قسم الرياضيات

الاحصاء الرياضي1

المرحلة الثالثة

الفصل الدراسي الاول

ا.م.د تسنيم حسن كاظم ا.م.د صادق ناجي ناصر

Lecture 1

Sampling Concepts

Definition**: Random sample**

The random variables $X_1, X_2, \dots, X_n$ are said to constitute a random sample of size n if

1. $X_1, X_2, \dots, X_n$ are independent random variables.
2. Every X_i has the same pdf $f(x)$; that is

$f_1(x_1) = f(x_1), f_2(x_2) = f(x_2) \dots, f_n(x_n) = f(x_n)$, so that the joint pdf:

$f(x_1, x_2, \dots, x_n) = f(x_1)f(x_2) \dots f(x_n) = \prod_{i=1}^{n} f(x_i)$

In other words, if the random variables $X_1, X_2, \dots, X_n$ are independent and identically distributed (iid), then these random variables constitute a random sample of size n from a common distribution.

Definition: **Statistic**

A function of one or more random variables that does not depend upon any unknown parameter is called a statistic. Therefore, a statistic $U(X) = U(X_1, X_2, \dots, X_n)$ is a function defined on the space of all possible sample points of the random variable X is also a random variable. Once the sample is drawn, a lowercase letter is used to represent the calculated or the observed value of the statistic

Example:

The sample mean $\bar{X}$ is a statistic

The sample variance S^2 is a statistic

$X_{(n)} = \max(X_1, X_2, \dots X_n)$ is a statistic

$X_{(1)} = \min(X_1, X_2, \dots X_n)$ is a statistic

The sample median is a statistic

But the random variable $Y = \frac{X-\mu}{\sigma}$ is not a statistic unless μ and σ are known numbers.

Definition: **Sampling Distribution**

The sampling distribution of a statistic is the probability distribution for the possible values of the statistic that results when random samples of size n are repeatedly drawn from the population.

Example 1:

Let X_1, X_2 and X_3 be independent random variables each have the pdf $f(x) = 2x$, $0 < x < 1$, zero elsewhere. The joint pdf $f(x_1, x_2, x_3)$ is $f(x_1).f(x_2).f(x_3) = 8x_1x_2x_3$, $0 < x_i < 1, i = 1,2,3$, zreo elsewhere. Let $Y = \max(X_1, X_2, X_3)$.

The distribution function of Y is

$$G(y) = P(Y \leq y) = P(X_1 \leq y, X_2 \leq y, X_3 \leq y)$$

$$= \int_0^y \int_0^y \int_0^y 8\,x_1x_2x_3\,dx_1\,dx_2dx_3$$

$= y^6 \quad 0 < y < 1$

Accordingly, the pdf of $y = \max(X_1, X_2, X_3)$ is

$g(y) = 6y^5, \quad 0 < y < 1$

$\quad = 0 \qquad elsewhere$

Example 2:

Let n be a positive integer and let the random variables X_i, $i = 1,2, \dots, n$, be independent, each having the same pdf $f(x) = p^x(1-p)^{1-x}$, $x = 0,1$ and zero elsewhere. If $Y = \sum_{i=1}^{n} X_i$, then Y is $b(n,p)$ with pdf $g(y) = \binom{n}{y} p^y (1-p)^{n-y} \quad y = 0,1, \dots, n$

It should be noted that the statistic $Y = \sum_{i=1}^{n} X_i$ does not depend upon the parameter p.

Definition: **The Sample Mean and The Sample Variance**

Let $X_1, X_2, \ldots., X_n$ denote a random sample of size n from a given distribution. The statistic

$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$ is called the mean of the random sample (sample mean).

And the statistic

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right]$$

is called the variance of the random sample (sample variance).

Theorem

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a population with mean μ and variance σ^2. Then $E(\bar{X}) = \mu$ and $var(\bar{X}) = \frac{\sigma^2}{n}$.

Proof:

$$E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i)$$

$$= \frac{1}{n}\sum_{i=1}^{n} \mu = \frac{1}{n} n\, \mu = \mu$$

and

$$var(\bar{X}) = var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} var(X_i)$$

$= \frac{1}{n^2} n\, \sigma^2 = \frac{\sigma^2}{n}$ because X_i's, i=1, 2, …, n are independent

The theorem states that regardless of the form of the population distribution, one can obtain the mean and standard deviation of the statistic $\bar{X}$ in terms of the mean and standard deviation of the population. Notice that the variance of each X_i is σ^2, where the variance of $\bar{X}$ is $\frac{\sigma^2}{n}$, which is smaller than σ^2 for $n \geq 2$.

Theorem

Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with mean μ and variance σ^2. Consider the sample variance

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

Show that $E(\,S^2) = \sigma^2$

$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right)$

$E(S^2) = \frac{1}{n-1}\left(\sum_{i=1}^{n} E\left(X_i^2\right) - n\,E(\bar{X}^2)\right)$

Using the fact that

$E(X^2) = var(X) + [E(X)]^2 = \sigma^2 + \mu^2$

Also

$E(\bar{X}^2) = var(\bar{X}) + \ [E(\bar{X})]^2 = \frac{\sigma^2}{n} + \mu^2$

We have the following

$E(S^2) = \frac{1}{n-1}\left[n\,\sigma^2 + n\,\mu^2 - n\left(\frac{\sigma^2}{n} + \mu^2\right)\right]$

$= \frac{1}{n-1}[n\,\sigma^2 + n\mu^2 - \sigma^2 - n\mu^2]$

$= \frac{1}{n-1}(n-1)\sigma^2 = \sigma^2$

This shows that the expected value of the sample variance is the same as the variance of the population under consideration. Hence S^2 is called an unbiased estimator of σ^2.

Lecture 2

Distributions of Functions of Random Variables

1. **The cumulative Distribution Function Technique**

Assume that a random variable X has a distribution function $F_X(x)$ and that $Y = U(X)$ is a function of X.

Then $F_Y(y) = P(Y \leq y) = P(U(X) \leq y)$

The pdf of Y is found by differentiating $F_Y(y)$.

Example:

Suppose that $f_X(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & o.w \end{cases}$

Consider $Y = e^x$. Find $f_Y(y)$.

$A = \{X: x \in R, 0 < x < \infty\}$

$B = \{Y: y \in R, 1 < y < \infty\}$

$F_Y(y) = P(Y \leq y) = P(e^x \leq y) = P(X \leq \ln y) = F_X(\ln y)$

$$= \int_0^{\ln y} 2\, e^{-2x} \;\; dx = 1 - e^{-2\ln y} = 1 - e^{\ln y^{-2}}$$

$$= 1 - y^{-2}$$

Hence

$$f_Y(y) = \frac{dF_Y(y)}{dy} = 2y^{-3} \quad 1 < y < \infty$$

Example:

Let X be a random variable with pdf $f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & o.w \end{cases}$

and let $U = 3X - 1$. Find the pdf of u.

$F_U(u) = P(U \leq u) = p(3X - 1 \leq u) = p\left(X \leq \frac{u+1}{3}\right)$

$$= \int_0^{\frac{u+1}{3}} f_X(x)dx = \int_0^{\frac{u+1}{3}} 2xdx = \left(\frac{u+1}{3}\right)^2$$

$A = \{X: x \in R, 0 \leq x \leq 1\}$

$B = \{U: u \in R, -1 \leq u \leq 2\}$

$$F_U(u) = \begin{cases} 0 & u < -1 \\ \left(\frac{u+1}{3}\right)^2 & -1 \leq u \leq 2 \\ 1 & u > 2 \end{cases}$$

$$\therefore f_U(u) = \frac{dF_U(u)}{du} = \begin{cases} \frac{2}{9}(u+1) & -1 \le u \le 2 \\ 0 & o.w. \end{cases}$$

Example

Let $f(x) = \frac{1}{2}, -1 < x < 1$ and zero elsewhere, be the pdf of a random variable X. Define the random variable Y= X^2. Find the pdf of Y.

$-1 < x < 1 \Rightarrow 0 < y < 1$

$F_Y(y) = p(Y \le y) = p(X^2 \le y) = p(-\sqrt{y} \le X \le \sqrt{y})$

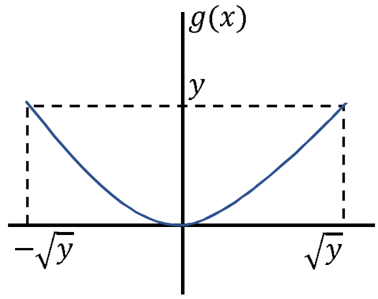
$$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2}\, dx = \frac{1}{2}[x]_{-\sqrt{y}}^{\sqrt{y}} = \sqrt{y}$$

The Distribution function is

$$F(y) = \begin{cases} 0 & y \le 0 \\ \sqrt{y} & 0 < y < 1 \\ 1 & 1 \le y \end{cases}$$

The pdf of Y is $f(y) = \frac{dF(y)}{dy} = \begin{cases} \frac{1}{2\sqrt{y}} & 0 < y < 1 \\ 0 & o.w. \end{cases}$

Let us consider the case $Y = g(x) = X^2$, where X is a random variable with distribution function $F_X(x)$ and pdf $f_X(x)$.



$F_Y(y) = P(Y \le y) = P(X^2 \le y)$

$= p(-\sqrt{y} \le X \le \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} f(x)dx$

$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$

In general

$$F_Y(y) = \begin{cases} F_X(\sqrt{y}) - F_X(-\sqrt{y}) & y > 0 \\ 0 & o.w. \end{cases}$$

On differentiating with respect to y,

$$f_Y(y) = \begin{cases} f_X(\sqrt{y})\left(\frac{1}{2\sqrt{y}}\right) + f_X(-\sqrt{y})\left(\frac{1}{2\sqrt{y}}\right) & y > 0 \\ 0 & o.w. \end{cases}$$

Or

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right] & y > 0 \\ 0 & o.w. \end{cases}$$

Example:

Let X be a random variable $\sim N(0,1)$ and let $Y = g(x) = X^2$. Find the pdf of Y .

$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ and $f_X(\sqrt{y}) = f_X(-\sqrt{y}) = \frac{1}{\sqrt{2\pi}}\, e^{-y/2}$

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left[\frac{1}{\sqrt{2\pi}}\, e^{\frac{-y}{2}} + \frac{1}{\sqrt{2\pi}}\, e^{\frac{-y}{2}}\right] = \frac{1}{2\sqrt{y}}\,\frac{1}{\sqrt{2\pi}}\left[2\, e^{\frac{-y}{2}}\right]$$

$$= \frac{y^{\frac{1}{2}-1}\; e^{-y/2}}{(2)^{\frac{1}{2}}\,\Gamma\left(\frac{1}{2}\right)},\; y > 0$$

Recall that $\sqrt{\pi} = \Gamma\left(\frac{1}{2}\right)$

Which is the pdf of the gamma distribution with $\alpha = \frac{1}{2}$ and $\beta = 2$.

Hence, $Y \sim \chi^2_{(1)}$.

Hence, if the random variable $X \sim N(0,1)$, then the random variable $Y = X^2 \sim \chi^2_{(1)}$.

Example:

Let the random variable X has the pdf

$$f_X(x) = \begin{cases} 2x\, e^{-x^2} & 0 < x < \infty \\ 0 & o.w. \end{cases}$$

Let $Y = X^2$. Find the pdf of Y.

$f_X(\sqrt{y}) = 2\sqrt{y}\, e^{-y}$ and $f_X(-\sqrt{y}) = 0$

$g_Y(y) = \frac{1}{2\sqrt{y}} 2\sqrt{y}\; e^{-y} = e^{-y},\; 0 < y < \infty$

Which is exponential with $\lambda = 1$.

Exercise :

Suppose that X have a continuous distribution with distribution $F(x)$ and pdf $f(x)$ prove the following:

1-If $Y = F(x)$ then show that $Y \sim U(0,1)$.

2- If $U = -\log(F(x))$, then show that $U \sim exp(1)$

3- If $V = -2\log(F(x))$, then show that $V \sim \chi^2_{(2)}$.

The Transformation of Variables Technique

This method is also called the change of variable technique.

1. Discrete case

   Let X be a discrete r.v. having pdf $f(x)$. Let A denote the set of discrete points, at each of which $f(x) > 0$, and let $y = u(x)$ define a one -to-one transformation that maps A onto B. Consider the r.v. $Y = u(x)$. If $y \in B$, then $x = w(y) \in A$. Accordingly, the pdf of Y is

   $g(y) = P(Y = y) = P(u(X) = y) = P\big(X = w(y)\big)$

   $= f\big(w(y)\big) \quad , y \in B$, and $g(y) = 0 \qquad , O.W.$

Example:

Let X have the passion pdf $f(x) = \begin{cases} \frac{\lambda^x e^{-\lambda}}{x!} & x = 0,1,2,\dots \\ 0 & o.w \end{cases}$

Define a new r.v. $Y = 4X$. Find the pdf of Y .

$A = \{x : x = 0,1,2,3,\dots\}$

$B = \{y : y = 0,4,8,12,\dots\}$

The function $y = 4x$ maps the space A onto space B such that there is one -to-one correspondence between the points of A and those of B.

