, \theta = 2000)$$

$$\beta = \int_{0}^{4000} \frac{1}{2000} e^{\frac{-x}{2000}}\, dx = -e^{\frac{-x}{2000}} \Big|_{0}^{4000} = -e^{\frac{-4000}{2000}} + e^{0} = 1 - e^{-2}$$

Example: Let $x_1, x_2, \cdots, x_n$ be a r. s. of size 25 from $N(\mu, 36)$. We shall reject $H_0: \mu = 75\ and\ accept\ H_1: \mu = 80$ if and only if $x > c,$ where c is constant. Find the value of c at ($\alpha = 0.01$) if it is known that ($Z_{0.99} = 2.33$).

Solution: since $X \sim N(\mu, \sigma^2) \Rightarrow f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\ e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty$

$$X \sim N(\mu, \sigma^2) \Rightarrow Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1/n)$$

$$\alpha = Pr(Type\ I\ Error) = Pr(reject\ H_0 \mid H_0\ is\ true) = Pr(x > c, \mu = 75)$$

$$= Pr\left(\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} > \frac{c - \mu}{\sigma/\sqrt{n}}, \mu = 75\right) = Pr\left(Z > \frac{c - 75}{6/5}\right)$$

$$= 1 - Pr\left(Z \le \frac{c - 75}{6/5}\right)$$

$$1 - Pr\left(Z \le \frac{c - 75}{6/5}\right) = 0.01 \Rightarrow Pr\left(Z \le \frac{c - 75}{6/5}\right) = 0.99 \Rightarrow \frac{c - 75}{6/5} = 2.33$$

$$\Rightarrow c = \frac{6}{5} \cdot 2.33 + 75 = 77.796$$

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$$\therefore c = 77.796$$

Exercises (6)

1. Let $x_1, x_2, \cdots, x_n$ be a random sample from Poisson (θ). The critical region for testing $\left(H_0 : \theta = \frac{1}{10}\right)$ against $\left(H_1 : \theta > \frac{1}{10}\right)$
a. Find the power of the test $K(\theta)$.
b. Find the level of significant (α).
2. Consider a normal distribution $N(\theta, 4)$. The simple hypothesis $(H_0 : \theta = 0)$ is rejected and the alternative composite hypothesis $(H_1 : \theta > 0)$ is accepted if and only if the observed mean $(\overline{X})$ of a random sample of size 25 is greater than or equal to$\left(\frac{3}{5}\right)$, find the power of the test.

2. Best Critical Region أفضل منطقة حرجة

يقال عن المنطقة C انها افضل منطقة حرجة (BCR) بحجم (α) عند اختبار فرضية العدم البسيطة $(H_0 : \theta = \theta_0)$ مقابل الفرضية البديلة $(H_1 : \theta = \theta_1)$ اذا وجدت منطقة حرجة اخرى مثل A بحيث ان الشرطين التالين متحققان:

1. $Pr(x_1, x_2, \cdots, x_n \in A \mid H_0) = Pr(x_1, x_2, \cdots, x_n \in C \mid H_0) = \alpha$
2. $Pr(x_1, x_2, \cdots, x_n \in A \mid H_1) \le Pr(x_1, x_2, \cdots, x_n \in C \mid H_0)$

Example: consider the r. v. $X \sim b(5,0)$ under $\left(H_0 : \theta = \frac{1}{2}\right)$ and $\left(H_1 : \theta = \frac{3}{4}\right)$, the following table gives the density values of (x) under $(H_0$ & $H_1)$

x	0	1	2	3	4	5
$f\left(x, \frac{1}{2}\right)$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$
$f\left(x, \frac{3}{4}\right)$	$\frac{1}{1024}$	$\frac{15}{1024}$	$\frac{90}{1024}$	$\frac{270}{1024}$	$\frac{405}{1024}$	$\frac{243}{1024}$

Assuming that $\left(\alpha = \frac{1}{32} = Pr(reject\ H_0 : H_0)\right)$, find the BCR.

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Solution: since $\left(\alpha = \frac{1}{32}\right)$, we have two critical regions which are:

$$A = \{x; x = 0\} \text{ and } C = \{x; x = 5\}$$

Now, we satisfied the two conditions:

$$Pr(x_1, x_2, \cdots, x_n \in A \mid H_0) = Pr(x_1, x_2, \cdots, x_n \in C \mid H_0) = \alpha$$

$$Pr(x \in A \mid H_0) = Pr(x \in C \mid H_0) = \frac{1}{32}$$

$$Pr(x \in A \mid H_1) = \frac{1}{1024}$$

$$Pr(x \in C \mid H_0) = \frac{243}{1024}$$

$$Pr(x \in A \mid H_1) = \frac{1}{1024} < \ Pr(x \in C \mid H_0) = \frac{243}{1024}$$

$$\therefore C \text{ is Best Critical Region } (BCR)$$

Example: consider the r. v. $X \sim b(5,0)$ under $\left(H_0 : \theta \ = \frac{1}{2}\right)$ and $\left(H_1 : \theta \ = \frac{3}{4}\right)$, the following table gives the density values of (x) under $(H_0 \ \& \ H_1)$

x	0	1	2	3	4	5
$f\left(x, \frac{1}{2}\right)$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$
$f\left(x, \frac{3}{4}\right)$	$\frac{1}{1024}$	$\frac{15}{1024}$	$\frac{90}{1024}$	$\frac{270}{1024}$	$\frac{405}{1024}$	$\frac{243}{1024}$

Assuming that $\left(\alpha = \frac{6}{32} = Pr(reject\ H_0 : H_0)\right)$, find the BCR.

Solution: since $\left(\alpha = \frac{6}{32}\right)$, we have four critical regions which are:

$$C_1 = \{x; x = 0,1\}, C_2 = \{x; x = 0,4\}, C_3 = \{x; x = 1,5\}, \text{and } C_4 = \{x; x = 4,5\}$$

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$$Pr(x \in C_1 \mid H_0) = Pr(x \in C_2 \mid H_0) = Pr(x \in C_3 \mid H_0) = Pr(x \in C_4 \mid H_0) = \frac{6}{32}$$

$$Pr(x \in C_1 \mid H_1) = Pr\left(x = 0,1 \mid \theta = \frac{3}{4}\right) = \frac{1}{1024} + \frac{15}{1024} = \frac{16}{1024}$$

$$Pr(x \in C_2 \mid H_0) = Pr\left(x = 0,4 \mid \theta = \frac{3}{4}\right) = \frac{1}{1024} + \frac{405}{1024} = \frac{406}{1024}$$

$$Pr(x \in C_3 \mid H_1) = Pr\left(x = 1,5 \mid \theta = \frac{3}{4}\right) = \frac{15}{1024} + \frac{243}{1024} = \frac{258}{1024}$$

$$Pr(x \in C_4 \mid H_0) = Pr\left(x = 4,5 \mid \theta = \frac{3}{4}\right) = \frac{405}{1024} + \frac{243}{1024} = \frac{648}{1024}$$

$\therefore C_4$ is Best Critical Region (BCR)

Solving of Exercises (1)

1. Given $X \sim Ber\left(1, \frac{1}{3}\right)$, find the following:

   iii. The p.m.f of x?

   iv. $M_x(t), \sigma_x^2,$ and μ_x?

Solution: since $X \sim Ber\left(1, \frac{1}{3}\right)$, then

i. $f(x) = \left(\frac{1}{3}\right)^x \left(1 - \frac{1}{3}\right)^{1-x} = \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^x, x = 0,1$

ii. $M_x(t) = 1 - p + p \cdot e^t = 1 - \frac{1}{3} + \frac{1}{3} e^t = \frac{2}{3} + \frac{1}{3} e^t$

$$\sigma_x^2 = p(1-p) = \frac{1}{3}\left(1 - \frac{1}{3}\right) = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$$

$$\mu_x = p = \frac{1}{3}$$

2. he m. g. f of ar.v x is $M_x(t) = \left(\frac{2}{3} + \frac{1}{3} e^t\right)^9$, find the following:

   iv. The p.mf of x?

   v. $\sigma_x^2,$ and μ_x?

   vi. Show that $Pr(\mu_x - 2\sigma_x < x < \mu_x + 2\sigma_x) = \sum_{i=1}^{n} x^9 \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{9-x}$?

Solution: since $M_x(t) = \left(\frac{2}{3} + \frac{1}{3} e^t\right)^9 \Rightarrow X \sim b\left(9, \frac{1}{3}\right), \& \; n = 9$

i. .

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x} = \binom{9}{x} \left(\frac{1}{3}\right)^x \left(1 - \frac{1}{3}\right)^{9-x} = \binom{9}{x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{9-x} \qquad x$$
$$= 0,1, \cdots, 9$$

ii. .

$$\mu_x = E(x) = np = 9 \cdot \frac{1}{3} = 3$$

$$\sigma_x^2 = np(1-p) = 3\cdot\left(1-\frac{1}{3}\right) = 3\cdot\frac{2}{3} = 2$$

3. Let $X \sim b(2,p)$ and $Y \sim b(4,p)$, if $Pr(X \geq 1) = \frac{5}{9}$, find $Pr(Y \geq 1)$?

Solution: $X \sim b(2,p), n = 2$

$$Pr(X \geq 1) = 1 - Pr(X < 1) = 1 - Pr(X = 0) = 1 - f(0) = 1 - \binom{2}{0} p^0 (1-p)^{2-0}$$

$$= 1 - (1-p)^2$$

And since $Pr(X \geq 1) = \frac{5}{9}$, then we have:

$$1-(1-p)^2 = \frac{5}{9} \Rightarrow (1-p)^2 = 1 - \frac{5}{9} \Rightarrow (1-p)^2 = \frac{4}{9} \Rightarrow 1-p = \frac{2}{3} \quad \Rightarrow p = \frac{1}{3}$$

$$Pr(Y \geq 1) = 1 - Pr(Y < 1) = 1 - Pr(Y = 1) = 1 - g(0)$$

$$= 1 - \binom{4}{0}\left(\frac{1}{3}\right)^0\left(1-\frac{1}{3}\right)^{4-0} = 1 - \left(\frac{2}{3}\right)^4 = 1 - \frac{16}{81} = \frac{65}{81}$$

4. Let X_1 and X_2 are independent r.vs such that $X_1 \sim P(4)$ and $X_2 \sim P(6)$, if $Y = X_1 + X_2$ find the following:

iv. The p.m.f of Y?

v. σ_y^2, and μ_y?

vi. $Pr(Y \leq 1)$?

Solution:

$$M_Y(t) = E(e^{tY}) = E\left(e^{t(X_1+X_{2,})}\right) = E(e^{tX_1}\cdot e^{tX_2}) = E(e^{tX_1})\cdot E(e^{tX_2})$$

$$= e^{4(e^t-1)}\cdot e^{6(e^t-1)} = e^{4(e^t-1)+6(e^t-1)} = e^{(4+6)(e^t-1)} = e^{10(e^t-1)}$$

$$\therefore Y \sim P(10)$$

i. $f(y) = \frac{e^{-10}\,10^y}{y!}, y = 0,1,2,\cdots$

ii. $\mu_x = \sigma_x^2 = 10$

iii.

$$Pr(Y \leq 1) = Pr(Y = 0) + Pr(Y = 1) = f(0) + f(1) = \frac{e^{-10}\,10^0}{0!} + \frac{e^{-10}\,10^1}{1!}$$

$$= e^{-10} + 10e^{-10} = 11e^{-10}$$

5. Given $X \sim P(\lambda)$ find the value of (λ), if you Know that $f(x) = \frac{4}{x} \cdot f(x-1), x \in \mathbb{N}$?

Solution: $X \sim P(\lambda) \Rightarrow f(x) = \frac{e^{-\lambda}\, 10^{x}}{x!}$ & $f(x-1) = \frac{e^{-\lambda}\, 10^{x-1}}{(x-1)!}$, then we have:

$$f(x) = \frac{4}{x} \cdot f(x-1)$$

$$\frac{e^{-\lambda}\,\lambda^{x}}{x!} = \frac{4}{x} \cdot \frac{e^{-\lambda}\,\lambda^{x-1}}{(x-1)!} \Rightarrow \frac{e^{-\lambda}\,\lambda^{x}}{x!} = 4 \cdot \frac{e^{-\lambda}\,\lambda^{x-1}}{x!} \Rightarrow \lambda^{x} = 4\lambda^{x-1} \Rightarrow \frac{\lambda^{x}}{\lambda^{x-1}} = 4 \Rightarrow \lambda = 4$$

6. Let $X_i \sim Nb(r_i, p), i = 1,2,\cdots,n$, show that $\sum_{i=1}^{n} X_i \sim Nb(\sum_{i=1}^{n} r_i\,, p)$?

Solution: let $Y = \sum_{i=1}^{n} X_i$

$$\begin{aligned} M_Y(t) &= E(e^{tY}) = E\left(e^{t\sum_{i=1}^{n} X_i}\right) = E\left(e^{t(X_1, X_2, \cdots, X_n)}\right) = E(e^{tX_1} \cdot e^{tX_2} \cdots e^{tX_n}) \\ &= E(e^{tX_1}) \cdot E(e^{tX_2}) \cdots E(e^{tX_n}) = M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_n}(t) \\ &= \left(\frac{p}{1-(1-p)e^{t}}\right)^{r_1} \cdot \left(\frac{p}{1-(1-p)e^{t}}\right)^{r_2} \cdots \left(\frac{p}{1-(1-p)e^{t}}\right)^{r_n} \\ &= \left(\frac{p}{1-(1-p)e^{t}}\right)^{r_1 + r_2 + \cdots + r_n} = \left(\frac{p}{1-(1-p)e^{t}}\right)^{\sum_{i=1}^{n} r_i} \end{aligned}$$

$$\therefore Y = \sum_{i=1}^{n} X_i \sim Nb\left(\sum_{i=1}^{n} r_i\,, p\right)$$

7. Let $X \sim Nb(4, 0.3)$, find the following:

iv. P.m.f of x?

v. $M_x(t), \sigma_x^2$, and μ_x?

vi. Let $y = 4 + 5x$, find σ_y^2, and μ_y?

Solution:

i. $f(x) = \binom{x+r-1}{x} P^{r} (1-P)^{x} = \binom{x+4-1}{x} (0.3)^4 (1-0.3)^{x} = \binom{x+3}{x} (0.3)^4 (0.7)^{x}$

ii. .

$$M_x(t) = \left(\frac{p}{1-(1-p)e^{t}}\right)^{r} = \left(\frac{0.3}{1-0.7e^{t}}\right)^{4}$$

$$\mu_x = \frac{r(1-p)}{p} = \frac{4 \cdot 0.7}{0.3} = \frac{28}{3} = 9.333$$

$$\sigma_x^2 = \frac{r(1-p)}{p^2} = \frac{4 \cdot 0.7}{(0.3)^2} = \frac{280}{9} = 31.111$$

iii.

$$\mu_y = E(y) = E(4+5x) = E(4) + E(5x) = 4 + 5E(x) = 4 + 5 \cdot \mu_x = 4 + 5 \cdot 9.333$$
$$= 50.665$$

$$\sigma_y^2 = var(y) = var(4+5x) = var(4) + var(5x) = 0 + 25var(x) = 25 \cdot 31.111$$
$$= 777.778$$

Solving Exercises (2)

1. Let $X \sim u(0,1)$, use the transformation method to find the distribution of $y = -2\,ln(x)$, then find μ_y and σ_y^2 ?