$g(y) = P(Y = y) = P(4X = y) = P\left(X = \frac{y}{4}\right)$

$= \frac{\lambda^{y/4} e^{-\lambda}}{(y/4)!} \quad y = 0,4,8,\dots$

$= 0 \quad o.w.$

Example: Let $X \sim b\left(3, \frac{2}{3}\right)$. Find the pdf of $Y = X^2$

We know that If x~b(n,p) then $f(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0,1,\dots,n$

So that

$f(x) = \frac{3!}{x!(3-x)!} \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{3-x} \qquad x = 0,1,2,3$

Lecture 3

The transformation $y = u(x) = x^2$ maps $A = \{x: x = 0,1,2,3\}$ onto $B = \{y: y = 0,1,4,9\}$. Since $x = w(y) = \sqrt{y}$,

$$g(y) = P(Y = y) = P(X^2 = y) = P\left(X = \sqrt{y}\right) = f\left(\sqrt{y}\right) =$$

$$\frac{3!}{(\sqrt{y})!(3-\sqrt{y})!}\left(\frac{2}{3}\right)^{\sqrt{y}}\left(\frac{1}{3}\right)^{3-\sqrt{y}} \qquad y = 0,1,4,9$$

In the bivariate case, let $f(x_1, x_2)$ be the joint pdf of two discrete r.v's X_1 and X_2 with A the set of points at which $f(x_1, x_2) > 0$
$A = \{(x_1, x_2): f(x_1, x_2) > 0\}$. Let $y_1 = u_1(x_1, x_2)$ and $y_2 = u_2(x_1, x_2)$ define a one-to-one transformation that maps A onto B. The joint pdf of the two new r.v's $Y_1 = u_1(x_1, x_2)$ and $Y_2 = u_2(x_1, x_2)$ is

$$g(y_1, y_2) = P(Y_1 = y_1, Y_2 = y_2) = P\left(Y_1 = u_1(x_1, x_2), Y_2 = u_2(x_1, x_2)\right)$$

$$= P\left(X_1 = w_1(y_1, y_2), X_2 = w_2(y_1, y_2)\right)$$

$$= f\left(w_1(y_1, y_2), w_2(y_1, y_2)\right) \quad (y_1, y_2) \in B$$

$$= 0 \qquad\qquad e.w$$

Where $x_1 = w_1(y_1, y_2)$ and $x_2 = w_2(y_1, y_2)$ are the single valued inverse of $y_1 = u_1(x_1, x_2)$ and $y_2 = u_2(x_1, x_2)$. Form the joint pdf $g(y_1, y_2)$ we may obtain the marginal pdf of Y_1 by summing on y_2 or the marginal pdf of Y_2 by summing on y_1.

Example:

Let X_1 and X_2 be two independents r.v.'s that have Poisson distributions with means μ_1 and μ_2, respectively. Find the pdf of $Y_1 = X_1 + X_2$

$$f(x_1, x_2) = \frac{\mu_1^{x_1}\, \mu_2^{x_2}\, e^{-\mu_1-\mu_2}}{x_1!\;\; x_2!} \qquad x_1 = 0,1,2,\ldots, x_2 = 0,1,2,\ldots$$

We need to define a second r.v. $y_2 = X_2$. Then $y_1 = x_1 + x_2$ and $y_2 = x_2$ represent a one-to-one transformation that maps A onto

$$B = \{(y_1, y_2): y_2 = 0,1,\ldots,y_1 \text{and } y_1 = 0,1,2,\ldots\}.$$

Note that if $(y_1, y_2) \in B$, then $0 \le y_2 \le y_1$. The inverse functions are given by $x_1 = y_1 - y_2$ and $x_2 = y_2$.

The joint pdf of Y_1 and Y_2 is

$g(y_1, y_2) = \frac{\mu_1{}^{y_1-y_2}\,\mu_2{}^{y_2}\,e^{-\mu_1-\mu_2}}{(y_1-y_2)!\;y_2!}\,(y_1, y_2) \in B, y_1 = 0,1,2,\cdots, y_1 =$
$0,1,2,\cdots y_1$

The marginal pdf of y_1 is

$$g_1(y_1) = \sum_{y_2=0}^{y_1} g(y_1, y_2) = \frac{e^{-\mu_1-\mu_2}}{y_1!}\sum_{y_2=0}^{y_1}\frac{y_1!}{(y_1-y_2)!\;y_2!}\,\mu_1{}^{y_1-y_2}\,\mu_2{}^{y_2}$$

$$= \frac{e^{-\mu_1-\mu_2}}{y_1!}\sum_{y_2=0}^{y_1} C_{y_2}^{y_1}\;\mu_1{}^{y_1-y_2}\,\mu_2{}^{y_2} = \frac{(\mu_1+\mu_2)^{y_1}\,e^{-\mu_1-\mu_2}}{y_1!}$$

Recall $(a+b)^n = \sum_{x=0}^{n} C_x^n\,a^x\,b^{n-x}$ $\qquad y_1 = 0,1,2,\ldots$

Hance $Y_1 = x_1 + x_2 \sim p(\mu_1 + \mu_2)$.

Example: Let the stochastically independent r.v.'s such that $X_1 \sim b(n_1, p)$ and $X_2 \sim b(n_2, p)$. Find the joint pdf of $Y_1 = X_1 + X_2$ and $Y_2 = X_2$. Find also the pdf of Y_1.

$f(x_1) = C_{x_1}^{n_1} p^{x_1}(1-p)^{n_1-x_1}$ $\quad x_1 = 0,1,\ldots,n_1$ and

$f(x_2) = C_{x_2}^{n_2} p^{x_2}(1-p)^{n_2-x_2}$ $\quad x_1 = 0,1,\ldots,n_2$

$$f(x_1, x_2) = C_{x_1}^{n_1}\,C_{x_2}^{n_2}\,p^{x_1+x_2}\,(1-p)^{n_1+n_2-x_1-x_2}$$

$y_1 = x_1 + x_2 \qquad y_2 = x_2$

$x_1 = y_1 - y_2 \qquad x_2 = y_2$

$f(y_1, y_2) = C_{y_1-y_2}^{n_1} C_{y_2}^{n_2} p^{y_1}(1-p)^{n_1+n_2-y_1}$ $\quad y_1 = 0,1,\ldots,n_1+n_2, y_2 =$
$0,1,\ldots,y_1$

$f(y_1) = \sum_{y_2=0}^{y_1} C_{y_1-y_2}^{n_1} C_{y_2}^{n_2} p^{y_1}(1-p)^{n_1+n_2-y_1}$

$= P^{y_i}(1-P)^{n_1+n_2-y_1} \sum_{y_2=0}^{y_1} C_{y_1-y_2}^{y_1} C_{y_2}^{n_2}$

(Since $\sum_{x=0}^{n} C_x^a \; C_{n-x}^b = C_{n-x}^b$) then

$f(y_1) = C_{y_1}^{n_1+n_2} P^{y_1} \; (1-P)^{n_1-n_2-y_1} \;\; y_1 = 0,1,\dots,n_1+n_2$

Hance $Y_1 \sim b(n_1+n_2, P)$

2. **Continuous case**

Let X be a continuous r.v. having pdf $f(x)$. Let A be the space where $f(x) > 0$. Consider the r.v. $Y = u(x)$, where $y = u(x)$ defines a one-to-one transformation that maps the set A onto the set B. Let the inverse of $y = u(x)$ be denoted by $x = w(y)$ and let the derivative $\frac{dx}{dy} = w(y)$ be continuous and not equal zero for all points y in B. Then the pdf of the r.v. $Y = u(x)$ is

$g(y) = f\big(w(y)\big) \;\; |w(y)| \qquad y \in B$

$= f\big(w(y)\big) \;\; |J|$

Where $J = \frac{dx}{dy} = w(y)$ is reffered to as the Jacobian of the transformation.

Example:

Let X be r.v. having pdf $f(x) = 2x, \quad 0 < x < 1.$ Define the r.v. $Y = 8X^3$. Find the pdf of Y.

$A = \{x: 0 < x < 1\}$

$B = \{y: 0 < x < 8\}$

$y = u(x) = 8x^3$

$x = w(y) = \frac{1}{2}\sqrt[3]{y} \qquad |J| = \left|\frac{dx}{dy}\right| = \frac{1}{6} y^{\frac{-2}{3}}$

$\therefore \; g(y) = f\big(w(y)\big)|J| = 2\frac{1}{2}\sqrt[3]{y}\frac{1}{6(\sqrt[3]{y})^2} = \frac{1}{6\sqrt[3]{y}}$

Example:

Let the r.v. $X \sim U(0,1)$ show that r.v. $Y = -2\ln x$ has a Chi square distribution with 2. d.f.

$y = u(x) = -2\ln x \therefore x = w(y) = e^{-y/2}$

$J = \frac{dx}{dy} = -\frac{1}{2}e^{-y/2}$

$\therefore g(y) = f(w(y))|J| = 1.\frac{1}{2}e^{-y/2} 0 < y < \infty$

$\therefore Y \sim \chi^2(2)$

Example:

Let $\chi \sim U\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Show that $Y = \tan X$ has a Cauchy distribution.

$f(x) = \frac{1}{\pi/2-(-\pi/2)} = \frac{1}{\pi}$ $with$ $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$y = u(x) = tanx$ $then$ $x = \tan^{-1}y$ if $x = -\pi/2$ $then$ $\tan(-\pi/2) = -\infty$, and if $x = \pi/2$ $then$ tan $\pi/2 = \infty$

$g(y) = f(\tan^{-1}y)|J|$

$J = \frac{dx}{dy} = \frac{1}{1+y^2}$

$\therefore g(y) = \frac{1}{\pi(1+y^2)} - \infty < y < \infty$

In the bivariate case, let $y_1 = u_1(x_1, x_2)$ and $y_2 = u_2(x_1, x_2)$ define a one -to-one transformation that maps a set A in the x_1x_2- plane onto a set B in the y_1y_2-plane if we express each of x_1 and x_2 in terms of y_1 and y_2, we can write $x_1 = w_1(y_1, y_2)$, $x_2 = w_2(y_1, y_2)$.

The Jacobian of the transformation will be

$$J = \begin{vmatrix} \frac{dx_1}{dy_1} & \frac{dx_1}{dy_2} \\ \frac{dx_2}{dy_1} & \frac{dx_2}{dy_2} \end{vmatrix}$$

Lecture 4

The joint pdf of $Y_1 = u_1(x_1, x_2)$ and $Y_2 = u_2(x_1, x_2)$ is $g(y_1, y_2) = h[w_1(y_1, y_2), w_2(y_1, y_2)]|J|$ $(y_1, y_2) \in B$

And the marginal pdf $g_1(y_1)$ of Y_1 can be obtained from $g(y_1, y_2)$ by integrating on y_2, and the marginal pdf $g_2(y_2)$ of Y_2 can be obtained from $g(y_1, y_2)$ by integrating on y_1

Example:

Let χ_1 and χ_2 denote a r.s. from $U(0,1)$. The joint pdf is then $f(x_1, x_2) = f(x_1)f(x_2) = 1$ $\quad with \quad 0 < x_1 < 1$

Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$ $\quad 0 < x_2 < 1$

Find the joint pdf of Y_1 and Y_2

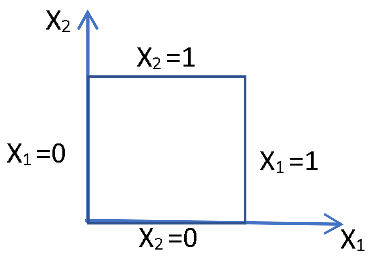
$A = \{(x_1, x_2): 0 < x_1 < 1, \quad 0 < x_2 < 1\}$

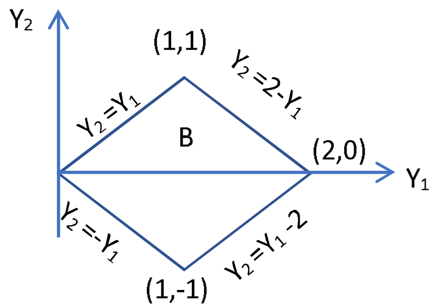
To determine the set B onto which A is mapped under the transformation, note that $y_1 + y_2 = x_1 + x_2 + X_1 - X_2 = 2\,x_1$

$y_1 - y_2 = x_1 + x_2 - x_1 + x_2 = 2x_2$

$x_1 = w_1(y_1, y_2) = \frac{1}{2}(y_1 + y_2)$

$x_2 = w_2(y_1, y_2) = \frac{1}{2}(y_1 - y_2)$





Now to determine the set B, the boundaries of A are transformed as follows:

$x_1 = 0 \Rightarrow 0 = \frac{1}{2}(y_1 + y_2) \Rightarrow y_2 = -y_1$

$x_1 = 1 \Rightarrow 1 = \frac{1}{2}(y_1 + y_2) \Rightarrow y_2 = 2 - y_1$

$x_2 = 0 \Rightarrow 0 = \frac{1}{2}(y_1 - y_2) \Rightarrow y_2 = y_1$

$x_2 = 1 \Rightarrow 1 = \frac{1}{2}(y_1 - y_2) \Rightarrow y_2 = y_1 - 2$

$$J = \begin{vmatrix} \frac{dx_1}{dy_1} & \frac{dx_1}{dy_2} \\ \frac{dx_2}{dy_1} & \frac{dx_2}{dy_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$g(y_1, y_2) = f\left[\frac{1}{2}(y_1 + y_2), \frac{1}{2}(y_1 - y_2)\right] |J|$

$= 1.\frac{1}{2} = \frac{1}{2} \qquad (y_1 - y_2) \in B$

$= \qquad 0 \qquad e.w.$

Where $B = \{(y_1, y_2): 0 < y_1 < 2, \quad -1 < y_2 < 1\}$

Example:

Let χ_1, χ_2 be a.r.s. of size $n = 2$ from $N(0,1)$.

Define $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$. Find the joint pdf of Y_1 and Y_2 and show that Y_1 and Y_2 are stochastically independent.

$f(x_1, x_2) = f(x_1).f(x_2) = \frac{1}{2\pi} exp\left[-\frac{1}{2}({x_1}^2 + {x_2}^2)\right] \qquad -\infty < x_i < \infty$

$y_1 = x_1 + x_2 \qquad\qquad i = 1,2$

$y_2 = x_1 - x_2 \qquad A = \{(x_1, x_2): -\infty < x_i < \infty, i = 1,2\}$

$B = \{(y_1, y_2): -\infty < y_i < \infty, i = 1,2\}$

$y_1 + y_2 = 2x_1 \Longrightarrow x_1 = \frac{1}{2}(y_1 + y_2)$

$y_1 - y_2 = 2x_2 \Longrightarrow x_2 = \frac{1}{2}(y_1 - y_2)$

$$J = \begin{vmatrix} \frac{dx_1}{dy_1} & \frac{dx_1}{dy_2} \\ \frac{dx_2}{dy_1} & \frac{dx_2}{dy_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

The joint pdf of Y_1 and Y_2 is

$$g(y_1,y_2) = f\left(\frac{1}{2}(y_1+y_2), \frac{1}{2}(y_1-y_2)\right)|J|$$

$$= \frac{1}{2\pi} exp\left[-\frac{1}{2}\left(\frac{1}{4}(y_1+y_2)^2 + \frac{1}{4}(y_1-y_2)^2\right)\right].\frac{1}{2}$$

$$= \frac{1}{2\pi} exp\left[-\frac{1}{2}\left(\frac{1}{4}({y_1}^2+2y_1y_2+{y_2}^2) + \frac{1}{4}(y_1-y_2)^2\right)\right].\frac{1}{2}$$

$$= \frac{1}{2\pi} exp\left[-\frac{1}{2}\left(\frac{1}{4}(2{y_1}^2+2{y_2}^2)\right)\right].\frac{1}{2}$$

$$= \frac{1}{2\pi} exp\left[-\frac{1}{2}\left(\frac{{y_1}^2+{y_2}^2}{2}\right)\right].\frac{1}{2}$$

$$= \frac{1}{4\pi} exp\left[-\frac{1}{4}({y_1}^2+{y_2}^2)\right] \qquad -\infty < y_i < \infty i = 1,2$$

$$g(y_1) = \int_{-\infty}^{\infty} g(y_1,y_2)dy_2 = \frac{1}{4\pi} e^{-\frac{1}{4}{y^2}_1} \int_{-\infty}^{\infty} e^{-\frac{1}{4}{y^2}_2} dy_2$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{1}{2}\frac{y_1^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{1}{2}\frac{y_2^2}{2}} dy_2$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{1}{2}\frac{y_1^2}{2}} \qquad -\infty < y_1 < \infty$$

That is $Y_1 \sim N(0,2)$ similarly $Y_2 \sim N(0,2)$ and $g(y_1,y_2) = g_1(y_1).g_2(y_2)$

Therefore Y_1 and Y_2 are stochastically independent.

Example: Let χ_1, χ_2 be a random sample of size $n = 2$ from exponential distribution with $\lambda = 1$. Define the random variables $Y_1 = \frac{x1}{x_1+x_2}$ and $Y_2 = x_1 + x_2$. Find the joint and marginal pdf's of Y_1 and Y_2 and show that Y_1 and Y_2 are stochastically independent

$$f(x_1,x_2) = e^{-x_1-x_2} = e^{-(x_1+x_2)} \qquad 0 < x_i < \infty \quad i = 1,2$$

$$A = \{(x_1,x_2): 0 < x_i < \infty, i = 1,2\}$$

$$A = \{(y_1,y_2): 0 < y_1 < 1, \qquad 0 < y_2 < \infty\}$$

$$y_1 = \frac{x1}{x_1+x_2} \Rightarrow y_1 = \frac{x1}{y_2} \Rightarrow x_1 = y_1y_2$$

$$y_2 = x_1 + x_2 \Rightarrow y_2 = y_1y_2 + x_2 \Rightarrow x_2 = y_2 - y_1y_2$$

$$= y_2(1-y_1)$$

$$J = \begin{vmatrix} \frac{dx_1}{dy_1} & \frac{dx_1}{dy_2} \\ \frac{dx_2}{dy_1} & \frac{dx_2}{dy_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ -y_2 & 1-y_2 \end{vmatrix} == y_2(1-y_1) + y_1y_2 = y_2 - y_1y_2 +$$

$$y_1y_2 = y_2$$

$\therefore g(y_1, y_2) \equiv f(y_1y_2, y_2 - y_1y_2)|J|$

$= y_2\, e^{-y_2} \qquad 0 < y_1 < 1, 0 < y_2 < \infty$

$g_1(y_1) = \int_0^\infty g(y_1, y_2) d\, y_2 = \int_0^\infty y_2 e^{-y_2}\, dy_2 = 1 \qquad 0 < y_1 < 1$

$g_2(y_2) = \int_0^1 g(y_1, y_2) d\, y_1 = \int_0^1 y_2 e^{-y_2}\, dy_1 = y_2 e^{-y_2} \qquad 0 < y_2 < \infty$

That is $Y_1 \sim V(0,1)$ and $Y_2 \sim G(1,2)$

$g_1(y_1). g_2(y_2) = y_2 e^{-y_2} = g(y_1, y_2)$

$\therefore$ Y_1 and Y_2 are stochastically independent

Example: Let χ_1 and χ_2 have the joint pdf

$f_{(\chi_1, \chi_2)}(x_1, x_2) = \lambda^2 e^{-\lambda(x_1 +, x_2)} \qquad x_1 > 0, x_2 > 0$

$= 0 \qquad e.w.$

Find the joint pdf of Y_1 and Y_2 if $Y_1 = \chi_1 + \chi_2$ and $Y_2 = \chi_2$

$A = \{(x_1, x_2): x_1 > 0, x_2 > 0\}$

$B = \{(y_1, y_2): 0 > y_2 < y_1, 0 < y_1 < \infty\}$

$x_1 > 0$

$y_1 = x_1 + x_2 \qquad y_2 = x_2 \qquad y_1 - y_2 > 0$

$x_1 = y_1 + y_2 \qquad x_2 = y_2 \qquad y_1 > y_2 \; or \; y_2 < y_1$

$$J = \begin{vmatrix} \frac{dx_1}{dy_1} & \frac{dx_1}{dy_2} \\ \frac{dx_2}{dy_1} & \frac{dx_2}{dy_2} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

$\therefore g(y_1, y_2) = f(y_1 - y_2, y_2). |J|$

$= \lambda^2 e^{-\lambda(y_1)}. 1 = \lambda^2 e^{-\lambda(y_1)} \qquad 0 < y_2 < y_1 < \infty$

The marginal pdf of Y_1 is

$$g_1(y_1) = \int_{y_2}^{y_1} g(y_1, y_2) d\, y_2 = \int_0^{y_1} \lambda^2 e^{-\lambda y_1} d\, y_2$$

$$= \lambda^2 e^{-\lambda y_1} \int_0^{y_1} dy_2 = \lambda^2 e^{-\lambda y_1} y_2]^{y_1}{}_0$$

$$= \lambda^2\, y_1\, e^{-\lambda y_1} \qquad y_1 > 0$$

Exercise :

Let χ_2 and have in dep gamma with parameters α, θ and β, θ respectively. Consider $Y_1 = \frac{x_1}{x_1+x_2}$ and $Y_2 = x_1 + x_2$. Find the joint and marginal pdf's of Y_1 and Y_2 and show that they are stochastically in dep.

$\therefore$ Y_1 and Y_2 are stochastically in dep.

The marginal pdf of Y_1 is

$$g_1(y_1) = \int_{y_2}^{y_1} g(y_1, y_2)d\,y_2 = \int_0^{y_1} \lambda^2 e^{-\lambda y_1} d\,y_2$$

$$= \lambda^2 e^{-\lambda y_1} \int_0^{y_1} dy_2 = \lambda^2 e^{-\lambda y_1} y_2]^{y_1}{}_0$$

$$= \lambda^2\, y_1\, e^{-\lambda y_1} \qquad y_1 > 0$$

Exercise :

Let χ_2 and have in dep gamma with parameters α, θ and β, θ respectively. Consider $Y_1 = \frac{x_1}{x_1+x_2}$ and $Y_2 = x_1 + x_2$. Find the joint and marginal pdf's of Y_1 and Y_2 and show that they are stochastically in dep.

$\therefore Y_1$ and Y_2 are stochastically in dep.

Gamma Distribution:

$$X \sim \Gamma(\alpha, \beta) \equiv \text{Gamma}(\alpha, \beta)$$

The corresponding probability density function in the shape-rate parametrization is

$$f(x; \alpha, \beta) = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \quad \text{for } x > 0 \quad \alpha, \beta > 0,$$

where $\Gamma(\alpha)$ is the gamma function. For all positive integers, $\Gamma(\alpha) = (\alpha - 1)!$.

The Beta Distribution

Let X_1 and X_2 be two independent random variables that have gamma distributions with parameters $(\alpha, 1)$ and $(\beta, 1)$ respectively. The joint pdf is

$$h\,(x_1, x_2) = \frac{1}{\Gamma(\alpha)\,\Gamma(\beta)}\, x_1{}^{\alpha-1}\, x_2{}^{\beta-1} e^{-x_1-x_2} \quad 0 < x_i < \infty\,, i = 1,2 \quad \alpha > 0,$$

$B > 0$.

Let $Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_1}{X_1+X_2}$. Show that $Y_2 \sim Beta(\alpha, \beta)$.

$A = \{(x_1, x_2): 0 < x_i < \infty, i = 1, 2\}$

$B = \{(y_1, y_2): 0 < y_1 < \infty, 0 < y_2 < 1\}$

$y_1 = u_1(x_1, x_2) = x_1 + x_2$

$y_2 = u_2(x_1, x_2) = \frac{x_1}{x_1+x_2}$

Hence,

$x_1 = y_1 y_2$ and $x_2 = y_1 - y_1 y_2 = y_1(1 - y_2)$

$$J = \begin{vmatrix} \frac{dx_1}{dy_1} & \frac{dx_1}{dy_2} \\ \frac{dx_2}{dy_1} & \frac{dx_2}{dy_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ 1 - y_2 & -y_1 \end{vmatrix} = -y_1 y_2 - y_1 + y_1 y_2 = -y_1$$

$$g(y_1, y_2) = y_1 \frac{1}{\Gamma(\alpha)\,\Gamma(\beta)} (y_1 y_2)^{\alpha-1} [y_1(1 - y_2)\,]^{\beta-1} e^{-y_1}$$

$$= \frac{{y_2}^{\alpha-1}\,(1-y_2)^{\beta-1}}{\Gamma(\alpha)\,\Gamma(\beta)}\; {y_1}^{\alpha+\beta-1}\, e^{-y_1} \qquad 0 < y_1 < \infty, 0 < y_2 < 1$$

$$g_2(y_2) = \frac{{y_2}^{\alpha-1}\,(1-y_2)^{\beta-1}}{\Gamma(\alpha)\,\Gamma(\beta)} \int_0^\infty {y_1}^{\alpha+\beta-1}\, e^{-y_1}\, dy_1$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\, {y_2}^{\alpha-1}(1 - y_2)^{\,\beta-1} \quad 0 < y_2 < 1$$

This pdf is that of a beta distribution with parameters α and β.

Since $g(y_1, y_2) = g_1(y_1). g_2(y_2)$, the pdf of Y_1 is

$$g_1(y_1) = \frac{1}{\Gamma(\alpha+\beta)}\, {y_1}^{\alpha+\beta-1}\, e^{-y_1} \quad 0 \,<\, y_1 < \infty$$

Which is that of a gamma distribution with parameter values of $\alpha + B$ and 1.

Assignment: Find the mean and the variance of the beta distribution.

Lecture 6

Definition;

Student's ***t*-distribution** has the probability density function given by

$$f(t) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\,\Gamma(\frac{\nu}{2})}\left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}},$$

where ν is the number of *degrees of freedom* and Γ is the gamma function.

Theorem

Let W denote a random variable that is $N(0,1)$; let V denote a random variable that is $\chi^2_{(n)}$; and let W and V be independent.

Then $T = \frac{W}{\sqrt{V/n}}$ has a t distribution with n degrees of freedom. Its pdf is

$$g_1(t) = \frac{\Gamma[(n+1)/2]}{\sqrt{\pi n}\Gamma(n/2)\,(1+t^2/n)^{(n+1)/2}} \qquad -\infty < t < \infty$$

Proof:

The joint pdf of W and V is

$$h(w,v) = \frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}w^2}\,\frac{1}{\Gamma(n/2)2^{n/2}}\,V^{\frac{n}{2}-1}\,e^{-\frac{v}{2}} \quad -\infty < w < \infty\,, 0 < v < \infty$$

Define a new random variable $\quad T = \frac{W}{\sqrt{V/n}}$

Let $t = \frac{w}{\sqrt{V/n}}$ and $u = v$ define a one-to-one transformation that maps

$A = \{(w,v): -\infty < w < \infty, 0 < v < \infty\}$ onto

$B = \{(t,u): -\infty < t < \infty, 0 < u < \infty\}$.