Solution: $f(x) = \frac{1}{b-a} = 1 \quad a \le x \le b$

Assuming that the space of X denoted by A and the space of Y denoted by B and are defined as follows:

$$A = \{X; a \le x \le b\}\ and\ B = \{Y; -\infty < Z < \infty\}$$

$$Y = u(x) = -2ln(x) \text{ is } (1-1) \text{ transformation maps A onto B}$$

$$X = u^{-1}(y) = e^{\frac{-y}{2}} \text{ is } (1-1) \text{ transformation maps B onto A}$$

$$|J| = \left|\frac{dx}{dy}\right| = \frac{-1}{2}e^{\frac{-y}{2}} \Rightarrow g(y) = f(u^{-1}(y)) \cdot |J|$$

$$g(y) = 1 \cdot \left|\frac{-1}{2} e^{\frac{-y}{2}}\right| = \frac{1}{2} e^{\frac{-y}{2}}$$

Then Y distributed as Exponential distribution with parameter $\lambda = \frac{1}{2}$.

$$\therefore Y \sim Exp\left(\frac{1}{2}\right)$$

2. Given that $X_1, X_2, \cdots, X_n$ are independent random variables where $X_i \sim G\ (\propto_i, \beta),\ \ i = 1,2,\cdots,n$, show that $Y = \sum_{i=1}^{n} X_i \sim G\ (\sum_{i=1}^{n} \propto_i, \beta)$?

Solution: $X_i \sim G\ (\propto_i, \beta) \Rightarrow f(x) = \frac{1}{\Gamma_{\propto_i}\ \beta^{\propto_i}}\ x^{\propto_i - 1}\ e^{-x/\beta},\ i = 1,2,\cdots,n$

$$\begin{aligned} M_Y(t) &= E(e^{tY}) = E\left(e^{t\sum_{i=1}^{n} X_i}\right) = E\left(e^{t(X_1, X_2, \cdots, X_n)}\right) = E(e^{tX_1} \cdot e^{tX_2} \cdots e^{tX_n}) \\ &= E(e^{tX_1}) \cdot E(e^{tX_2}) \cdots E(e^{tX_n}) = M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_n}(t) \\ &= (1-\beta t)^{-\propto_1} \cdot (1-\beta t)^{-\propto_2} \cdots (1-\beta t)^{-\propto_n} \\ &= (1-\beta t)^{-(\propto_1 + \propto_2 + \cdots + \propto_n)} = (1-\beta t)^{-\sum_{i=1}^{n} \propto_i} \end{aligned}$$

$$\therefore Y = \sum_{i=1}^{n} X_i \sim G\left(\sum_{i=1}^{n} \propto_i, \beta\right)$$

4. Let $X \sim \beta(\propto, \beta)$ and $Y = ln\left(\frac{X}{1-X}\right)$, find $M_Y(t)$?

Solution: $f(x) = \frac{\Gamma_{\propto+\beta}}{\Gamma_{\propto}\ \Gamma_{\beta}}\ x^{\propto-1}\ (1-x)^{\beta-1}, 0 < x < 1$

$$\begin{aligned} M_Y(t) &= E\left(e^{t ln\left(\frac{X}{1-X}\right)}\right) = E\left(e^{ln\left(\frac{X}{1-X}\right)^t}\right) = E\left(\left(\frac{X}{1-X}\right)^t\right) \\ &= \frac{\Gamma_{\propto+\beta}}{\Gamma_{\propto}\ \Gamma_{\beta}} \int_0^1 \frac{x^t}{(1-x)^t} \cdot x^{\propto-1}\ (1-x)^{\beta-1}\ dx \\ &= \frac{\Gamma_{\propto+t}\ \Gamma_{\beta-t}}{\Gamma_{\propto}\ \Gamma_{\beta}} \int_0^1 \frac{\Gamma_{\propto+\beta}}{\Gamma_{\propto+t}\ \Gamma_{\beta-t}} x^{(\propto+t)-1}\ (1-x)^{(\beta-t)-1}\ dx = \frac{\Gamma_{\propto+t}\ \Gamma_{\beta-1}}{\Gamma_{\propto}\ \Gamma_{\beta}} \end{aligned}$$

5. Let X be a r. v. that follow the Beta distribution, find the constant C in each following cases:

i) $f(x) = C\, x^2(1-x)^5$?

ii) $f(x) = C\,(x - x^2)^{0.5}$?

Solution:

i. $X \sim \beta(\propto, \beta) \Rightarrow f(x) = \frac{\Gamma_{\propto+\beta}}{\Gamma_{\propto}\,\Gamma_{\beta}}\, x^{\propto-1}\,(1-x)^{\beta-1}, 0 < x < 1$

$$\text{since } f(x) = C\, x^2(1-x)^5$$

$$\frac{\Gamma_{\propto+\beta}}{\Gamma_{\propto}\,\Gamma_{\beta}}\, x^{\propto-1}\,(1-x)^{\beta-1} = C\, x^2(1-x)^5$$

$$\propto -1 = 2 \Rightarrow \propto = 3 \;\&\; \beta - 1 = 5 \Rightarrow \beta = 6$$

$$C = \frac{\Gamma_{\propto+\beta}}{\Gamma_{\propto}\,\Gamma_{\beta}} = \frac{\Gamma_9}{\Gamma_3\,\Gamma_6} = \frac{8!}{2!\cdot 5!} = \frac{8\times 7\times 6}{2} = 168$$

8. let $X_1, X_2, \cdots, X_n$ are independent r. vs. where $X_i \sim N(\mu_i, \sigma_i^2), i = 1,2,\cdots,n$, show that $Y = \sum_{i=1}^{n} X_i \sim N\left(\sum_{i=1}^{n}\mu_i, \sum_{i=1}^{n}\sigma_i^2\right)$?

Solution:

$$M_Y(t) = E(e^{tY}) = E\left(e^{t\sum_{i=1}^{n}X_i}\right) = E\left(e^{t(X_1,X_2,\cdots,X_n)}\right) = E(e^{tX_1}\cdot e^{tX_2}\cdots e^{tX_n})$$

$$= E(e^{tX_1})\cdot E(e^{tX_2})\cdots E(e^{tX_n}) = M_{X_1}(t)\cdot M_{X_2}(t)\cdots M_{X_n}(t)$$

$$= e^{t\mu_1+\frac{t^2}{2}\sigma_1^2}\cdot e^{t\mu_2+\frac{t^2}{2}\sigma_2^2}\cdots e^{t\mu_n+\frac{t^2}{2}\sigma_n^2} = e^{t(\mu_1+\mu_2+\cdots+\mu_n)+\frac{t^2}{2}(\sigma_1^2+\sigma_2^2+\cdots+\sigma_n^2)}$$

$$= e^{t\sum_{i=1}^{n}\mu_i+\frac{t^2}{2}\sum_{i=1}^{n}\sigma_i^2}$$

$$\therefore Y = \sum_{i=1}^{n} X_i \sim N\left(\sum_{i=1}^{n}\mu_i, \sum_{i=1}^{n}\sigma_i^2\right)$$

10. Let $X \sim X^2(n)$ and $X+Y \sim X^2(n+m)$, where X and Y are independent r. vs. use m.g.f to find the distribution of Y ?

Solution: $X \sim X^2(n) \Rightarrow M_x(t) = (1-2t)^{\frac{-n}{2}}$ and since $X+Y \sim X^2(n+m)$, then

$$M_{x+y}(t) = (1-2t)^{\frac{-(n+m)}{2}} = (1-2t)^{\frac{-n-m}{2}} = (1-2t)^{\frac{-n}{2}} \cdot (1-2t)^{\frac{-m}{2}}$$
$$= M_x(t) \cdot M_y(t)$$

$$\therefore M_y(t) = (1-2t)^{\frac{-m}{2}}$$

$$\therefore Y \sim X^2(m)$$

11. Let $X \sim N(0,2)$, find $E\left(x^{k/2}\right)$, where k is even positive number, then find $E(X^2)$?

Solution: $X \sim N(0,2) \Rightarrow f(x) = \frac{1}{\sqrt{4\pi}} e^{\frac{-x^2}{4}}, -\infty < x < \infty,$

$$E\left(x^{k/2}\right) = \int_{-\infty}^{\infty} x^{k/2} \cdot \frac{1}{\sqrt{4\pi}} e^{\frac{-x^2}{4}} \, dx$$

Let $z = \frac{x^2}{4} \Rightarrow x^2 = 4z \Rightarrow x = 2\sqrt{z} \Rightarrow dx = \frac{2dz}{2\sqrt{z}} \Rightarrow dx = \frac{dz}{\sqrt{z}}$, then we have

$$E\left(x^{k/2}\right) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} (2\sqrt{z})^{k/2} \cdot e^{-z} \frac{dz}{\sqrt{z}} = \frac{2^{k/2}}{2\sqrt{\pi}} \int_{-\infty}^{\infty} (\sqrt{z})^{k/2-1} \cdot e^{-z} \, dz$$
$$= \frac{2^{k/2-1}}{\sqrt{\pi}} \int_{-\infty}^{\infty} z^{\frac{k}{4}-\frac{1}{2}} \cdot e^{-z} \, dz = \frac{2^{k/2-1}}{\sqrt{\pi}} \, 2 \int_{0}^{\infty} z^{\left(\frac{k}{4}+\frac{1}{2}\right)-1} \cdot e^{-z} \, dz$$
$$= \frac{2^{k/2}}{\sqrt{\pi}} \Gamma_{\frac{k}{4}+\frac{1}{2}}$$

Now, to find $E(X^2) \Rightarrow \frac{k}{2} = 2 \Rightarrow k = 4$

$$E(X^2) = \frac{2^{4/2}}{\sqrt{\pi}}\ \Gamma_{\frac{4}{4}+\frac{1}{2}} = \frac{2^2}{\sqrt{\pi}}\ \Gamma_{1+\frac{1}{2}} = \frac{4}{\sqrt{\pi}}\ \Gamma_{\frac{3}{2}} = \frac{4}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = 2$$

$$E(X^2) = var(x) + \big(E(x)\big)^2 = 2 + 0 = 2$$

13. Let (t) be a r. v. with p.d.f $f(t) = C\ \left(1 + \frac{1}{5}\ t^2\right)^{-3}$, find the value of (C) such that the r. v. follows t distribution?

Solution: $t \sim t(n-1)$

$$f(t) = \frac{\Gamma_{(r+1)/2}}{\sqrt{\pi r}\ \Gamma_{r/2}} \cdot \left(1 + \frac{t^2}{r}\right)^{\frac{-(r+1)}{2}} , -\infty < t < \infty$$

Since $f(t) = C\ \left(1 + \frac{1}{5}\ t^2\right)^{-3} \Rightarrow r = 5$, then

$$C\ \left(1 + \frac{1}{5}\ t^2\right)^{-3} = \frac{\Gamma_{(5+1)/2}}{\sqrt{5\pi}\ \Gamma_{5/2}} \cdot \left(1 + \frac{t^2}{5}\right)^{\frac{-(5+1)}{2}}$$

$$C\ \left(1 + \frac{t^2}{5}\right)^{-3} = \frac{\Gamma_3}{\sqrt{5\pi}\ \Gamma_{5/2}} \cdot \left(1 + \frac{t^2}{5}\right)^{-3}$$

$$C\ = \frac{\Gamma_3}{\sqrt{5\pi}\ \Gamma_{5/2}} = \frac{2!}{\sqrt{5\pi}\ \frac{3}{2}\Gamma_{3/2}} = \frac{2}{\sqrt{5\pi}\ \frac{3}{2}\frac{\sqrt{\pi}}{2}} = \frac{8}{3\pi\sqrt{5}}$$

14. Let $= \frac{w}{\sqrt{v/2}}$, where $W \sim N(0,1)$ and $V \sim X^2(2)$, show that (t^2) has F distribution with parameters $n_1 = 1$ and $n_2 = 2$?

Solution: since $w \sim N(0,1) \Rightarrow w^2 \sim X^2(1)$, (by theorem (2))

$$t^2 = \frac{w^2}{v/2} = \frac{w^2/1}{v/2}\ , where\ w^2 \sim X^2(1)\ \&\ V \sim X^2(2)$$

$$t^2 \sim f(1,2), \text{by the definition of } f \text{ distribution}$$

19. Let $x_1, x_2, \cdots, x_n$ be random samples from $G(\propto, \beta)$, show that $\overline{X} \sim G\left(n \propto, \frac{\beta}{n}\right)$, then show that $E(\overline{X}) = \propto \beta$ and $var(\overline{X}) = \frac{\propto \beta^2}{n}$?
use $M_{\overline{X}}(t) = E\left(e^{t\overline{X}}\right) = E\left(e^{\frac{t}{n}(x_1, x_2, \cdots, x_n)}\right)$

Solution: if $X \sim G(\propto, \beta) \Rightarrow M_x(t) = (1 - \beta t)^{-\propto},\ t > \frac{1}{\beta}$

$$M_{\overline{X}}(t) = E\left(e^{t\overline{X}}\right) = E\left(e^{\frac{t}{n}(x_1 + x_2 + \cdots + x_n)}\right) = E\left(e^{\frac{tx_1}{n} + \frac{tx_2}{n} + \cdots + \frac{tx_n}{n}}\right)$$

$$= E\left(e^{\frac{tx_1}{n}} \cdot e^{\frac{tx_2}{n}} \cdots e^{\frac{tx_n}{n}}\right) = E\left(e^{\frac{tx_1}{n}}\right) \cdot E\left(e^{\frac{tx_2}{n}}\right) \cdots E\left(e^{\frac{tx_n}{n}}\right)$$

$$= M_{x_1}\left(\frac{t}{n}\right) \cdot M_{x_2}\left(\frac{t}{n}\right) \cdots M_{x_n}\left(\frac{t}{n}\right)$$

$$= \left(1 - \beta\frac{t}{n}\right)^{-\propto} \cdot \left(1 - \beta\frac{t}{n}\right)^{-\propto} \cdots \left(1 - \beta\frac{t}{n}\right)^{-\propto} = \left(1 - \frac{\beta}{n}t\right)^{-n\propto}$$

$$\therefore \overline{X} \sim G\left(n \propto, \frac{\beta}{n}\right)$$

Solving of Exercises (3)

1. Let $x_1, x_2, \cdots, x_n$ be a r. s. from distribution has p.d.f.

$$f(x,\theta) = \begin{cases} \frac{1}{\theta} e^{\frac{-x}{\theta}} & 0 < x < \infty, 0 < \theta < \infty \\ 0 & O.W \end{cases}$$

Show that $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is unbiased statistic for (θ)?

Solution: since $x \sim G(\propto, \beta), where \ \propto = 1,\ \beta = \theta \ and \ E(x) = \propto \beta = \theta$, then

$$E(\overline{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n}E\left(\sum_{i=1}^{n} x_i\right) = \frac{1}{n}E\left(\sum_{i=1}^{n} E(x_i)\right) = \frac{1}{n}E\left(\sum_{i=1}^{n} \theta\right)$$
$$= \frac{1}{n} \cdot n\theta = \theta$$

$\therefore \overline{X}$ is an unbiased statistic for θ.

2. Let $y_1 < y_2 < y_3$ be the order statistics of a r. s. of size 3 from the uniform distribution having p.d.f. $f(x, \theta) = \frac{1}{\theta}, 0 < x < \theta, 0 < \theta < \infty$. Show that $(4y_1)$, $(2y_2)$ and $\left(\frac{4}{3}y_3\right)$ are all unbiased statistics for (θ) and find the variance of each of these unbiased statistics?

Solution: $n = 3$

$$F(y_1) = \int_0^{y_1} \frac{1}{\theta} du = \frac{1}{\theta} \cdot u \Big|_0^{y_1} = \frac{1}{\theta}(y_1 - 0) = \frac{y_1}{\theta}$$

$$g(y_k) = \frac{n!}{(k-1)!\,(n-k)!} \left(F(y_k)\right)^{k-1} \cdot \left(1 - F(y_k)\right)^{n-k} \cdot f(y_k)$$

$$g(y_1) = \frac{3!}{0!\,2!} \left(F(y_1)\right)^0 \cdot \left(1 - F(y_1)\right)^2 \cdot f(y_1) = 3\left(1 - \frac{y_1}{\theta}\right)^2 \cdot \frac{1}{\theta} = \frac{3}{\theta}\left(1 - \frac{y_1}{\theta}\right)^2$$

$$E(4y_1) = 4E(y_1) = 4\int_0^\theta \frac{3}{\theta} y_1 \left(1 - \frac{y_1}{\theta}\right)^2 dy_1 = \frac{12}{\theta}\int_0^\theta y_1 \left(1 - \frac{2y_1}{\theta} + \frac{y_1^2}{\theta^2}\right) dy_1$$

$$= \frac{12}{\theta}\int_0^\theta \left(y_1 - \frac{2y_1^2}{\theta} + \frac{y_1^3}{\theta^2}\right) dy_1 = \frac{12}{\theta}\left(\frac{y_1^2}{2} - \frac{2y_1^3}{3\theta} + \frac{y_1^4}{4\theta^2}\right)\Big|_0^\theta$$

$$= \frac{12}{\theta}\left(\frac{\theta^2}{2} - \frac{2\theta^3}{3\theta} + \frac{\theta^4}{4\theta^2}\right) = \frac{12}{\theta}\left(\frac{\theta^2}{2} - \frac{2\theta^2}{3} + \frac{\theta^2}{4}\right) = 6\theta - 8\theta + 3\theta = \theta$$