Since $w = t\sqrt{u/n}$ and $v = u$

$$J = \begin{vmatrix} \frac{dw}{dt} & \frac{dw}{du} \\ \frac{dv}{dt} & \frac{dv}{du} \end{vmatrix} = \begin{vmatrix} \sqrt{\frac{u}{n}} & \frac{t}{\sqrt{n}}\frac{1}{2\sqrt{u}} \\ 0 & 1 \end{vmatrix} = \frac{\sqrt{u}}{\sqrt{n}}$$

Accordingly, the joint pdf of T and U is

$$g(t,u) = h\left(\frac{t\sqrt{u}}{\sqrt{n}}, u\right).|J|$$

$$= \frac{1}{\sqrt{2\pi}\Gamma\left(\frac{n}{2}\right)2^{n/2}} \; u^{\frac{n}{2}-1} \exp\left[-\frac{1}{2}\left(\frac{t^2u}{n}+u\right)\right]\frac{\sqrt{u}}{\sqrt{n}}$$

$$= \frac{1}{\sqrt{2\pi n}\Gamma\left(\frac{n}{2}\right)2^{n/2}} \; u^{\frac{n}{2}+\frac{1}{2}-1} \exp[-\frac{u}{2}\left(1+\frac{t^2}{n}\right)] \quad -\infty < t < \infty \; , 0 < u < \infty$$

The marginal pdf of T is

$$g_1(t) = \int_0^\infty g(t,u)du$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi n}\Gamma\left(\frac{n}{2}\right)2^{n/2}} \; u^{\frac{(n+1)}{2}-1} \exp\left[-\frac{u}{2}\left(1+\frac{t^2}{n}\right)\right] du$$

Let $z = \frac{u}{2}\left[1+\frac{t^2}{n}\right]$ then $u = \frac{2z}{1+\frac{t^2}{n}}$ and $du = \frac{2}{1+\frac{t^2}{n}}\, dz$

$$g_1(t) = \int_0^\infty \frac{1}{\sqrt{2\pi n}\Gamma\left(\frac{n}{2}\right)2^{n/2}} \left(\frac{2z}{1+\frac{t^2}{n}}\right)^{\frac{(n+1)}{2}-1} e^{-z} \left(\frac{2}{1+\frac{t^2}{n}}\right) dz$$

$$= \frac{1}{\sqrt{\pi n}\Gamma\left(\frac{n}{2}\right)2^{(n+1)/2}} \; 2^{(n+1)/2} \; \frac{1}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} \int_0^\infty z^{\frac{(n+1)}{2}-1} e^{-z}\, dz$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\Gamma\left(\frac{n}{2}\right)\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} \qquad -\infty < t < \infty$$

Thus, if $W \sim N(0,1)$, $V \sim \chi^2_{(n)}$, and if W and V are independent. Then

$$T = \frac{W}{\sqrt{V/n}} \sim t_{(n)}$$

It is, in general, difficult to evaluate the distribution function of T. Some approximate values of $p(T \le t) = \int_{-\infty}^{t} g_1(w)dw$ are found for selected values of n and t in special tables. The t distribution is symmetric about $t = 0$. That is $E(T) = 0$ where $n \ge 2$. When $n = 1$ the $t - dist$ribution reduced to the Cauchy distribution.

Lecture 7

Example

Let $X \sim t_{(7)}$,then

$P(X \leq 1.415) = 0.90$

And $P(X \leq -1.415) = 1 - P(X \leq 1.415) = 0.10$

Theorem

Let $T \sim t_{(n)}$.Then $E(T) = 0, n \geq 2$ and $Var(T) = \frac{n}{n-2}, n \geq 3$

Proof

Using the definition of T and the independence of W and V

$$E(T) = E\left[\frac{W}{\sqrt{\frac{V}{n}}}\right] = E(W)E\left(\frac{\sqrt{n}}{\sqrt{V}}\right) = 0$$

Since $W \sim N(0,1)$, $E(W) = 0, Var(W) = 1$

$Var\ (T) = E(T^2) - [E(T)]^2$

$$E(T^2) = E\left(\frac{W}{\sqrt{V/n}}\right)^2 = n\,E(W^2)E\left(\frac{1}{V}\right)$$

$E(w^2) = 1$

$$E(V^{-1}) = \int_0^\infty \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{n/2}}\, V^{-1}\, V^{\frac{n}{2}-1}\ \ e^{-\frac{V}{2}}\, dv$$

$$= \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{n/2}} \int_0^\infty V^{\left(\frac{n}{2}-1\right)-1}\ \ e^{-\frac{V}{2}}\, dv$$

Let $y = \frac{v}{2}$, then $\ v = 2y\ and \quad dv = 2dy$

$$E(V^{-1}) = \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{n/2}} \int_0^\infty (2y)^{\left(\frac{n}{2}-1\right)-1}\ \ e^{-y}\ 2dy = \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{n/2}}\ 2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}-1\right)$$

$$E(V^{-1}) = \frac{2^{-1}}{\Gamma\left(\frac{n}{2}\right)}\ \Gamma\left(\frac{n}{2}-1\right) = \frac{2^{-1}\Gamma\left(\frac{n}{2}-1\right)}{\left(\frac{n}{2}-1\right)\Gamma\left(\frac{n}{2}-1\right)}$$

Recall that $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$

$$= \frac{1}{2\frac{n-2}{2}} = \frac{1}{n-2}$$

$$E(T^2) = n\,E(W^2)\,E\left(\frac{1}{V}\right)$$

$$\therefore E(T^2) = n.1.\frac{1}{n-2} = \frac{n}{n-2} = Var(T) \qquad\qquad n \geq 3$$

The F- distribution

Theorem:

If U and V are independent chi-square random variables with n and m degrees of freedom respectively, then

$F = \frac{U/n}{V/m}$ has an F- distribution with n and m d.f.

Proof:

The joint pdf of U and V is

$$h(u,v) = \frac{1}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)2^{(n+m)/2}}\ u^{\frac{n}{2}-1}V^{\frac{m}{2}-1}e^{-\frac{u+v}{2}} \qquad 0 < u < \infty\ ,0 < v < \infty$$

Define the new random variable $W = \frac{U/n}{V/m}$

The equations $w = \frac{u/n}{v/m}$ and $z = v$ define a one-to-one transformation that maps the set $A = \{(u,v): 0 < u < \infty, 0 < V < \infty\}$ onto the set

$B = \{(w,z): 0 < w < \infty, 0 < z < \infty\}$.

Since $\frac{u}{n} = w\frac{v}{m}$ then $u = \frac{n}{m}wz$ and $v = z$. The Jacobian is

$$J = \begin{vmatrix} \frac{du}{dw} & \frac{du}{dz} \\ \frac{dv}{dw} & \frac{dv}{dz} \end{vmatrix} = \begin{vmatrix} \frac{n}{m}z & \frac{n}{m}w \\ 0 & 1 \end{vmatrix} = \frac{n}{m}z$$

The joint pdf of the random variables W and Z is

$$g(w,z) = \frac{1}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)2^{(n+m)/2}}\left(\frac{n}{m}wz\right)^{\frac{n}{2}-1} z^{\frac{m}{2}-1}e^{-\frac{z}{2}\left(\frac{n}{m}w+1\right)}\frac{n}{m}z$$

The marginal pdf of W is $g_1\,(w) = \int_0^{\infty} g(w,z)\,dz$

$$= \int_0^\infty \frac{\left(\frac{n}{m}\right)^{n/2} w^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right) 2^{(n+m)/2}} \; z^{\frac{n+m}{2}-1} \; e^{-\frac{z}{2}\left(\frac{n}{m}w+1\right)} \; dz$$

Let $y = \frac{z}{2} \left(\frac{n}{m} \; w + 1 \right)$ then $z = \frac{2y}{\frac{n}{m}w+1}$

$$\therefore dz = \frac{2}{\left(\frac{n}{m}w+1\right)} \; dy$$

$$g_1(w) = \int_0^\infty \frac{\left(\frac{n}{m}\right)^{\frac{n}{2}} w^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right) 2^{\frac{(n+m)}{2}}} \left(\frac{2y}{\frac{n}{m}w+1}\right)^{\frac{(n+m)}{2}-1} e^{-y} \left(\frac{2}{\frac{n}{m}w+1}\right) dy$$

$$= \frac{\left(\frac{n}{m}\right)^{\frac{n}{2}} w^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)\left(\frac{n}{m}w+1\right)^{\frac{(n+m)}{2}}} \int_0^\infty y^{\frac{(n+m)}{2}-1} e^{-y} \, dy$$

$$= \frac{\Gamma\left(\frac{n+m}{2}\right)\left(\frac{n}{m}\right)^{\frac{n}{2}} w^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)\left(1+\frac{n}{m}w\right)^{\frac{(n+m)}{2}}} \qquad 0 < w < \infty$$

This pdf is usually called an F-distribution and the ratio $F = \frac{U/n}{V/m}$ has an F-distribution with n and m d.f. Approximate values of $P(F \le b) = \int_0^b g_1(w)dw$ are available for selected values of n, m and b.

Example:

When $n = 7, m = 8, P(F \le 3.50) = 0.95$

When $n = 9, m = 4, P(F \le 14.7) = 0.99$

Remark:

Since $F = \frac{U/n}{V/m} \sim F(n,m)$, then $\frac{1}{F} = \frac{V/m}{U/n} \sim F(m,n)$

For example, if $F \sim F(4,9)$ such that $P(F(4,9) \le c) = 0.01$

Then $P\left(\frac{1}{F(4,9)} \ge \frac{1}{c}\right) = 0.01$ or $P\left(\frac{1}{F(4,9)} \le \frac{1}{c}\right) = 0.99$

Which is equivalent to $P\left(F(9,4) \le \frac{1}{c}\right) = 0.99$

From F tables $\frac{1}{c} = 14.7 \quad \therefore \; c = \frac{1}{14.7} = 0.0682$

Theorem

If $X \sim F(n,m)$, then $E(X^r) = \left(\frac{m}{n}\right)^r \frac{\Gamma\left(\frac{n}{2}+r\right)\Gamma\left(\frac{m}{2}-r\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)} \qquad m > 2r$

Proof

Since $X \sim F(n,m)$, then $X = \frac{U/n}{V/m}$ where $U \sim \chi^2_{(n)}$ and $V \sim \chi^2_{(m)}$

$$E(X^r) = E\left(\frac{U/n}{V/m}\right)^r = \left(\frac{m}{n}\right)^r E(U^r)\, E(V)^{-r}$$

$$E(U^r) = \int_0^\infty u^r\, f(u) du = \int_0^\infty u^r \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{\left(\frac{n}{2}\right)}}\, u^{\frac{n}{2}-1}\, e^{-\frac{u}{2}}\, du$$

$$= \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{\left(\frac{n}{2}\right)}} \int_0^\infty u^{\frac{n}{2}+r-1}\, e^{-\frac{u}{2}}\, du$$

Let $y = \frac{u}{2}$ $\quad then\; u = 2y \quad and \quad du \;\Rightarrow du = 2dy$

$$E(U^r) = \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{\left(\frac{n}{2}\right)}} \int_0^\infty (2y)^{\frac{n}{2}+r-1}\, e^{-y}\; 2dy$$

$$= \frac{2^r}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty y^{\frac{n}{2}+r-1}\, e^{-y}\, dy = \frac{2^r}{\Gamma\left(\frac{n}{2}\right)}\, \Gamma\left(\frac{n}{2}+r\right)$$

$$E(V^{-r}) = \int_0^\infty v^{-r}\, f(v)\, dv$$

$$= \int_0^\infty \frac{1}{\Gamma\left(\frac{m}{2}\right) 2^{\left(\frac{m}{2}\right)}}\, v^{-r} v^{\frac{m}{2}-1} e^{-\frac{v}{2}}\, dv$$

$$= \frac{1}{\Gamma\left(\frac{m}{2}\right) 2^{\left(\frac{m}{2}\right)}} \int_0^\infty v^{\frac{m}{2}-r-1}\, e^{-\frac{v}{2}}\, dv$$

Let $y = \frac{v}{2}$ $\quad then\; v = 2y$ and $\quad dv = 2dy$

$$E(V^{-r}) = \frac{1}{\Gamma\left(\frac{m}{2}\right) 2^{\left(\frac{m}{2}\right)}} \int_0^\infty (2y)^{\frac{m}{2}-r-1}\, e^{-y}\, 2dy$$

$$= \frac{2^{-r}}{\Gamma\left(\frac{m}{2}\right)} \Gamma\left(\frac{m}{2} - r\right)$$

$$\therefore E(X^r) = \left(\frac{m}{n}\right)^r \frac{2^r}{\Gamma\left(\frac{n}{2}\right)} \Gamma\left(\frac{n}{2} + r\right) \frac{2^{-r}}{\Gamma\left(\frac{m}{2}\right)} \Gamma\left(\frac{m}{2} - r\right)$$

$$= \left(\frac{m}{n}\right)^r \frac{\Gamma\left(\frac{n}{2}+r\right)\Gamma\left(\frac{m}{2}-r\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)}$$

Now

$$E(X) = \frac{m}{n} \frac{\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{m}{2}-1\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)} = \frac{m}{n} \frac{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}-1\right)}{\Gamma\left(\frac{n}{2}\right)\left(\frac{m}{2}-1\right)\Gamma\left(\frac{m}{2}-1\right)}$$

$$= \frac{m}{n} \frac{\frac{n}{2}}{\frac{m-2}{2}} = \frac{m}{m-2} \qquad , \ m > 2$$

$$E(X^2) = \left(\frac{m}{n}\right)^2 \frac{\Gamma\left(\frac{n}{2}+2\right)\Gamma\left(\frac{m}{2}-2\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)}$$

$$= \frac{m^2}{n^2} \frac{\left(\frac{n}{2}+1\right)\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{m}{2}-2\right)}{\Gamma\left(\frac{n}{2}\right)\left(\frac{m}{2}-1\right)\Gamma\left(\frac{m}{2}-1\right)}$$

$$= \frac{m^2}{n^2} \frac{\left(\frac{n}{2}+1\right)\left(\frac{n}{2}\right)\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}-2\right)}{\Gamma\left(\frac{n}{2}\right)\left(\frac{m}{2}-1\right)\left(\frac{m}{2}-2\right)\Gamma\left(\frac{m}{2}-2\right)} = \frac{m^2}{n} \frac{\frac{1}{2}\left(\frac{n+2}{2}\right)}{\left(\frac{m-2}{2}\right)\left(\frac{m-4}{2}\right)}$$

$$= \frac{m^2(n+2)}{n\,(m-2)\,(m-4)}$$

$$\therefore Var\,(X) = E(X^2) - [E(X)]^2 = \frac{m^2(n+2)}{n\,(m-2)\,(m-4)} - \frac{m^2}{(m-2)^2}$$

$$= \frac{m^2(n+2)(m-2) - m^2 n(m-4)}{n(m-2)^2(m-4)}$$

$$= \frac{m^2[(n+2)(m-2) - n(m-4)]}{n(m-2)^2(m-4)} = \frac{m^2(nm+2m-2n-4-nm+4n)}{n(m-2)^2(m-4)}$$

$$= \frac{m^2(2m+2n-4)}{n(m-2)^2(m-4)} = \frac{2m^2(m+n-2)}{n(m-2)^2(m-4)}$$

Lecture 9

Order statistics

Let the random variables $X_1, X_2, \dots, X_n$ form a random sample of size n from a distribution for which the pdf is $f(x)$ and the distribution function is $F(x)$.