$\therefore$ $(4y_1)$ is unbiased statistics for θ.

$$F(y_2) = \int_0^{y_2} \frac{1}{\theta} du = \frac{1}{\theta} \cdot u \Big|_0^{y_2} = \frac{1}{\theta}(y_2 - 0) = \frac{y_2}{\theta}$$

$$g(y_2) = \frac{3!}{1!\,1!} \left(F(y_2)\right)^1 \cdot \left(1 - F(y_2)\right)^1 \cdot f(y_2)$$

$$g(y_2) = 6\left(\frac{y_2}{\theta}\right) \cdot \left(1 - \frac{y_2}{\theta}\right) \cdot \frac{1}{\theta} = 6\left(\frac{y_2}{\theta^2} - \frac{y_2^2}{\theta^3}\right)$$

$$E(2y_2) = 2E(y_2) = 2\int_0^\theta 6y_2 \left(\frac{y_2}{\theta^2} - \frac{y_2^2}{\theta^3}\right) dy_2 = 12\int_0^\theta \left(\frac{y_2^2}{\theta^2} - \frac{y_2^3}{\theta^3}\right) dy_2$$

$$= 12\left(\frac{y_2^3}{3\theta^2} - \frac{y_2^4}{4\theta^3}\right)\Big|_0^\theta = 12\left(\frac{\theta^3}{3\theta^2} - \frac{\theta^4}{4\theta^3}\right) = 4\theta - 3\theta = \theta$$

$\therefore$ $(2y_2)$ is unbiased statistics for θ.

$$F(y_3) = \int_0^{y_3} \frac{1}{\theta} du = \frac{1}{\theta} \cdot u \Big|_0^{y_3} = \frac{1}{\theta}(y_3 - 0) = \frac{y_3}{\theta}$$

$$g(y_3) = \frac{3!}{2!\,0!} \left(F(y_3)\right)^2 \cdot \left(1 - F(y_3)\right)^0 \cdot f(y_3) = \frac{3}{\theta}\left(\frac{y_3}{\theta}\right)^2 = \frac{3y_3^2}{\theta^3}$$

$$E\left(\frac{4}{3}y_3\right) = \frac{4}{3}E(y_3) = \frac{4}{3}\int_0^\theta y_3 \frac{3y_3^2}{\theta^3}dy_3 = \frac{4}{\theta^3}\int_0^\theta y_3^3\, dy_3 = \frac{1}{\theta^3}y_3^4 \Big|_0^\theta = \frac{1}{\theta^3}\theta^4 = \theta$$

$\therefore \left(\frac{4}{3}y_3\right)$ is unbiased statistics for θ.

Now, to find the variance of $(4y_1)$.

$$E(y_1^2) = \int_0^\theta y_1^2 \frac{3}{\theta}\left(1-\frac{y_1}{\theta}\right)^2 dy_1 = \frac{3}{\theta}\int_0^\theta y_1^2\left(1-\frac{2y_1}{\theta}+\frac{y_1^2}{\theta^2}\right)dy_1$$

$$= \frac{3}{\theta}\int_0^\theta \left(y_1^2-\frac{2y_1^3}{\theta}+\frac{y_1^4}{\theta^2}\right)dy_1 = \frac{3}{\theta}\left(\frac{y_1^3}{3}-\frac{2y_1^4}{4\theta}+\frac{y_1^5}{5\theta^2}\right)\Big|_0^\theta$$

$$= \frac{3}{\theta}\left(\frac{\theta^3}{3}-\frac{\theta^4}{2\theta}+\frac{\theta^5}{5\theta^2}\right) = \frac{3}{\theta}\left(\frac{\theta^3}{3}-\frac{\theta^3}{2}+\frac{\theta^3}{5}\right) = \frac{3}{\theta}\cdot\frac{\theta^3}{30} = \frac{\theta^2}{10}$$

$$E(4y_1) = \theta \Rightarrow 4E(y_1) = \theta \Rightarrow E(y_1) = \frac{\theta}{4}$$

$$var(4y_1) = 16\, var(y_1) = 16\left(E(y_1^2) - \left(E(y_1)\right)^2\right) = 16\left(\frac{\theta^2}{10}-\frac{\theta^2}{16}\right)$$

$$= 16\left(\frac{8\theta^2-5\theta^2}{80}\right) = \frac{3\theta^2}{5}$$

Now, to find the variance of $(2y_2)$.

$$E(y_2^2) = E(y_2^2) = \int_0^\theta 6y_2^2\left(\frac{y_2}{\theta^2}-\frac{y_2^2}{\theta^3}\right)dy_2 = 6\int_0^\theta \left(\frac{y_2^3}{\theta^2}-\frac{y_2^4}{\theta^3}\right)dy_2 = 6\left(\frac{y_2^4}{4\theta^2}-\frac{y_2^5}{5\theta^3}\right)\Big|_0^\theta$$

$$= 6\left(\frac{\theta^4}{4\theta^2}-\frac{\theta^5}{5\theta^3}\right) = 6\left(\frac{\theta^2}{4}-\frac{\theta^2}{5}\right) = 6\frac{\theta^2}{20} = \frac{3\theta^2}{10}$$

$$E(2y_2) = \theta \Rightarrow 2E(y_2) = \theta \Rightarrow E(y_2) = \frac{\theta}{2}$$

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$$var(2y_2) = 4\,var(y_2) = 4\left(E(y_2^2) - \left(E(y_2)\right)^2\right) = 4\left(\frac{3\theta^2}{10} - \left(\frac{\theta}{2}\right)^2\right) = 4\left(\frac{3\theta^2}{10} - \frac{\theta^2}{4}\right)$$

$$= 4\left(\frac{6\theta^2 - 5\theta^2}{20}\right) = 4\frac{\theta^2}{20} = \frac{\theta^2}{5}$$

Now, to find the variance of $\left(\frac{4}{3}y_3\right)$.

$$E(y_3^2) = E(y_3^2) = \int_0^\theta y_3^2 \frac{3y_3^2}{\theta^3} dy_3 = \frac{3}{\theta^3}\int_0^\theta y_3^4\, dy_3 = \frac{3}{5\theta^3} y_3^5 \Big|_0^\theta = \frac{3}{5\theta^3}\theta^5 = \frac{3}{5}\theta^2$$

$$E\left(\frac{4}{3}y_3\right) = \theta \Rightarrow \frac{4}{3}E(y_3) = \theta \Rightarrow E(y_3) = \frac{3}{4}\theta$$

$$var\left(\frac{4}{3}y_3\right) = \frac{4}{3}\,var(y_3) = \frac{16}{9}\left(E(y_3^2) - \left(E(y_3)\right)^2\right) = \frac{16}{9}\left(\frac{3}{4}\theta^2 - \left(\frac{3}{5}\theta\right)^2\right)$$

$$= \frac{16}{9}\left(\frac{3}{4}\theta^2 - \frac{9}{25}\theta^2\right) = \frac{4}{3}\theta^2 - \frac{16}{25}\theta^2 = \frac{\theta^2}{15}$$

3. Let $x_1, x_2, \cdots, x_n$ be a r. s. from $P(\theta)$. Show that $\sum_{i=1}^n x_i$ is a sufficient statistics for θ?