We denote the ordered random variables $Y_1 < Y_2 < \cdots < Y_n$ the order statistics of that sample. That is:

Y_1 is the smallest of $X_1, X_2, \dots, X_n$

Y_2 is the second smallest of $X_1, X_2, \dots, X_n$

$\vdots$

Y_n is the largest of $X_1, X_2, \dots, X_n$

The sample range R is the distance between the smallest and the largest observation $R = Y_n - Y_1$ is an important statistic which is defined using order statistics.

The joint p.d.f of $Y_1, Y_2, \dots, Y_n$ is

$$g(y_1, y_2, \dots, y_n) = \begin{cases} n!\, f(y_1)f(y_2)\cdots f(y_n) & -\infty < y_1 \le y_2 \le \cdots \le y_n \le \infty \\ 0 & o.w \end{cases}$$

The multiplier $n!$ arises because $y_1, \dots, y_n$ can be arranged among themselves in $n!$ ways and the p.d.f for any such single arrangement amounts to $\prod_{i=1}^{n} f(y_i)$.

Definition

The largest value Y_n in the random sample is defined as follows

$Y_n = \max\{X_1, X_2, \dots, X_n\}$

For every given value of $y(-\infty < y < \infty)$

$$G_n(y) = P(Y_n \leq y) = P(X_1 \leq y, \dots, X_n \leq y)$$

$$= P(X_1 \leq y)P(X_2 \leq y) \dots P(X_n \leq y) \qquad X_i \text{ independent}$$

$$= [F(y)]^n$$

The p.d.f of Y_n is

$$g_n(y_n) = n[F(y_n)]^{n-1}f(y_n) \qquad -\infty < y_n < \infty$$

The smallest value Y_1 in the random sample is defined as follows

$$Y_1 = \min[X_1, X_2, \dots, X_n]$$

For every given value of $y(-\infty < y < \infty)$

$$G_1(y) = P(Y_1 \leq y) = 1 - P(Y_1 > y)$$

$$= 1 - P(X_1 > y, X_2 > y, \dots, X_n > y)$$

$$= 1 - [1 - F(y)]^n$$

The p.d.f of Y_1 is

$$g_1(y_1) = n[1 - F(y_1)]^{n-1}f(y_1) \quad -\infty < y_1 < \infty$$

Definition

Let $Y_1 < Y_2 < \cdots < Y_n$ be the order statistics of a random sample of size n from a distribution of a continuous type with distribution function $F(x)$ and p.d.f $f(x) = F'(x)$. If Y_r denote the rth order statistic, then the pdf of Y_r is

$$g_r(y_r) = \frac{n!}{(r-1)!\,(n-r)!}\,[F(y_r)]^{r-1}\,[1-F(y_r)]^{n-r} f(y_r)$$

Theorem:

For a random sample of size n the distribution function of the rth order statistic is

$$G_r(y_r) = \sum_{j=r}^{n} \binom{n}{j} [F(y_r)]^j [1-F(y_r)]^{n-j}$$

Example:

Let $Y_1 < Y_2 < Y_3 < Y_4 < Y_5$ be the order statistics of a random sample X_1, X_2, X_3, X_4, X_5 of size $n = 5$ from a distribution with pdf $f(x) = 2x, 0 < x < 1$, then $F_X(x) = \int_0^x f(t)dt = 2\frac{t^2}{2}]_0^x = x^2, 0 < x < 1.$

That is $F_X(y) = P(X \leq y) = y^2$. Find:

1. $g_1(y_1) = n[1-F(y_1)]^{n-1} f(y_1) = 5[1-y_1^2]^4\, 2y_1 = 10y_1[1-y_1^2]^4$

$G_1(y_1) = 1 - [1-F(y_1)]^n = 1-[1-y_1^2]^5 \qquad 0 < y_1 < 1$

2. $g_5(y_5) = 5[F(y_5)]^{5-1} f(y_5) = 5[y_5^2]^4 2y_5 = 10y_5^9 \qquad 0 < y_5 < 1$

$G_5(y_5) = [F(y_5)]^5 = [y_5^2]^5 = y_5^{10}$

3. $g_r(y_r) = \frac{n!}{(r-1)!(n-r)!}\,[F(y_r)]^{r-1}[1-F(y_r)]^{n-r} f(y_r)$

$g_4(y_4) = \frac{5!}{3!1!}\,[y_4^2]^3[1-y_4^2]\,(2y_4) = 40y_4^7(1-y_4^2), \quad 0 < y_4 < 1$

$G_4(y_4) = \sum_{j=4}^{5} \binom{5}{j}\; [F(y_4)]^j\, [1-F(y_4)]^{5-j}$

$=\binom{5}{4}[y_4^2]^4[1-y_4^2]^1+\binom{5}{5}[y_4^2]^5=5y_4^8(1-y_4^2)+y_4^{10}$

4. $P\left(Y_4\le\frac{1}{2}\right)=5\left(\frac{1}{2}\right)^8\left(\frac{3}{4}\right)+\left(\frac{1}{2}\right)^{10}=\frac{15}{4}\frac{1}{256}+\frac{1}{1024}=\frac{16}{1024}=\frac{1}{64}$

Example:

Let $X_1\ and\ X_2$ be a random sample from a distribution with pdf

$f(x)=e^{-x}, 0\le x<\infty$. What is the density of $Y_1=\min(X_1,X_2)$.

$F(x)=\int_0^x e^{-t}dt=1-e^{-x}$

$g_1(y)=n[1-F(y)]^{n-1}f(y)$

$=2[1-1+e^{-y_1}]e^{-y_1}=2e^{-2y_1}\quad 0<y_1<y_2$

Finally, the joint pdf of any two order statistics say $Y_i<Y_j$ is

$$g_{ij}(y_i,y_j)=\frac{n!}{(i-1)!(j-i-1)!(n-j)!}[F(y_i)]^{i-1}\left[F(y_j)-F(y_i)\right]^{j-i-1}\left[1-F(y_j)\right]^{n-j}f(y_i)f(y_j)$$

The joint pdf of (Y_1,Y_n) would be given by

$$g_{1n}(y_1,y_n)=\frac{n!}{(n-2)!}[F(y_n)-F(y_1)]^{n-2}f(y_1)f(y_n)\qquad -\infty<y_1<y_n<\infty$$

Example

Let $Y_1<Y_2<Y_3$ be the order statistics of a random sample of size n=3 from a $U(0,1)$. Find the pdf of $Z_1=Y_3-Y_1$; the sample range.

Since $X\sim U(0,1)\ \therefore F(x)=x,\ 0<x<1$

The joint pdf of Y_1 and Y_3 is

$$g_{13}(y_1, y_3) = \frac{3!}{1!}\,[F(y_3) - F(y_1)]^{3-2}\, f(y_1).f(y_3)$$

$$= 6[y_3 - y_1] \qquad\qquad 0 < y_1 < y_3 < 1$$

In addition to $Z_1 = Y_3 - Y_1$, let $Z_2 = Y_3$.

The inverse function of $z_1 = y_3 - y_1$ and $z_2 = y_3$ are

$y_1 = z_2 - z_1$ and $y_3 = z_2$

The corresponding Jacobian of the one-to-one transformation is

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial z_1} & \frac{\partial y_1}{\partial z_2} \\ \frac{\partial y_3}{\partial z_1} & \frac{\partial y_3}{\partial z_2} \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = -1$$

Thus, the joint p.d.f of Z_1 and Z_2 is

$h(z_1, z_2) = 6z_1\,|-1| = 6z_1 \qquad 0 < z_1 < z_2 < 1$

Accordingly, the pdf of the range $Z_1 = Y_3 - Y_1$ is

$$h_1(z_1) = \int_{z_1}^{1} 6z_1 dz_2 = 6z_1[z_2]_{z_1}^{1} = 6z_1[1 - z_1],\ \ 0 < z_1 < 1$$

Definition

The sample median is defined to be the middle order statistic if n is odd and the average of the middle two order statistics if n is even. That is

$$m = \begin{cases} Y_{\left(\frac{n+1}{2}\right)} & when\, n\, is\, odd \\ \dfrac{Y_{(n/2)} + Y_{(n/2)+1}}{2} & when\, n\, is\, even \end{cases}$$

Example:

Let $Y_1 < Y_2 < Y_3$ be order statistics having pdf $f(x) = e^{-x}, 0 < x < \infty$. Find

1.The joint pdf of $Y_1 < Y_2 < Y_3$

$$g(y_1, y_2, y_3) = 3!\, f(y_1). f(y_2). f(y_3) = 6\, e^{-y_1}\, e^{-y_2} e^{-y_3}$$

$$= 6\, e^{-(y_1+y_2+y_3)}$$

2.The marginal p.d.f's of Y_1 and Y_3

$$g_1(y_1) = n[1 - F(y_1)]^{n-1} f(y_1) = 3[1 - (1 - e^{-y_1})]^2 e^{-y_1}$$

$$= 3e^{-3y_1} \qquad y_1 > 0$$

$$g_3(y_3) = n[F(y_n)]^{n-1} f(y_n) = 3[1 - e^{-y_3}]^2 e^{-y_3}$$

$$= 3e^{-y_3}[1 - 2e^{-y_3} + e^{-2y_3}] \qquad y_3 > 0$$

3.The joint p.d.f of Y_1 and Y_3

$$g(y_1, y_3) = \frac{3!}{1!}\, [F(y_3) - F(y_1)]\, f(y_1). f(y_3)$$

$$= 6[1 - e^{-y_3} - 1 + e^{-y_1}] e^{-y_1}\, e^{-y_3} = 6e^{-(y_1+y_3)}[e^{-y_1} - e^{-y_3}] \qquad 0 < y_1 < y_3 < \infty$$

4.The p.d.f of the median and the value of the median.

$$Y_{\frac{n+1}{2}} = Y_2 = m$$

$$g_2(y_2) = \frac{3!}{1!1!}\, [F(y_2)][1 - F(y_2)] f(y_2) = 6[1 - e^{-y_2}\,][1 - 1 + e^{-y_2}] e^{-y_2}$$

$$= 6e^{-2y_2}(1 - e^{-y_2}) \qquad\qquad 0 < y_2 < \infty$$

Lecture 11

$F(m) = F(Y_2) = \frac{1}{2}$

$1 - e^{-y_2} = \frac{1}{2} \Longrightarrow e^{-y_2} = \frac{1}{2} \Longrightarrow -y_2 = \ln\frac{1}{2} = \ln 1 - \ln 2$

$\therefore y_2 = m = \ln 2 \quad (median)$

$P(Y_1 > m) = \int_m^\infty g(y_1)dy_1 = \int_{\ln 2}^\infty 3e^{-3y_1}\, dy_1 = -e^{-3y_1}\big]_{ln2}^{\infty}$

$= -\left[0 - e^{-3\ln 2}\right] = e^{\ln 2^{-3}} = \frac{1}{2^3} = \frac{1}{8}$

Example:

Find the probability that the range of a random sample of size $n = 4$ from a $U(0,1)$ is less than $\frac{1}{2}$.

We have $f(x) = 1, \quad 0 < x < 1$. Then $F(x) = x$

Let $Z_1 = Y_4 - Y_1$ denote the sample range and we will find $P\left(Z_1 < \frac{1}{2}\right)$.

$g(y_1, y_4) = \frac{4!}{2!}\,[F(y_4) - F(y_1)]^2\, f(y_1)f(y_4)$

$= 12\,[y_4 - y_1]^2 \qquad 0 < y_1 < y_4 < 1$

Let $Z_1 = Y_4 - Y_1$ and let $Z_2 = Y_4$. The inverse functions of $z_1 = y_4 - y_1$ and $z_2 = y_4$ are $y_1 = z_2 - z_1$ and $y_4 = z_2$

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial z_1} & \frac{\partial y_1}{\partial z_2} \\ \frac{\partial y_4}{\partial z_1} & \frac{\partial y_4}{\partial z_2} \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = -1$$

$h(z_1, z_2) = 12\,[z_2 - z_2 + z_1]^2 . |-1| = 12\, z_1^2 \qquad 0 < z_1 < z_2 < 1$

$\therefore g(z_1) = \int_{z_1}^{1} 12\, z_1^2\, dz_2 = 12z_1^2\, [z_2]_{z_1}^{1} = 12\, z_1^2[1 - z_1], \quad 0 < z_1 < 1$

Hence

$$P\left(z_1 < \frac{1}{2}\right) = \int_0^{1/2} 12z_1^2(1 - z_1)dz_1 = 12\int_0^{\frac{1}{2}} z_1^2 - z_1^3\, dz_1$$

$$= 12\left[\frac{z_1^3}{3} - \frac{z_1^4}{4}\right]_0^{\frac{1}{2}} = 12\left[\frac{4z_1^3 - 3z_1^4}{12}\right]_0^{\frac{1}{2}}$$

$$= 4\left(\frac{1}{8}\right) - 3\left(\frac{1}{6}\right) = \frac{8-3}{16} = \frac{5}{16}.$$

Assignment

1. Let $Y_1 < Y_2 < Y_3 < Y_4$ be the order statistics of a random sample of size n=4 from a uniform distribution with pdf $f(x) = 1, 0 < x < 1$. Find the pdf of Y_3 then find

$p\left(\frac{1}{3} < Y_3 < \frac{2}{3}\right)$.