Solution: since $(x_1, x_2, \cdots, x_n) \sim P(\theta)$, then $f(x_i, \theta) = \frac{e^{-\theta}\theta^{x_i}}{x_i!}, i = 1,2,\cdots,n.$

$$L(x_1, x_2, \cdots, x_n, \theta) = \prod_{i=1}^n f(x_i, \theta) = \prod_{i=1}^n \frac{e^{-\theta}\theta^{x_i}}{x_i!} = \frac{e^{-\theta}\theta^{x_1}}{x_1!} \cdot \frac{e^{-\theta}\theta^{x_2}}{x_2!} \cdots \frac{e^{-\theta}\theta^{x_n}}{x_n!}$$

$$= \frac{e^{-n\theta}\theta^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!} = \left(e^{-n\theta}\theta^{\sum_{i=1}^n x_i}\right) \cdot \left(\frac{1}{\prod_{i=1}^n x_i!}\right)$$

Now applying The Factorization Theorem

$$L(x_1, x_2, \cdots, x_n, \theta) = K_1(T, \theta) \cdot K_2(x_1, x_2, \cdots, x_n)$$

$$K_1(T, \theta) = e^{-n\theta}\theta^{\sum_{i=1}^n x_i}$$

$$K_2(x_1, x_2, \cdots, x_n) = \frac{1}{\prod_{i=1}^{n} x_i!}$$

$\therefore \sum_{i=1}^{n} x_i$ is a sufficient statistics for θ.

4. Show that the (n^{th}) order statistic of a r. s. of size n from the uniform distribution having p.d.f. $f(x,\theta) = \frac{1}{\theta}, 0 < x < \theta, 0 < \theta < \infty$, is sufficient statistic for (θ)?

Solution: $n = k$

$$F(y_n) = \int_0^{y_n} u\,du = \int_0^{y_n} \frac{1}{\theta}\,du = \frac{1}{\theta}\,u\Big|_0^{y_n} = \frac{y_n}{\theta}$$

$$g(y_n, \theta) = \frac{n!}{(n-1)!\,(n-n)!}\left(F(y_n)\right)^{n-1} \cdot \left(1 - F(y_n)\right)^{n-n} \cdot f(y_n)$$

$$= \frac{n(n-1)!}{(n-1)!}\left(\frac{y_n}{\theta}\right)^{n-1}\frac{1}{\theta} = \frac{n}{\theta^n}\,y_n^{n-1}$$

$$L(x_1, x_2, \cdots, x_n, \theta) = \prod_{i=1}^{n} f(x_i, \theta) = \prod_{i=1}^{n} \frac{1}{\theta} = \frac{1}{\theta^n}$$

Now, applying Fisher Neyman Theorem

$$H(x_1, x_2, \cdots, x_n, \theta) = \frac{L(x_1, x_2, \cdots, x_n, \theta)}{g(y_n, \theta)} = \frac{\frac{1}{\theta^n}}{\frac{n}{\theta^n}\,y_n^{n-1}} = \frac{1}{\theta^n} \cdot \frac{\theta^n}{n\,y_n^{n-1}} = \frac{1}{n\,y_n^{n-1}}$$

$\therefore H(x_1, x_2, \cdots, x_n, \theta)$ dose not depend upon (θ).

$\therefore y_n$ is sufficient statistic for (θ).

Solving of Exercises (4)

1. Let $x_1, x_2, \cdots, x_n$ be a r. s. from $P(\theta)$, find MLE estimator for $Pr(x > 0)$?

Solution: since $(x_1, x_2, \cdots, x_n) \sim P(\theta)$, then $f(x_i, \theta) = \frac{e^{-\theta}\theta^{x_i}}{x_i!}, i = 1,2, \cdots, n.$

$$L(x_1, x_2, \cdots, x_n, \theta) = \prod_{i=1}^{n} f(x_i, \theta) = \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{x_i}}{x_i!} = \frac{e^{-\theta}\theta^{x_1}}{x_1!} \cdot \frac{e^{-\theta}\theta^{x_2}}{x_2!} \cdots \frac{e^{-\theta}\theta^{x_n}}{x_n!}$$

$$= \frac{e^{-n\theta}\theta^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!} = \left(e^{-n\theta}\theta^{\sum_{i=1}^{n} x_i}\right) \cdot \left(\frac{1}{\prod_{i=1}^{n} x_i!}\right)$$

$$ln(L) = ln\left(\frac{e^{-n\theta}\theta^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}\right) = ln\left(e^{-n\theta}\right) + ln\left(\theta^{\sum_{i=1}^{n} x_i}\right) - ln\left(\prod_{i=1}^{n} x_i!\right)$$

$$= -n\theta + \sum_{i=1}^{n} x_i \cdot ln(\theta) - \sum_{i=1}^{n} ln(x_i!)$$

$$\frac{\partial ln(L)}{\partial \theta} = -n + \frac{1}{\theta}\sum_{i=1}^{n} x_i$$

$$\frac{\partial ln(L)}{\partial \theta} = 0 \Rightarrow -n + \frac{1}{\theta}\sum_{i=1}^{n} x_i = 0 \Rightarrow \frac{1}{\theta}\sum_{i=1}^{n} x_i = n \Rightarrow \theta = \frac{1}{n}\sum_{i=1}^{n} x_i \Rightarrow \hat{\theta} = \overline{X}$$

$$Pr(x > 0) = 1 - Pr(x \leq 0) = 1 - Pr(x = 0) = 1 - \frac{e^{-\theta}\theta^{0}}{0!} = 1 - e^{-\theta}$$

MLE for $Pr(x > 0)$ is given as:

$$Pr(x > 0) = 1 - e^{-\hat{\theta}} = 1 - e^{-\overline{X}}$$

2. Let $x_1, x_2, \cdots, x_n$ be a r. s. from the following p.d.f.

$$f(x, \theta_1, \theta_2) = \frac{1}{\theta_2}\, e^{\frac{-(x-\theta_1)}{\theta_2}}, \theta_1 < x < \infty, -\infty < \theta_1 < \infty, \& \; 0 < \theta_2 < \infty$$

Find the MLE for θ_1 and θ_2?

Solution: to find MLE for θ_1, we have:

$$L(x_1, x_2, \cdots, x_n, \theta_1, \theta_2) = \prod_{i=1}^{n} f(x_i, \theta_1, \theta_2)$$

$$= \frac{1}{\theta_2} e^{\frac{-(x_1-\theta_1)}{\theta_2}} \cdot \frac{1}{\theta_2} e^{\frac{-(x_2-\theta_1)}{\theta_2}} \cdots \frac{1}{\theta_2} e^{\frac{-(x_n-\theta_1)}{\theta_2}} = \frac{1}{\theta_2^n} e^{\frac{-1}{\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)}$$

We can't use the differentiation method because the range of (x) depend upon (θ_1), but it is clear that (L) has maximum value at the largest value of (θ_1) which coincide with the smallest value of (x). Hence, $\hat{\theta}_1 = min(x_i)$ the smallest order statistic of the sample.

Now, to find the MLE for (θ_2).

$$ln(L) = ln\left(\theta_2^{-n} e^{\frac{-1}{\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)}\right) = ln(\theta_2^{-n}) + ln\left(e^{\frac{-1}{\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)}\right)$$

$$= -n\, ln(\theta_2) - \frac{1}{\theta_2}\sum_{i=1}^{n}(x_i - \theta_1)$$

$$\frac{\partial ln(L)}{\partial \theta} = \frac{-n}{\theta_2} + \frac{1}{\theta_2^2}\sum_{i=1}^{n}(x_i - \theta_1)$$

$$\frac{\partial ln(L)}{\partial \theta} = 0 \Rightarrow \frac{-n}{\theta_2} + \frac{1}{\theta_2^2}\sum_{i=1}^{n}(x_i - \theta_1) = 0 \Rightarrow \frac{n}{\theta_2} = \frac{1}{\theta_2^2}\left(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n}\theta_1\right)$$

$$\Rightarrow \frac{\theta_2^2}{\theta_2} = \frac{1}{n}\left(\sum_{i=1}^{n} x_i - n\theta_1\right) \Rightarrow \hat{\theta}_2 = \frac{1}{n}\sum_{i=1}^{n} x_i - \hat{\theta}_1$$

$$\therefore \hat{\theta}_2 = \overline{X} - min(x_i)$$

$\therefore \hat{\theta}_2$ is MLE estimator for θ_2.