2. Let $X_1, X_2, \dots, X_n$ be a random sample from a U (0,1).

a. Find the pdf of the kth order statistic Y_k.

b. Find the joint pdf of Y_2 and Y_5.

3. Let $Y_1 < Y_2 < Y_3$ be the order statistics of a random sample of size n=3 from a uniform distribution with pdf $f(x) = \frac{1}{\theta}, 0 < x < \theta$. Find

1. The joint pdf of $Y_1, Y_2, and\ Y_3$
2. The marginal pdf of $Y_1\ and\ Y_3$.
3. The joint pdf of $Y_1\ and\ Y_3$.
4. The pdf of the median and the value of the median.

The Moment Generating Function(mgf) Technique

The moment generating function method is based on the following uniqueness theorem.

Theorem

Let $M_X(t)$ and $M_y(t)$ denote the mgf's X and Y, respectively. If both mgf's exist and $M_X(t) = M_Y(t)$ for all values of t, then X and Y have the same pdf.

This method can also be used to find the sum of two or more independent random variables. For example, if X and Y are independent random variables then $M_{X+Y}\,(t) = Ee^{t(X+Y)} = Ee^{tX}.Ee^{tY} = M_X(t).M_y(t)$

Example:

Let $X \sim Poisson(\lambda_1)$ and $Y \sim Poisson(\lambda_2)$. If X and Y are independent, what is the pdf of $Z = X + Y$?

$M_X(t) = E\ e^{tX} = e^{\lambda_1(e^t-1)}$ and $M_Y(t) = E\ e^{tY} = e^{\lambda_2(e^t-1)}$

Further X and Y are independent, then

$$M_{X+Y}\,(t) = M_X(t).M_y(t) = e^{\lambda_1(e^t-1)}\,e^{\lambda_2(e^t-1)}$$

$$= e^{(\lambda_1+\lambda_2)(e^t-1)}$$

That is $X + Y \sim Poisson(\lambda_1 + \lambda_2)$. Hence the pdf of $Z = X + Y$ is

$$h(z) = \begin{cases} \dfrac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^z}{z!}, & z = 0, 1, 2, \ldots \\ 0 & o.w \end{cases}$$

Lecture 12

Example

What is the pdf of the sum of two independent random variables each of which is gamma (α, θ)?

Let $X \sim gamma\ (\alpha, \theta)$ and $Y \sim gamma\ (\alpha, \theta)$

$M_X(t) = (1 - \theta t)^{-\alpha}$ and $M_Y(t) = (1 - \theta t)^{-\alpha}$

Since X and Y are independent

$M_{X+Y}\ (t) = M_X(t)\ M_y(t) = (1 - \theta t)^{-\alpha}\ (1 - \theta t)^{-\alpha} = (1 - \theta t)^{-2\alpha}$

$\therefore X + Y \sim gamma\ (2\alpha, \theta)$

Example

Let $X \sim binomial(n, p)$, find the probability distribution of $Y = n - X$

$M_Y(t) = Ee^{tY} = Ee^{t(n-X)} = e^{nt}\ E\ e^{-tX} = e^{nt}\ M_X(-t)$

Since $M_X(t) = (q + pe^t)^n$ and $q = 1 - p$

$M_X(-t) = (q + pe^{-t})^n$

Hence

$M_Y(t) = (e^t)^n\ (q + pe^{-t})^n = (qe^t + p)^n$

$\therefore Y \sim binomial(n, q)$

Example

Let X_1 and X_2 be independent random variables with $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ respectively. Let $Y = X_1 - X_2$, find the pdf of Y.

$M_Y(t) = Ee^{tY} = Ee^{t(X_1 - X_2)} = Ee^{tX_1}\ Ee^{-tX_2} \qquad X_1, X_2\ independent$

$$= exp\left(\mu_1 t + \frac{\sigma_1^2 t^2}{2}\right)\ exp\left(-\mu_2 t + \frac{\sigma_2^2 t^2}{2}\right)$$

$$= exp\left[(\mu_1 - \mu_2)t + \frac{(\sigma_1^2 + \sigma_2^2)t^2}{2}\right]$$

Hence $Y \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$

Theorem-1

Let $X_1, X_2, \ldots, X_n$ be independent random variables having respectively, the normal distribution $N(\mu_i, \sigma_i^2), i = 1, \ldots, n$. The random variable $Y = a_1X_1 + a_2X_2 + \cdots + a_nX_n$, where $a_1, a_2, \ldots, a_n$ are real constants, is normally distributed with mean $a_1\mu_1 + a_2\mu_2 + \cdots + a_n\mu_n$, and variance $a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \cdots + a_n^2\sigma_n^2$ $i.e\ Y \sim N\left(\sum_{i=1}^n a_i\mu_i,\ \sum_{i=1}^n a_i^2\sigma_i^2\right)$

Proof

$$M_Y(t) = E\ e^{tY} = Ee^{t(a_1X_1 + a_2X_2 + \cdots + a_nX_n)}$$

$$= E\ e^{ta_1X_1}.E\ e^{ta_2X_2} \ldots.E\ e^{ta_nX_n} = \prod_{i=1}^n E\ e^{ta_iX_i} \qquad X_i are\ independent$$

Since $X \sim N(\mu, \sigma^2)$

$$M_X(t) = Ee^{tX} = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$

Hence

$$Ee^{ta_iX_i} = \exp\left(\mu_i(a_it) + \frac{\sigma_i^2(a_it)^2}{2}\right)$$

$$\therefore M_Y(t) = \prod_{i=1}^n \exp\left[(a_i\mu_i)t + \frac{\sigma_i^2(a_it)^2}{2}\right]$$

$$= \exp\left[\left(\sum_{i=1}^n a_i\mu_i\right)t + \frac{\left(\sum_{i=1}^n a_i^2\sigma_i^2\right)t^2}{2}\right]$$

But this is the mgf of a distribution that is $N\left(\sum_{i=1}^{n} a_i\mu_i, \sum_{i=1}^{n} a_i^2\sigma_i^2\right)$. Thus Y has this normal distribution.

The next theorem is a generalization of theorem (1).

Theorem - 2

If $X_1, X_2, \dots, X_n$ are independent random variables with respective mgf's $M_{X_i}(t), i = 1, \dots, n$, then the mgf of $Y = \sum_{i=1}^{n} a_iX_i$, where $a_1, a_2, \dots, a_n$ are real constants, is $M_Y(t) = \prod_{i=1}^{n} M_{X_i}(a_it)$

Proof

$$M_Y(t) = Ee^{tY} = Ee^{t(a_1X_1+a_2X_2+\cdots+a_nX_n)}$$

$$= Ee^{a_1tX_1}Ee^{a_2tX_2} \dots . Ee^{a_ntX_n} \qquad X_i \text{ are independent}$$

Since

$Ee^{tX_i} = M_{X_i}(t)$, also $Ee^{a_itX_i} = M_{X_i}(a_it)$

Thus, we have that

$$M_Y(t) = M_{X_1}(a_1t)\, M_{X_2}(a_2t) \dots .\ M_{X_n}(a_nt) = \prod_{i=1}^{n} M_{X_i}(a_it)$$

Corollary

If $X_1, \dots, X_n$ are observations of a random sample from a distribution with mgf $M_X(t)$, then the mgf of $Y = \sum_{i=1}^{n} a_iX_i$, where $a_1, a_2, \dots, a_n$ are real constants, is $M_Y(t) = \prod_{i=1}^{n} M_{X_i}(a_it)$.

a. Let $a_i = 1, i = 1, \dots, n$, then the mgf of $Y = \sum_{i=1}^{n} X_i$ is
   $M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t) = [M_X(t)]^n$
b. Let $a_i = \frac{1}{n}, i = 1, \dots, n$, then the mgf of $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ is

$M_{\bar{X}}(t) = \prod_{i=1}^{n} M_{X_i}\left(\frac{t}{n}\right) = \left[M_X\left(\frac{t}{n}\right)\right]^n$

Example

Let $X_1, X_2, \dots, X_n$ denote the outcomes of n Bernoulli trials. The mgf of $X_i, i = 1, \dots, n$, is

$M_{X_i}(t) = (1-p) + pe^t = q + pe^t$, where $q = 1 - p$. If $Y = \sum_{i=1}^{n} X_i$, then

$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n}(1 - p + pe^t) = \prod_{i=1}^{n}(q + pe^t) = [q + pe^t]^n$

Hence, $M_Y(t) = [M_X(t)]^n = [q + pe^t]^n$

Thus $Y \sim binomial(n, p)$

Example

Let X_1, X_2, X_3 be the observations of a random sample of size $n = 3$ form the exponential distribution having mean β.

$f(x) = \frac{1}{\beta}\, e^{-x/\beta} \quad x > 0$

$M_X(t) = \frac{1}{1-\beta t}, t < \frac{1}{\beta}$

1.The mgf of $Y = X_1 + X_2 + X_3$ is

$M_Y(t) = [M_X(t)]^n = [(1 - \beta t)^{-1}]^3 = (1 - \beta t)^{-3}$

Which is that of a gamma distribution with $\alpha = 3$ and β i.e $Y \sim gamma(3, \beta)$

2.The mgf of $\bar{X} = (X_1 + X_2 + X_3)/3$ is

$M_{\bar{X}}(t) = \left[M_X\left(\frac{t}{n}\right)\right]^n = \left[\left(1 - \frac{\beta t}{3}\right)^{-1}\right]^3 = \left(1 - \frac{\beta t}{3}\right)^{-3}, t < 3/\beta$

Hence $\bar{X} \sim gamma(3, \beta/3\ )$.

Lecture 13

Theorem - 3

If $X_1, X_2, \dots, X_n$ are observations of a random sample of size n from the normal distribution $N(\mu, \sigma^2)$, then the distribution of the sample mean

$\bar{X} = \sum_{i=1}^{n} X_i / n$ is $N(\mu, \sigma^2/n)$.

Proof

$M_X(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$. From theorem (2)

$$M_{\bar{X}}(t) = \left[M_X\left(\frac{t}{n}\right)\right]^n = \left\{\exp\left[\mu\left(\frac{t}{n}\right) + \frac{\sigma^2 (t/n)^2}{2}\right]\right\}^n$$

$$= \exp\left\{\mu t + \frac{(\sigma^2/n)t^2}{2}\right\}$$

Hence $\bar{X} \sim N(\mu, \sigma^2/n)$

Theorem - 4

Let $X_1, X_2, \dots, X_n$ be independent random variables that have respectively the chi-square distributions $\chi^2_{(r_1)}, \chi^2_{(r_2)}, \dots, \chi^2_{(r_n)}$. Then the random variable $Y = X_1 + X_2 + \dots + X_n$ has a chi-square distribution with $r_1 + r_2 + \dots + r_n$ degrees of freedom. That is $Y \sim \chi^2(r_1 + r_2 + \dots + r_n)$.

Proof

$$M_Y(t) = Ee^{tY} = Ee^{t(X_1 + X_2 + \dots + X_n)} = E\, e^{tX_1} e^{tX_2} \dots e^{tX_n}$$

$$= E\, e^{tX_1} Ee^{tX_2} \dots Ee^{tX_n} \qquad X_i \text{ are independent}$$

$$= (1-2t)^{-\frac{r_1}{2}} (1-2t)^{-\frac{r_2}{2}} \dots (1-2t)^{-\frac{r_n}{2}} \qquad , \ t < \frac{1}{2}$$

Thus

$M_Y(t) = (1-2t)^{-(r_1+r_2+\cdots+r_n)/2}$

But this is the mgf of a distribution that is $\chi^2(r_1 + r_2 + \cdots + r_n)$. Accordingly, $Y \sim \chi^2(\sum_{i=1}^{n} r_i)$

Example

Let the random variable $Z \sim N(0,1)$. Use the method of mgf to find the pdf of Z^2.

$$M_{Z^2}(t) = Ee^{tZ^2} = \int_{-\infty}^{\infty} e^{tz^2} f(z)dz = \int_{-\infty}^{\infty} e^{tz^2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2(\frac{1}{2}-t)}\, dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}(1-2t)}\, dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2(1-2t)^{-1}}}\, dz$$

$$= \frac{1}{(1-2t)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,(1-2t)^{-\frac{1}{2}}} e^{-\frac{z^2}{2(1-2t)^{-1}}} dz$$

The integrand of the integral is a normal pdf with mean zero and variance $(1-2t)^{-1}$ and the integral is equal to one. Hence

$$M_{Z^2}(t) = \frac{1}{(1-2t)^{1/2}} = (1-2t)^{-\frac{1}{2}}$$

$\therefore Z^2 \sim gamma\left(\frac{1}{2}, 2\right) or\ \chi^2_{(1)}$. And for $Y = Z^2$

$$f_Y(y) = \begin{cases} \dfrac{y^{\frac{1}{2}-1}\, e^{-y/2}}{\Gamma\left(\frac{1}{2}\right)(2)^{\frac{1}{2}}} & y \geq 0 \\ 0 & o.w \end{cases}$$

Theorem - 5

Let $X_1, X_2, \ldots, X_n$ denote a random sample of size n from a distribution that is $N(\mu, \sigma^2)$. Then the random variable $Y = \sum_{i=1}^{n} \left(\frac{X_i-\mu}{\sigma}\right)^2$ has a chi- square distribution with n degrees of freedom.

Proof

Recall that if the random variable $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$, then $Z^2 \sim \chi^2_{(1)}$.