3. Let $x_1, x_2, \cdots, x_n$ be a r. s. from $N(\mu, \sigma^2)$, find the moment estimator for μ & σ^2?

Solution: $M_1 = E(x^1) = E(x) = \mu$ and $m_1 = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{X}$

$$M_1 = m_1 \Rightarrow \hat{\mu} = \overline{X}$$

$$M_2 = E(x^2) = var(x) + \left(E(x)\right)^2 = \sigma^2 + \hat{\mu}^2 \Rightarrow M_2 = \sigma^2 + \left(\overline{X}\right)^2$$

$$m_2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

$$M_2 = m_2 \Rightarrow \sigma^2 + \left(\overline{X}\right)^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 \Rightarrow \sigma^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\overline{X}\right)^2$$

$$\therefore \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \overline{X}\right)^2$$

4. Let $x_1, x_2, \cdots, x_n$ be a r. s. from $G(\propto, \beta)$, find the moment estimator for $\propto$ & β?

Solution: $M_1 = E(x^1) = E(x) = \propto \beta$ and $m_1 = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{X}$

$$M_1 = m_1 \Rightarrow \propto \beta = \overline{X} \Rightarrow \hat{\beta} = \frac{\overline{X}}{\propto}$$

$$M_2 = E(x^2) = var(x) + \left(E(x)\right)^2 = \propto \beta^2 + \propto^2 \beta^2 = \propto \beta^2(1+\propto)$$

$$m_2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

$$M_2 = m_2 \Rightarrow \propto \hat{\beta}^2(1+\propto) = \frac{1}{n}\sum_{i=1}^{n} x_i^2 \Rightarrow \propto \frac{\left(\overline{X}\right)^2}{\propto^2}(1+\propto) = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

$$\frac{\left(\overline{X}\right)^2}{\propto}(1+\propto) = \frac{1}{n}\sum_{i=1}^{n} x_i^2 \Rightarrow \frac{\left(\overline{X}\right)^2}{\propto} + \left(\overline{X}\right)^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 \Rightarrow \frac{\left(\overline{X}\right)^2}{\propto} = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\overline{X}\right)^2$$

$$\therefore \hat{\propto} = \frac{\left(\overline{X}\right)^2}{\frac{1}{n}\sum_{i=1}^{n}\left(x_i - \overline{X}\right)^2}$$

Solving Exercises (5)

1. If it is known that $n = 17$ is the size of r.s. from $N(\mu, \sigma^2)$ with $\overline{X} = 5.3$, $S^2 = 6.2$, Find 95% C.I. for both μ & σ^2. The tabulated values are: $t_{0.025}(16) = 2.120, X^2_{0.975}(16) = 28.8, X^2_{0.025}(16) = 6.91$.

Solution: $n < 30$

Find C.I. for (μ), when (σ^2) is unknown

We have $1 - \alpha = 0.95 \Rightarrow \alpha = 0.05 \Rightarrow \frac{\alpha}{2} = 0.025 \Rightarrow 1 - \frac{\alpha}{2} = 0.975$

$$Pr\left(\overline{X} - t_{\frac{\alpha}{2}}(n-1) \cdot \frac{S}{\sqrt{n-1}} < \mu < \overline{X} + t_{\frac{\alpha}{2}}(n-1) \cdot \frac{S}{\sqrt{n-1}}\right) = 1 - \alpha$$

$$C.I.\text{ for both } \mu = \overline{X} \mp t_{\frac{\alpha}{2}}(16) \cdot \frac{S}{\sqrt{n-1}} = 5.3 \mp (2.120) \cdot \frac{\sqrt{6.2}}{\sqrt{16}} = 5.3 \mp 1.32$$

$$C.I.\text{ for both } \mu = (3.98, 6.62)$$

Lower bound ($CL = 3.98$) and upper bound ($CU = 6.62$).

Now, we find C.I. for (σ^2), when(μ) is unknown.

$$Pr\left(\frac{(n-1)\,S^2}{X^2_{1-\frac{\alpha}{2}}(n-1)} < \sigma^2 < \frac{(n-1)\,S^2}{X^2_{\frac{\alpha}{2}}(n-1)}\right) = 1 - \alpha$$

$$Pr\left(\frac{(16)(6.2)}{28.8} < \sigma^2 < \frac{(16)(6.2)}{6.91}\right) = 0.95 \Rightarrow Pr(3.444 < \sigma^2 < 14.356) = 0.95$$

$$C.I\text{ for } \sigma^2 = \left(\frac{(n-1)\,S^2}{X^2_{1-\frac{\alpha}{2}}(n-1)}, \frac{(n-1)\,S^2}{X^2_{\frac{\alpha}{2}}(n-1)}\right) = (3.444, 14.356)$$

Lower bound ($CL = 3.444$) and upper bound ($CU = 14.356$)

2. Given $\overline{X} = 18$, is the mean of a r. s. of size 20 from $N(\mu, \sigma^2)$. Find 99% C.I. for μ if it is known that $Z_{0.005} = 2.58$.

Solution: we have $1 - \alpha = 0.99 \Rightarrow \alpha = 0.01 \Rightarrow \frac{\alpha}{2} = 0.005$

$$Pr\left(\overline{X} - Z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + Z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

$$Pr\left(18 - (2.58) \cdot \frac{5}{\sqrt{20}} < \mu < 18 + (2.58) \cdot \frac{5}{\sqrt{20}}\right) = 0.99$$

$$\Rightarrow Pr(18 - 2.885 < \mu < 18 + 2.885) = 0.99$$

$$\Rightarrow Pr(15.115 < \mu < 20.885) = 0.99$$

$$C.I \text{ for } \mu = (15.115, 20.885)$$

Lower bound ($CL = 15.115$) and upper bound ($CU = 20.885$).

3. A r. s. of size 10 is drawn from $N(\mu, \sigma^2)$. The values of individuals are 10.7, 12.6, 9.3, 9.5, 11.3, 12.2, 11.5, 11.1, 10.4 and 10.2. Find 95% C.I. for both μ & σ^2, $t_{0.025}(9) = 2.262$, $X^2_{0.975}(9) = 19$ & $X^2_{0.025}(9) = 2.7$

Solution: we have $n < 30$, $1 - \alpha = 0.95 \Rightarrow \alpha = 0.05 \Rightarrow \frac{\alpha}{2} = 0.025 \Rightarrow \frac{\alpha}{2} = 0.975$

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{10}(10.7 + 12.6 + 9.3 + 9.5 + 11.3 + 12.2 + 11.5 + 11.1 + 10.4 + 10.2)$$

$$= \frac{108.8}{10} = 10.88$$

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \overline{X}\right)^2 = \frac{1}{n}\sum_{i=1}^{n} {x_i}^2 - \frac{1}{n}\sum_{i=1}^{n}\left(\overline{X}\right)^2 = \frac{1}{n}\sum_{i=1}^{n} {x_i}^2 - \frac{1}{n}n\left(\overline{X}\right)^2 = \frac{1}{n}\sum_{i=1}^{n} {x_i}^2 - \left(\overline{X}\right)^2$$

$$= \frac{1194.18}{10} - 118.3744 = 119.418 - 118.3744 = 1.0436$$

$$S = 1.0216$$

Find C.I. for (μ) when $(\sigma^2)$is unknown

$$Pr\left(\overline{X} - t_{\frac{\alpha}{2}} \cdot \frac{S}{\sqrt{n-1}} < \mu < \overline{X} + t_{\frac{\alpha}{2}} \cdot \frac{S}{\sqrt{n-1}}\right) = 1 - \alpha$$

$$Pr\left(10.88 - (2.262) \cdot \frac{1.0216}{3} < \mu < 10.88 + (2.262) \cdot \frac{1.0216}{3}\right) = 0.95$$

$$\Rightarrow Pr\left(10.88 - (2.262) \cdot \frac{1.0216}{3} < \mu < 10.88 + (2.262) \cdot \frac{1.0216}{3}\right) = 0.95$$

$$\Rightarrow Pr(10.88 - 0.77 < \mu < 10.88 + 0.77) = 0.95$$

$$Pr(10.11 < \mu < 11.65) = 0.95$$

$$C.I \text{ for } \mu = (10.11, 11.65)$$

Lower bound $(CL = 10.11)$ and upper bound $(CU = 11.65)$.

Now, to find C.I. for (σ^2) when (μ) is unknown

$$Pr\left(\frac{(n-1)\,S^2}{X^2_{1-\frac{\alpha}{2}}(n-1)} < \sigma^2 < \frac{(n-1)\,S^2}{X^2_{\frac{\alpha}{2}}(n-1)}\right) = 1 - \alpha$$

$$Pr\left(\frac{(9)(1.0436)}{19} < \sigma^2 < \frac{(9)(1.0436)}{2.7}\right) = 0.95$$

$$Pr(0.4943 < \sigma^2 < 3.4787) = 0.95$$

$$C.I \text{ for } \sigma^2 = (0.4943, 3.4787)$$

Lower bound $(CL = 0.4943)$ and upper bound $(CU = 3.4787)$.

4. Two random samples each of size 10 from $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$ yield $\overline{X}_1 = 4.8$, $\overline{X}_2 = 5.6$, $S_1^2 = 8.64$, $S_2^2 = 7.88$, find 95% for C.I. $(\mu_1 - \mu_2)$ if it's known $t_{0.025}(18) = 2.101$.

Solution: we have $n_1, n_2 < 30$, $1-\alpha = 0.95 \Rightarrow \alpha = 0.05 \Rightarrow \frac{\alpha}{2} = 0.025$

$$S_p^2 = \frac{(n_1-1)\,S_1^2 + (n_2-1)\,S_2^2}{n_1+n_2-2} = \frac{9\cdot 8.64 + 9\cdot 7.88}{18} = 8.26 \;\Rightarrow S_p = 2.874$$

$$\text{C.I. for } (\mu_1-\mu_2) = \left(\overline{X}_1 - \overline{X}_2\right) \mp t_{\frac{\alpha}{2}}(n_1+n_2-2)\cdot S_p \cdot \sqrt{1/n_1 + 1/n_2}$$

$$= (4.8-5.6) \mp 2.101\cdot 2.874\cdot \sqrt{1/5}$$

$$C.I \text{ for } (\mu_1-\mu_2) = (-3.5, 1.9)$$

Lower bound $(CL = -3.5)$ and upper bound $(CU = 1.9)$.

5. let $x_1, x_2, \cdots, x_{10}$ be a r. s. from normal population from $N(\mu, \sigma^2)$, let $0 < a < b$, show that the mathematical expectation of the length of random interval

$$\left[\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{b}, \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{a}\right] \text{ is } (b-a)\left(\frac{n\,\sigma^2}{ab}\right)$$

Solution:

$$length\ (L) = \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{a} - \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{b} = \frac{b\sum_{i=1}^{n}(x_i-\mu)^2 - a\sum_{i=1}^{n}(x_i-\mu)^2}{ab}$$

$$= \frac{(b-a)\sum_{i=1}^{n}(x_i-\mu)^2}{ab}$$

$$E(L) = E\left(\frac{(b-a)\sum_{i=1}^{n}(x_i-\mu)^2}{ab}\right) = \frac{(b-a)}{ab}E\left(\sum_{i=1}^{n}(x_i-\mu)^2\right)$$

$$= \frac{(b-a)}{ab}E\left(\sum_{i=1}^{n}\left(x_i^2 - 2x_i\,\mu + \mu^2\right)\right)$$

$$= \frac{(b-a)}{ab}E\left(\sum_{i=1}^{n}x_i^2 - 2\mu\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\mu^2\right)$$

$$= \frac{(b-a)}{ab}E\left(\sum_{i=1}^{n}x_i^2 - 2n\,\mu^2 + n\,\mu^2\right) = \frac{(b-a)}{ab}E\left(\sum_{i=1}^{n}x_i^2 - n\,\mu^2\right)$$

$$= \frac{(b-a)}{ab}\left(\sum_{i=1}^{n}E\left(x_i^2\right) - \sum_{i=1}^{n}\mu^2\right) = \frac{(b-a)}{ab}\left(\sum_{i=1}^{n}\left(E\left(x_i^2\right) - \mu^2\right)\right)$$

$$= \frac{(b-a)}{ab}\left(\sum_{i=1}^{n}\sigma^2\right) = \frac{(b-a)}{ab}(n\sigma^2) = (b-a)\left(\frac{n\,\sigma^2}{ab}\right)$$

Solving Exercises (6)

1. Let $x_1, x_2, \cdots, x_n$ be a random sample from $P(\theta)$. The critical region for testing $\left(H_0\colon \theta = \frac{1}{10}\right)$ against $\left(H_1\colon \theta > \frac{1}{10}\right)$ is $C = \{x_1, x_2, \cdots, x_n, \sum_{i=1}^{n} x_i \geq 1\}$

a. Find the power of the test $K(\theta)$.

b. Find the level of significant (α).

Solution: since $X \sim P(\lambda)$, then $f(x) = \frac{e^{-\lambda}\,\lambda^x}{x!}$ $x = 0,1,2,\cdots$ & $\lambda > 0$

$x_i \sim P(\lambda), i = 1,2,\cdots,n$, then let $y = \sum_{i=1}^{n} x_i \sim P(\sum_{i=1}^{n}\lambda) = P(n\lambda)$

$$K(\theta) = Pr(reject\ H_0 \mid H_1\ is\ true) = Pr(x_1, x_2, \cdots, x_n \in C \mid H_1)$$

$$= Pr\left(y \geq 1 \mid \theta > \frac{1}{10}\right) = 1 - Pr\left(y < 1 \mid \theta > \frac{1}{10}\right) = 1 - Pr(y = 0)$$

$$= 1 - \frac{e^{-\lambda}\,\lambda^0}{0!} = 1 - e^{-\lambda}, \theta > \frac{1}{10}$$

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$$\alpha = Pr(Type\ I\ Error) = Pr(reject\ H_0 \mid H_0\ is\ true)$$
$$= Pr\left(x_1, x_2, \cdots, x_n \in C \mid \theta = \frac{1}{10}\right) = Pr\left(y \geq 1 \mid \theta = \frac{1}{10}\right)$$
$$= Pr\left(y < 1 \mid \theta > \frac{1}{10}\right) = 1 - Pr(y = 0) = 1 - \frac{e^{-\frac{1}{10}}\lambda^0}{0!} = 1 - e^{-\frac{1}{10}}$$

2. Consider a normal distribution $N(\theta, 4)$. The simple hypothesis $(H_0\colon \theta = 0)$ is rejected and the alternative composite hypothesis $\left(H_1\colon \theta = \frac{1}{10}\right)$ is accepted if and only if the observed mean $(\overline{X})$ of a random sample of size 25 is greater than or equal to$\left(\frac{3}{5}\right)$, find the power of the test. now that $Z_{0.6915} = \frac{1}{2}$.

Solution: since $X \sim N(\mu, \sigma^2) \Rightarrow f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\ e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty$

$$X \sim N(\mu, \sigma^2) \Rightarrow \overline{X} \sim N(\mu, {\sigma^2}/{n}) \Rightarrow Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

$$\frac{\overline{X} - \theta}{2/\sqrt{25}} \sim N(0,1) \Rightarrow Z = \frac{5(\overline{X} - \theta)}{2} \sim N(0,1)$$

$$K(\theta) = Pr(reject\ H_0 \mid H_1\ is\ true) = Pr(x_1, x_2, \cdots, x_n \in C \mid H_1)$$
$$= Pr\left(\overline{X} \geq \frac{3}{5}, \theta = \frac{1}{10}\right) = Pr\left(\frac{5(\overline{X} - \theta)}{2} \geq \frac{5\left(\frac{3}{5} - \theta\right)}{2}, \theta = \frac{1}{10}\right)$$
$$= Pr\left(Z \geq \frac{3 - 5\theta}{2}, \theta = \frac{1}{10}\right) = Pr\left(Z \geq \frac{1}{2}\right) = 1 - Pr\left(Z < \frac{1}{2}\right)$$
$$= 1 - 0.6915 = 0.3085$$

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