Since X_i's are independent. Hence by theorem (4) with $r_i = 1, i = 1, \dots n$ the random variable $Y = \sum_{i=1}^{n} Z_i^2 = \sum_{i=1}^{n} \left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2_{(n)}$

Example

Let X_1 and X_2 be two independent standard normal random variables. Let $Y_1 = X_1 + X_2$ and $Y_2 = X_2 - X_1$. Use the mgf method to find the joint pdf of Y_1 and Y_2.

$$M_{(Y_1,Y_2)}(t_1,t_2) = Ee^{Y_1t_1+Y_2t_2} = Ee^{(X_1+X_2)t_1+(X_2-X_1)t_2}$$

$$= Ee^{X_1t_1+X_2t_1+ X_2t_2-X_1t_2}$$

$$= Ee^{(t_1-t_2)X_1}\, Ee^{(t_1-t_2)X_2} \qquad X_1 and\ X_2\ are\ independent$$

$$= M_{X_1}(t_1 - t_2). M_{X_2}(t_1 + t_2)$$

Since $X_1\ and\ X_2 \sim N(0,1)$, we have $M_X(t) = \exp\left(\frac{t^2}{2}\right)$

$$M_{(Y_1,Y_2)}(t_1,t_2) = \exp\left[\frac{(t_1-t_2)^2}{2}\right]. \exp\left[\frac{(t_1+t_2)^2}{2}\right]$$

$$= \exp\left(\frac{t_1^2-2t_1t_2+t_2^2+t_1^2+2t_1t_2+t_2^2}{2}\right)$$

$$= \exp\left(\frac{2t_1^2+2t_2^2}{2}\right) = \exp\left(\frac{2t_1^2}{2}\right) . \exp\left(\frac{2t_2^2}{2}\right)$$

$$= M_{Y_1}(t_1)\, M_{Y_2}(t_2)$$

Hence Y_1 and Y_2 are independent random variables and each $\sim N(0, 2)$

Chapter Two

Limiting Distributions

Sequences of Random Variables

We denote a sequence of random variables $X_1, X_2, \ldots$ by $\{X_n\}_{n=1}^{\infty}$, with a corresponding sequence of distribution functions $F_n(x) = P(X_n \leq x)$ for each $n = 1,2, \ldots$. The subscript n make the dependence on the sample size n more explicit.

When the distribution of a random variable depends upon a positive integer n, clearly the pdf, cdf and mgf are all depend upon n. For example

- If the random variable $X \sim b(n,p)$, then $f(x)$, $F(x)$ and $M_X(t)$ are all involve n
- If $\bar{X}$ is the mean of a random sample of size n from a distribution that is $N(\mu, \sigma^2)$, then $\bar{X} \sim N(\mu, \sigma^2/n)$ depends upon n.

Also, the distribution of the random variable $Y = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$ depends upon n, where S^2 is the sample variance of this random sample from the normal distribution.

In the previous chapter we considered various methods of determining the distribution of a function of random variables, but sometimes, we may face difficulties in using a particular method.

Example

If $\bar{X}$ is the mean of a random sample of size n from $U(0,1)$ distribution, then

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & o.w \end{cases}$$

The mgf of X is given by $M_X(t) = Ee^{tX} = \int_0^1 e^{tx} f(x)dx = \frac{e^t - 1}{t}, t \neq 0$

$$= 1 \qquad , t = 0$$

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The mgf of $\bar{X}$ is

$$M_{\bar{X}}(t) = E\left(e^{t\bar{X}}\right) = \left[M_X\left(\frac{t}{n}\right)\right]^n = \left[\frac{e^{\frac{t}{n}}-1}{\frac{t}{n}}\right]^n \qquad ,t \neq 0$$

$$= 1 \qquad ,t = 0$$

Since $M_{\bar{X}}(t)$ depends upon n, the distribution of $\bar{X}$ depends upon n. But the pdf of $\bar{X}$ could not be easily derived. Hence, one of the purposes of limiting distributions is to approximate, for large values of n, some of the complicated pdf's.

Convergence in distribution

Definition

The sequence of random variables $\{X_n\}_{i=1}^{\infty}$ is said to converge in distribution to the random variable X if: $\lim_{n\to\infty} F_n(x) = F(x)$

for all values x at which $F(x)$ is continuous. The distribution of X is called the limiting distribution of X_n. Or $X_n \xrightarrow{D} X$.

Note that by saying $X_n \xrightarrow{D} X$, we mean that the distribution of X is the asymptotic distribution or the limiting distribution of the sequence $\{X_n\}$. Or we may say that X_n has a limiting distribution with distribution function $F(x)$.

Example

Let $X_1, X_2, \dots, X_n$ be a random sample from $U(0,\theta)$ and let Y_n be the nth order statistic. Find the limiting distribution of Y_n.

$$f(x) = \begin{cases} \frac{1}{\theta} & 0 < x < \theta,\ \theta > 0 \\ 0 & o.w. \end{cases}$$

The pdf of Y_n is $g_n(y_n) = n[F(y_n)]^{n-1} f(y_n) = n\left(\frac{y_n}{\theta}\right)^{n-1} \frac{1}{\theta}$

$$g_n(y_n) = \begin{cases} \frac{n y_n^{n-1}}{\theta^n} & 0 < y_n < \theta \\ 0 & o.w. \end{cases}$$

The distribution function of Y_n is

$$F_n(y_n) = \begin{cases} 0 & y_n < 0 \\ \int_0^{y_n} \frac{nz^{n-1}}{\theta^n}\, dz = \left(\frac{y_n}{\theta}\right)^n & 0 \leq y_n < \theta \\ 1 & \theta \leq y_n < \infty \end{cases}$$

Since $y_n < \theta$,

$$\lim_{n\to\infty} F_n(y_n) = \begin{cases} 0 & -\infty < y_n < \theta \\ 1 & \theta \leq y_n < \infty \end{cases}$$

Now,

$$\text{F(y)} = \begin{cases} 0 & -\infty < y < \theta \\ 1 & \theta \leq y < \infty \end{cases}$$

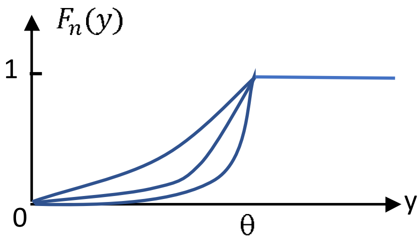
is a distribution function, and $\lim_{n\to\infty} F_n(y_n) = F(y)$ at each point of continuity of $F(y)$. Thus $Y_n,\ n = 1,2,\ldots \xrightarrow{D} Y$ a random variable that has a degenerate distribution at the point $y = \theta$.

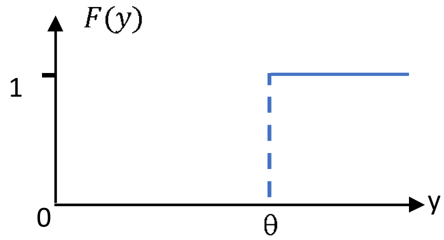
Definition

The function $F(y)$ is the distribution function of a degenerate distribution at the value $y = c$ if

$$F(y) = \begin{cases} 0 & y < c \\ 1 & y \geq c \end{cases}$$

That is; $F(y)$ is the distribution function of a discrete distribution that assigns probability one at the value $y = c$ and zero otherwise.





Example

Let $X_1, X_2, \ldots, X_n$ be a random sample from a standard normal $N(0,1)$, then $\bar{X}_n \sim N\left(0, \frac{1}{n}\right)$. Find the limiting distribution of $\bar{X}$.

The distribution function of $\bar{X}$ is

$$F_n(\bar{x}) = \int_{-\infty}^{\bar{x}} \frac{1}{\sqrt{2\pi}\sqrt{1/n}} e^{-nw^2/2}\, dw$$

Let $v = \sqrt{n}w$ then $dv = \sqrt{n}\, dw$

Hence, $F_n(\bar{x}) = \int_{-\infty}^{\sqrt{n}\bar{x}} \frac{1}{\sqrt{2\pi}} e^{-v^2/2} dv$

It is clear that

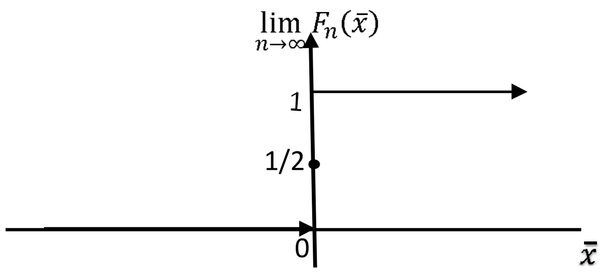
$$\lim_{n\to\infty} F_n(\bar{x}) = \begin{cases} 0 & \bar{x} < 0 \\ \frac{1}{2} & \bar{x} = 0 \\ 1 & \bar{x} > 0 \end{cases}$$

The function

$$F(\bar{x}) = \begin{cases} 0 & \bar{x} < 0 \\ 1 & \bar{x} \geq 0 \end{cases}$$

Is a distribution function and $\lim_{n\to\infty} F_n(\bar{x}) = F(\bar{x})$ at every point of continuity of $F(\bar{x})$. (Note that $F(\bar{x})$ is not continuous at $\bar{x} = 0$)

Accordingly, the sequence $\{\bar{X}_n\}_{i=1}^{\infty}$ converges in distribution to a random variable that has a degenerate distribution at $\bar{x} = 0$.



Example

Let $X_1, X_2, \dots, X_n$ be a random sample from $U(0, \theta)$ and let Y_n be the nth order statistic. If $Z_n = n(\theta - Y_n)$, find the limiting distribution of Z_n.

$$g_n(y_n) = n\left(\frac{y_n}{\theta}\right)^{n-1} \frac{1}{\theta} \qquad 0 \leq y_n < \theta$$

$$Z_n = n(\theta - Y_n) \Longrightarrow \frac{Z_n}{n} = \theta - Y_n$$

$$\therefore Y_n = \theta - \frac{Z_n}{n}$$

$$J = \frac{\partial y}{\partial z_n} = -\frac{1}{n}$$

$$|J| = \left|-\frac{1}{n}\right| = \frac{1}{n}$$

The pdf of Z_n is

$$h_n(z_n) = n\left(\frac{\theta - \frac{z_n}{n}}{\theta}\right)^{n-1} \frac{1}{n\theta} = \frac{1}{\theta^n}\left(\theta - \frac{z_n}{n}\right)^{n-1} \qquad 0 \le z_n < n\theta$$

And the distribution function of Z_n is

$$G_n(z_n) = \int_0^{z_n} \frac{1}{\theta^n}\left(\theta - \frac{w}{n}\right)^{n-1} dw = -\frac{n}{\theta^n} \int_0^{z_n} \left(\theta - \frac{w}{n}\right)^{n-1} - \frac{1}{n}\, dw$$

$$= -\frac{n}{\theta^n} \frac{\left[\theta - \frac{w}{n}\right]^n}{n}\Bigg]_0^{z_n} = -\left[\left(\frac{\theta - \frac{z_n}{n}}{\theta}\right)^n - \left(\frac{\theta}{\theta}\right)^n\right]$$

$$= 1 - \left(1 - \frac{z_n}{n\theta}\right)^n \qquad 0 \le z_n < n\theta$$

$$\therefore G_n(z_n) = \begin{cases} 0 & z < 0 \\ 1 - \left(1 - \frac{z_n}{n\theta}\right)^n & 0 \le z_n < n\theta \\ 1 & n\theta \le z_n \end{cases}$$

Hence

$$\lim_{n\to\infty} G_n(z_n) = \begin{cases} 0 & z_n < 0 \\ 1 - e^{-\frac{z_n}{\theta}} & 0 \le z_n < \infty \end{cases}$$

$$Recall\ that: \lim_{n\to\infty}\left(1 - \frac{z/\theta}{n}\right)^n = e^{-z/\theta}$$

Now

$$G(z) = \begin{cases} 0 & z < 0 \\ 1 - e^{-z/\theta} & 0 < z \end{cases}$$

is a distribution function that is everywhere continuous and $\lim_{n\to\infty} G_n(z_n) = G(z)$ at all points of continuity of $G(z)$.

Thus Z_n has a limiting distribution with distribution function $G(z)$; i.e., $Z_n \xrightarrow{D} Z$, where Z is an exponentially distributed random variable.

Convergence in Probability

Theorem Markov Inequality

If X is a random variable that takes only nonnegative values, then for any value $t > 0$

$$p(X \geq t) \leq \frac{E(X)}{t}$$

Proof

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

$$= \int_{-\infty}^{t} xf(x)dx + \int_{t}^{\infty} xf(x)dx$$

$$\geq \int_{t}^{\infty} x\, f(x)dx$$

$$\geq \int_{t}^{\infty} t\, f(x)dx \qquad because\ \ x \in [t, \infty)$$

Hence, E(X)$\geq t\int_{t}^{\infty} f(x)dx = tP(X \geq t)$

And

$$P(X \geq t) \leq \frac{E(X)}{t}$$

Theorem: Chebyshev's Inequality

Let X be a random variable with mean μ and variance σ^2, then for any value $k > 0$

$$p(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

Proof

By Markov inequality, we have $p((X-\mu)^2 \geq t^2) \leq \frac{E(X-\mu)^2}{t^2} \quad for\ all\ t > 0$

Since $(X-\mu)^2 \geq t^2$ if and only if $|X - \mu| \geq t$, we get

$$p((X-\mu)^2 \geq t^2) = p(|X-\mu| \geq t) \leq \frac{E(X-\mu)^2}{t^2} \qquad for\ all\ t > 0$$

Hence $P(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2}$

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Letting $t = k\sigma$, we see that

$$P(|X-\mu| \geq k\sigma) \leq \frac{\sigma^2}{k^2\sigma^2} = \frac{1}{k^2}$$

Hence $[1 - P(|X-\mu| < k\sigma)] \leq \frac{1}{k^2}$

Or

$$P(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

Definition: Convergence in Probability

A sequence of random variables $X_1, X_2, \ldots$ converges in probability to a random variable X if, for every $\epsilon > 0$,

$$\lim_{n\to\infty} P(|X_n - X| < \epsilon) = 1$$

Or equivalently $\lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = 0$

That is, we say that $X_n \xrightarrow{P} X$ if one of the above limits is true.

Remark:

$\lim_{n\to\infty} P(|X_n - c| \geq \epsilon) = 0$ is often used for the convergence of a random variable X_n to a constance c and we write $X_n \xrightarrow{P} c$

Theorem: The Weak Law of Large Numbers

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with $\mu = E(X_i)$ and $\sigma^2 = Var\,(X_i) < \infty$ for $i = 1,2,\ldots\infty$.

Then

$$\lim_{n\to\infty} P\left(\left|\frac{X_1+X_2+\cdots+X_n}{n} - \mu\right| \geq \epsilon\right) = 0 \qquad \text{for every } \epsilon > 0$$

Or equivalently, $\bar{X}_n \xrightarrow{P} \mu$

Proof

Let $\bar{X}_n = \frac{X_1+X_2+\cdots+X_n}{n}$

Recall that $E(\bar{X}_n) = \mu$ and $Var(\bar{X}_n) = \frac{\sigma^2}{n}$

By Chebyshev's inequality

$P(|\bar{X}_n - \mu| \geq \epsilon) \leq \frac{Var(\bar{X}_n)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2}$

Taking the limit as $n \to \infty$

$\lim_{n\to\infty} P(|\bar{X}_n - \mu| \geq \epsilon) \ \leq \lim_{n\to\infty} \frac{\sigma^2}{n\epsilon^2}$

Which yields

$\lim_{n\to\infty} P(|\bar{X}_n - \mu| \geq \epsilon) = 0$

Hence $\bar{X}_n, n = 1,2,3, \dots$ converges in probability to μ if σ^2 is finite which is written as $\bar{X}_n \xrightarrow{P} \mu$.

The weak law of large numbers states that the sample mean $\bar{X}$ converges in probability to the population mean μ when n is large and $0 < \sigma^2 < \infty$.

Definition: The Strong Law of Large Numbers

Let $X_1, X_2, \dots$ be a sequence of independent and identically distributed random variables with a finite mean $E(X_i) = \mu$ for $i = 1,2, \dots \infty$. Then

$P(\lim_{n\to\infty} \bar{X}_n = \mu) = 1$

In other words, as n approaches infinity $\bar{X}_n$ converge to μ with probability 1. This type of convergence is called almost sure convergence.

Example

Let $Y_n \sim b(n,p)$, show that $\frac{Y_n}{n} \xrightarrow{P} p$

$P\left(\left|\frac{Y_n}{n} - p\right| \geq \epsilon\right) = P(|Y_n - np| \geq n\epsilon)$

$$= P\left(|Y_n - np| \geq \frac{n\epsilon}{\sigma}\sigma\right) \ \leq \ \frac{1}{\left(\frac{n\epsilon}{\sigma}\right)^2} = \frac{\sigma^2}{n^2\epsilon^2}$$

$$\lim_{n\to\infty} P\left(\left|\frac{Y_n}{n} - p\right| \geq \epsilon\right) = \lim_{n\to\infty} \frac{npq}{n^2\ \epsilon^2}$$

$$= \frac{pq}{\epsilon^2} \lim_{n\to\infty} \frac{1}{n} = 0$$

Hence, $\frac{Y_n}{n} \xrightarrow{P} p$.

The Central Limit Theorem (C.L.T)

The central limit theorem is one of the most important results in probability. We have seen earlier that if $X_1, X_2, \dots, X_n$ is a random sample from $N(\mu, \sigma^2)$, then $\bar{X} \sim N(\mu, \sigma^2/n)$, and as n increases, the variance of $\bar{X}$ decreases. Consequently, the distribution of $\bar{X}$ depends on n. If we let $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$, then $Z \sim N(0,1)$. The C.L.T states that even though the population distribution is far from begin normal, still for large sample size n, the distribution of the standardized sample mean is approximately standard normal.

Theorem: C.L.T

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a distribution with mean μ and finite positive variance σ^2. Then the random variable

$$Y_n = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} = \frac{\sum X_i - n\mu}{\sqrt{n}\,\sigma}$$

Has a limiting distribution that is $N(0,1)$. That is

$$\lim_{n\to\infty} P(Y_n \le y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2} dt$$

A practical use of the C.L.T is approximating. Usually, a value of $n > 30$ will ensure that the distribution of Y_n can be closely approximated by a normal distribution; namely

$$P(Y_n \le y) \approx \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{t^2}{2}} dt = \Phi(y)$$

Example

Let $\bar{X}$ denote the mean of a random sample of size $n = 75$ from $U(0,1)$. Approximate $P(0.45 < \bar{X} < 0.55)$.

For the uniform distribution, $E(X) = \mu = \frac{1}{2}$, $Var(X) = \sigma^2 = \frac{1}{12}$.

The approximate value of

$$P(0.45 < \bar{X} < 0.55) = P\left[\frac{\sqrt{n}(0.45-\mu)}{\sigma} < \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} < \frac{\sqrt{n}(0.55-\mu)}{\sigma}\right]$$

$$= P\left[\frac{\sqrt{75}(0.45-0.50)}{1/\sqrt{12}} < \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} < \frac{\sqrt{75}(0.55-0.50)}{1/\sqrt{12}}\right]$$

$$= P(30(-0.05) < Z < 30(0.05))$$

$$= P(-1.5 < Z < 1.5) = \Phi(1.5) - \Phi(-1.5) = \Phi(1.5) - [1 - \Phi(1.5)]$$

$$= 2\Phi(1.5) - 1 = 2(0.9332) - 1$$

$$= 1.8664 - 1 = 0.8664$$

Example

Let $\bar{X}$ denote the mean of a random sample of size $n = 15$ from a distribution whose pdf is $f(x) = \frac{3}{2}x^2;\ -1 < x < 1$. Approximate $P(0.03 \le \bar{X} \le 0.15)$.

$$\mu = E(X) = \int_{-1}^{1} x\left(\frac{3}{2}x^2\right)dx = \frac{3}{2}\frac{x^4}{4}\Big]_{-1}^{1} = \frac{3}{8}[1-1] = 0$$

$$E(X^2) = \int_{-1}^{1} x^2\left(\frac{3}{2}x^2\right)dx = \frac{3}{2}\frac{x^5}{5} = \frac{3}{10}[1+1] = \frac{3}{5}$$

$$\therefore Var(X) = E(X^2) - [E(X)]^2 = \frac{3}{5}$$

$$P(0.03 \le \bar{X} \le 0.15) = P\left(\frac{0.03-0}{\sqrt{3/75}} \le \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \le \frac{0.15-0}{\sqrt{3/75}}\right)$$

$$= P(5(0.03) \le Z \le 5(0.15)) = P(0.15 \le Z \le 0.75)$$

$$= \Phi(0.75) - \Phi(0.15) = 0.7743 - 0.5596 = 0.2138$$

Example

Let $X_1, X_2, \dots, X_n$ be a random sample of size $n = 100$ from $b\left(1, \frac{1}{2}\right)$. Approximate $P(48 < \sum X_i < 52)$.

We have $\mu = E(X) = \frac{1}{2}$, and $\sigma^2 = Var(X) = p(1-p) = \frac{1}{4}$

Since $X \sim b\left(1, \frac{1}{2}\right)$ then $\sum X_i \sim b\left(100, \frac{1}{2}\right)$

$$P(48 < \sum X_i < 52) = P\left(\frac{(48-0.5) - 100\left(\frac{1}{2}\right)}{\sqrt{100}\sqrt{\frac{1}{4}}} < \frac{\sum X_i - n\mu}{\sqrt{n}\sigma} < \frac{(52+0.5) - 100\left(\frac{1}{2}\right)}{\sqrt{100}\sqrt{\frac{1}{4}}}\right)$$

$$= P\left(\frac{47.5-50}{5} < \frac{\sum X_i - n\mu}{\sqrt{n}\sigma} < \frac{52.5-50}{5}\right) = P(-0.5 < Z < 0.5)$$

$$= \Phi(0.5) - \Phi(-0.5) = \Phi(0.5) - [1 - \Phi(0.5)]$$

$$= 2\Phi(0.5) - 1 = 2(0.691) - 1 = 0.382$$

Some Useful Theorems on Limiting Distributions

1. If the random variable $U_n \xrightarrow{P} c$, then $\frac{U_n}{c} \xrightarrow{P} 1 \qquad c \neq 0$
2. If the random variable $U_n \xrightarrow{P} c$, then $\sqrt{U_n} \xrightarrow{P} \sqrt{c} \qquad c > 0$
3. If the random variable $U_n \xrightarrow{P} c$, and the random variable $V_n \xrightarrow{P} d$, then

- $U_n + V_n \xrightarrow{P} c + d$
- $\frac{U_n}{V_n} \xrightarrow{P} \frac{c}{d} \qquad d \neq 0$
- $U_n . V_n \xrightarrow{P} c . d$

4. If the random variable U_n has a limiting distribution and the random variable $V_n \xrightarrow{P} 1$, then $W_n = \frac{U_n}{V_n}$ has a limiting distribution as that of U_n.

Lemma

Let $X_1, X_2, \dots X_n$ be a random sample of size n from X with EX^{2k} exists, then $\frac{1}{n}\sum_{i=1}^{n} X_i^k \xrightarrow{P} EX^k, k = 1,2,3, \dots$

Lemma

Let $X_1, X_2, \dots X_n$ be a random sample of size n fom X with $E(X^4)$ exists and $Var\,(X) = \sigma^2$, then

1. $S_n^2 \xrightarrow{P} \sigma^2$ where $S_n^2 = \frac{1}{n}\sum(X_i - \bar{X})^2$
2. $S_{n-1}^2 \xrightarrow{P} \sigma^2$ where $S_{n-1}^2 = \frac{1}{n-1}\sum(X_i - \bar{X})^2$

Proof

1. $S_n^2 = \frac{1}{n}\sum X_i^2 - \bar{X}_n^2$

   Since $\frac{1}{n}\sum X_i^2 \xrightarrow{P} E(X^2)$ and $\bar{X}_n = \frac{1}{n}\sum X_i \xrightarrow{P} E(X)$

   Hence $(\bar{X}_n)^2 \xrightarrow{P} [E(X)]^2$

   Then

   $$S_n^2 = \frac{1}{n}\sum X_i^2 - \bar{X}_n^2 \xrightarrow{P} E(X^2) - [E(X)]^2 = \sigma^2$$

   $\therefore S_n^2 \xrightarrow{P} \sigma^2$

2. $S_{n-1}^2 = \frac{1}{n-1}\sum(X_i - \bar{X})^2 = \frac{n}{n-1}\frac{1}{n}\sum(X_i - \bar{X})^2 = \frac{n}{n-1}S_n^2$

   Since

   $\frac{n}{n-1} \xrightarrow{P} 1$ as $n \to \infty$ then $S_{n-1}^2 = \frac{n}{n-1}S_n^2 \xrightarrow{P} 1.\sigma^2$

   Hence, $S_{n-1}^2 \xrightarrow{P} \sigma^2$

Theorem

Let $X_1, X_2, \dots, X_n$ be a random sample from X with $E(X) = \mu$ and $Var(X) = \sigma^2$. Then

$T_n = \frac{\bar{X}_n - \mu}{s/\sqrt{n}} \sim N(0,1)$ as $n \to \infty$

Proof

By the C.LT $\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$ as $n \to \infty$.

Since $S^2 \xrightarrow{P} \sigma^2$ as $n \to \infty$

and $\frac{S^2}{\sigma^2} \xrightarrow{P} 1$ as $n \to \infty$

and $\sqrt{\frac{S^2}{\sigma^2}} \xrightarrow{P} 1$ as $n \to \infty$

Then

$$T_n = \frac{\bar{X}_n - \mu/\sigma/\sqrt{n}}{\sqrt{s^2/\sigma^2}} = \frac{\bar{X}_n - \mu}{s/\sqrt{n}} \sim N(0,1) \text{ as } n \to \infty$$

Theorem

Let $X_1, X_2, \dots, X_n$ and $Y_1, Y_2, \dots, Y_m$ be the items of two independent random samples of sizes n and m with $E(X) = \mu_x$, $E(Y) = \mu_y$, $Var(X) = \sigma_x^2$ and $Var(Y) = \sigma_y^2$. Then

1. $\frac{(\bar{X}_n - \bar{Y}_m) - (\mu_x - \mu_y)}{\sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}} \sim N(0,1)$ as $n, m \to \infty$

2. $\frac{(\bar{X}_n - \bar{Y}_m) - (\mu_x - \mu_y)}{\sqrt{\frac{S_x^2}{n} + \frac{S_y^2}{m}}} \sim N(0,1)$ as $n, m \to \infty$

Proof

1. By the C.L.T $\bar{X}_n \sim N\left(\mu_x, \frac{\sigma_x^2}{n}\right)$ as $n \to \infty$

and $\bar{Y}_m \sim N\left(\mu_y, \frac{\sigma_y^2}{m}\right)$ as $m \to \infty$

Then $\bar{X}_n - \bar{Y}_m \sim N\left(\mu_x - \mu_y, \frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}\right)$ as $n, m \to \infty$

Hence $\frac{(\bar{X}_n - \bar{Y}_m) - (\mu_x - \mu_y)}{\sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}} \sim N(0,1)$ as $n, m \to \infty$

2. We have already shown that

$$\frac{(\bar{X}_n - \bar{Y}_m) - (\mu_x - \mu_y)}{\sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}} \sim N(0,1) \text{ as } n, m \to \infty$$

And since $\sqrt{\frac{s_x^2}{n} + \frac{s_y^2}{m}} \Big/ \sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}} \xrightarrow{P} 1$ as $n, m \to \infty$

We have that

$$\frac{(\bar{X}_n - \bar{Y}_m) - (\mu_x - \mu_y) \Big/ \sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}}{\sqrt{\frac{S_x^2}{n} + \frac{S_y^2}{m}} \Big/ \sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}} \sim N(0,1) \text{ as } n, m \to \infty.$$